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THE LOCAL PRODUCT STRUCTURE OF EXPANSIVE AUTOMORPHISMS OF SOLENOIDS AND THEIR ASSOCIATED C^*-ALGEBRAS

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ABSTRACT. An explicit description of a hyperbolic canonical coordinate system for an expansive automorphism of a compact connected abelian group is given. These dynamical systems are factors of subshifts of finite type. Some properties of the associated crossed product C^*-algebra are discussed. In these examples, the C^*-algebras of Ruelle are crossed product algebras.

1. **Introduction.** A further investigation of the dynamical systems underlying a certain family of crossed product C^*-algebras provides our motivation for this paper. Various families of crossed product C^*-algebras arising from automorphisms of compact connected abelian groups, are introduced and studied in [3, 4, 5]. The (typical) algebra occuring in these families (and the natural extension of these families described below) is separable, amenable, non-type I (in fact antiliminal), finite and embeddable in an AF-algebra. It is known that the K-groups of these algebras are, in general, not sufficient to distinguish their isomorphism classes, as the ideal structure must play a role. The algebras possess a separating family of finite dimensional quotients and a count of the finite dimensional irreducible representations provides an isomorphism invariant related to the entropy of the given group automorphism ([10, 3, 4, 5]). These algebras may also be described in terms of a universal property involving unitary generators. Here a connection with aspects of 'wavelet theory' appears with potentially interesting points of view.

In the following we explicitly describe a local product structure—a topological Smale space structure—for expansive automorphisms of compact connected (abelian) groups. Many of the above stated properties of the crossed product algebras formed from these systems follow from this observation. As shown in [17, 19] there are several important C^*-algebras formed from various equivalence relations that one can associate to such a system in addition to the usual crossed-product algebra. We point out that the dynamical systems considered here naturally include both the prototype Anasov diffeomorphism of the 2-torus and the solenoid example considered in [17]. Note that the dimension of the group can be an arbitrary finite number.

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We also point out that the dynamical systems we consider are viewed in [11] as examples of higher dimensional analogues of subshifts of finite type, where the symbol space may be a compact Lie group (in our case, a finite product of copies of the circle $\mathbb{T}$). A reference to [8] is made to conclude that the expansive systems considered have a local product structure. To strictly apply [8] one seems to require that the dynamical system is a factor of a subshift of finite type, which is not immediately clear here. That it is such a factor does however follow (from [8]) once we know that the dynamical system has a local product structure. In any case, more is shown. By describing a system of canonical coordinates along with a metric which expands and contracts on the appropriate sets, a Smale space structure is explicitly given.

The following is an overview of the paper (with detailed references found in the main body of the text). With a pair of nonsingular integral $d \times d$ matrices, F and M, is associated a subshift automorphism of an (explicitly defined) compact, abelian, and finite dimensional group G. Restricting the automorphism to the connected component, K, of the identity of G, one obtains a model for a class of automorphisms of solenoids which include all expansive automorphisms of compact, connected (abelian) groups. The identical dynamical system K may be obtained from more than one system G, indeed, the rational matrix $Q = M^{-1}F$ determines K. The point of view taken is that both G and K are principle fibre bundles (over the same base space, the d-torus) with O-dimensional groups as fibres. Using this viewpoint it is seen that K is a Bohr compactification of $\mathbb{R}^d$ (d-dimensional real space) with fibre a Bohr compactification of $\mathbb{Z}^d$. The local bundle charts later provide a convenient vantage point to view the product structure. The characteristic class of the bundle K is also identified. That this context is a natural extension of the results in [5] is explicitly seen after the dual group $\hat{K}$ is computed. We also consider the (dual) dynamical system defined by a complex matrix whose spectrum consists of algebraic numbers.

In Section 3, by considering the smallest sublattice of $\mathbb{Z}^d$ invariant under both Q and Q^{-1}, we obtain a basis of $\mathbb{Z}^d$ in which the matrix F has a useful block upper triangular form. This leads to some easily verifiable conditions that assist in determining whether the fibre of K is a finite group (so K is a covering space) or a Cantor group. Conditions on the matrices F and M which ensure that G is connected (*i.e.*, $G = K$) are also examined, which is useful, in as far as G is explicitly defined.

In Section 4, the bundle charts are used to show the existence of a local product structure for hyperbolic matrices Q. As expected the group structure on K is reflected in some additional coordinate like behaviour. The local charts also enable an intuitive description of the local stable and unstable sets.

In conjunction with some ergodic type results for our expansive systems, the existence of a product structure has some particularly nice consequences: the dynamical systems occur as elementary parts of the Bowen spectral decomposition and so have both the shadowing property and the specification property. The later in turn implies that the entropy of an expansive automorphism of a compact, connected, abelian group is given by the growth rate of the number of periodic points, giving an alternate approach to

one of the results in [15], and also yielding isomorphism invariants for the associated crossed product algebra. Along with the crossed product algebra one can associate many other C^*-algebras with our dynamical system. We consider the stable, unstable and asymptotic algebras (which, in our context, are also crossed product algebras) and show, in certain cases, that we obtain examples of a class of algebras of much recent interest, namely purely infinite, simple, unital, nuclear, algebras. A potentially interesting pair of (isomorphic) crossed product algebras defined by the characteristic class of the bundle K is also considered. These later algebras are also direct limits of finite dimensional algebras over the base space.

In the final section a universal property for the crossed product algebra $C(K) \rtimes \mathbb{Z}$ is given. One consequence, for example, is an identification of the universal C^*-algebra generated by unitaries satisfying classical wavelet identities with the crossed product algebra we consider. We also identify the quotient of a Laurent polynomial ring, considered in earlier work, with rings occuring in our present context, thus allowing an extension of earlier results.

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NOTATION. Denote the nonnegative integers, positive integers, integers, rationals, reals, complexes and the unit circle in the complex plane by $\mathbb{N}$, $\mathbb{N}_+$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{T}$ respectively. For $d \in \mathbb{N}$ identify $\mathbb{T}^d$ (with unit e) with $\mathbb{R}^d/\mathbb{Z}^d$ by $(e^{2\pi i x_1}, \ldots, e^{2\pi i x_d}) = q(x_1, \ldots, x_d)$ where $q: \mathbb{R}^d \longrightarrow \mathbb{R}^d/\mathbb{Z}^d$ is the natural quotient map. The group operation in $\mathbb{T}^d$ is therefore written additively. An endomorphism of $\mathbb{T}^d$ is viewed as an element of $\mathcal{M}_d(\mathbb{Z})$, the $d \times d$ matrix ring over $\mathbb{Z}$. An element $A \in \mathcal{M}_d(\mathbb{Z})$ maps $q(x)$ to $q(Ax)$, $x \in \mathbb{R}^d$. For R a ring and $A \in \mathcal{M}_d(R)$, $\operatorname{char}(A)$ denotes the characteristic polynomial of A, $\det(\lambda - A)$, an element of $R[\lambda]$. The transpose of A is A_t. For R a subring of $\mathbb{C}$, the spectrum of A is denoted $\operatorname{sp}(A)$. For K a group, $|K|$ denotes the cardinality of K. For K compact, $\operatorname{Aut}(K)$ is the group of continuous automorphisms of K. The entropy of a $\sigma \in \operatorname{Aut}(K)$ is denoted $h(\sigma)$. Where convenient $\langle\ ,\ \rangle: \hat{G} \times G \longrightarrow \mathbb{T}$ denotes the duality between the locally compact abelian groups G and $\hat{G}$. Given two elements $a, b \in \mathbb{Z}$, (a, b) denotes their greatest common divisor.

2. **Solenoids and fibrebundles.** A solenoid is a compact connected abelian group of finite dimension [16]. We consider the class of topological dynamical systems arising from a (continuous) automorphism σ of a solenoid K with the property that the dual group $\hat{K}$ is finitely generated under the dual automorphism $\hat{\sigma}$, *i.e.*, that $\hat{K}$ is generated (as a group) by a finite union of orbits under $\hat{\sigma}$. We call these dynamical systems (K, σ) σ-finitely generated (σ-f.g.) solenoids. These systems were studied, for example, in [13]. It is known ([13]) that the dynamical systems given by expansive automorphisms of compact, connected groups are included in this class (the fact that the group must be abelian is in [12]). The condition that $\hat{K}$ is finitely generated under $\hat{\sigma}$ is equivalent to the condition (definable in a more general context) that (K, σ) satisfies the descending chain condition, in other words, every decreasing sequence of closed σ-invariant subgroups has only finitely many strict inclusions ([11]).

The class of σ-f.g. solenoids may also be described by replacing the assumption that $\dim K < \infty$ with the assumption that $h(\sigma)$, the entropy of σ, is finite ([11]). In one direction, the entropy of an automorphism of a solenoid is (computable and) finite by a result of Yuzvinskii ([16]). In the other direction, finite entropy of σ yields that $\hat{K}$ has "locally finite rank under $\hat{\sigma}$" ([13]) and this, coupled with the assumption that $\hat{K}$ is finitely generated under $\hat{\sigma}$, shows rank $\hat{K}(= \dim K)$ is finite.

Given a triple (d, F, M) with d a positive integer and $F, M \in \mathcal{M}_d(\mathbb{Z})$ with $\det F, \det M \neq 0$ we construct a σ-f.g. solenoid (K, σ). Viewing F, M as surjective endomorphisms of the d-torus $\mathbb{T}^d$, let K be the connected component of the identity of the compact abelian group $G = \{y \in (\mathbb{T}^d)^{\mathbb{Z}} \mid My_{n+1} = Fy_n, n \in \mathbb{Z}\}$. The continuous group automorphism σ (shift to the left) defined on $(\mathbb{T}^d)^{\mathbb{Z}}$ by $(\sigma y)_n = y_{n+1}$, $(y \in (\mathbb{T}^d)^{\mathbb{Z}}, n \in \mathbb{Z})$ restricts to a group automorphism of both G and K. Denote both of these again by σ. This class of examples, in fact, exhaust all of the possibilities; any σ-f.g. solenoid (K, σ) is topologically conjugate to one formed from such a triple, where in fact M can be taken to be multiplication by a positive integer a ([13]).

Let (K, σ) be a σ-f.g. solenoid described by the triple (d, F, M). Let G be the shift invariant subgroup of $(\mathbb{T}^d)^{\mathbb{Z}}$ defined above. We first examine the continuous group homomorphism $\pi: G \longrightarrow \mathbb{T}^d$ where π denotes the map $y \longrightarrow y_0$, $(y \in G)$. Note that since both M and F are surjective maps of $\mathbb{T}^d$, the equation $Mx = Fy$ always has a solution x (respectively y) given y (respectively x) in $\mathbb{T}^d$. It follows that $\pi: G \longrightarrow \mathbb{T}^d$ is surjective and $\mathbb{T}^d \simeq G/H$ with $H = \pi^{-1}(e)$ a closed subgroup of G.

PROPOSITION 2.1. *The closed subgoup $H = \pi^{-1}(e)$ of G is totally disconnected, and so a 0-dimensional group.*

PROOF. It is enough to show that $P_n = \{y_n \mid y \in H\}$ is finite for each $n \in \mathbb{Z}$, for then H is a subspace of a countable product of finite discrete spaces. The set P_0 consists of the single element e. For $A \in \mathcal{M}_d(\mathbb{Z})$ with $\det A \neq 0$, the set $A^{-1}(t)$ is a finite subset of $\mathbb{T}^d$ for each $t \in \mathbb{T}^d$; it has $|A^{-1}(\mathbb{Z}^d)/\mathbb{Z}^d| = |\mathbb{Z}^d/A(\mathbb{Z}^d)| = |\det A|$ elements. Thus $A^{-1}(P)$ is a finite subset of $\mathbb{T}^d$ for $P \subseteq \mathbb{T}^d$ finite. The result follows by induction. ∎

The next proposition shows that the map $\pi: G \longrightarrow \mathbb{T}^d$ has a local cross section, so defines a (locally trivial) fibre bundle with fibre $\pi^{-1}(e) = H$ and structure group H, *i.e.*, a principle H-bundle ([21]). Although a local cross section can be constructed directly using induction and the homotopy lifting property for both the covering projections M and F, there is a shorter argument.

PROPOSITION 2.2. *There is a neighbourhood U of e in $\mathbb{T}^d$ and a continuous map $s: U \longrightarrow G$ with $\pi \circ s(u) = u$, $(u \in U)$.*

PROOF. Denote by Q the isomorphism of $\mathbb{R}^d$ given by the element $M^{-1}F \in \mathrm{GL}(d, \mathbb{Q})$. Define a continuous group homomorphism $\theta: \mathbb{R}^d \longrightarrow G$ by $\big(\theta(x)\big)_n = q(Q^n x)$ for $x \in \mathbb{R}^d$, $n \in \mathbb{Z}$. Since $q: \mathbb{R}^d \longrightarrow \mathbb{T}^d$ is a covering projection one can choose a neighbourhood U of e in $\mathbb{T}^d$ and a homeomorphism $h: U \longrightarrow \mathbb{R}^d$ with $q \circ h(u) = u$, $(u \in U)$. Clearly $\pi \circ \theta = q$, so the map $\theta \circ h = s$ is a local cross section for π. ∎

Let Σ denote the subgroup $p^{-1}(e)$ of K where $p: K \longrightarrow \mathbb{T}^d$ is the restriction of π to K.

COROLLARY 2.3. *The map $p: K \longrightarrow \mathbb{T}^d$ defines a principle Σ-bundle.*

PROOF. If w is a path in $\mathbb{T}^d$ with initial point e it has a (unique) lifting $\tilde{w}$ in $\mathbb{R}^d$ (with initial point 0). Since $\mathbb{R}^d$ is (path) connected, $\operatorname{Im}\theta \subseteq K$, where $\theta: \mathbb{R}^d \longrightarrow G$ is defined in the previous proposition. The path $\theta \circ \tilde{w}$ thus lies in K and covers the path w with respect to p. Therefore $p: K \longrightarrow \mathbb{T}^d$ is a surjection and the local cross section s defined above has image in K. ∎

Henceforth θ will denote the homomorphism $\theta: \mathbb{R}^d \longrightarrow K$ defined above. Note that $p \circ \theta = q$ and that θ is the exponential map for the compact abelian group K.

For future reference we describe the local homeomorphism ϕ describing $K \xrightarrow{p} \mathbb{T}^d$ as a fibre bundle near the identity $\tilde{e}$ of K. Choosing U as in Proposition 1.2 define $\phi: U \times \Sigma \longrightarrow p^{-1}(U)$ by $\phi(u, g) = s(u) + g = \theta\big(h(u)\big) + g$, $u \in U$, $g \in \Sigma$. For $k \in p^{-1}(U)$, $\phi^{-1}(k) = \big(p(k), k - s\big(p(k)\big)\big)$. By means of a similar chart $\psi: U \times H \longrightarrow \pi^{-1}(U)$ for the H-bundle G, $\psi(u, g) = \theta\big(h(u)\big) + g$, $u \in U$, $g \in H$, we see that $H + K = G$. Since $\Sigma = H \cap K$ it follows that $G/K \simeq H/\Sigma$. The group G/K is Hausdorff, compact and totally disconnected, so 0-dimensional. It follows that $0 \longrightarrow \widehat{G/K} \longrightarrow \hat{G} \longrightarrow \hat{K} \longrightarrow 0$ is an exact sequence of discrete abelian groups with $\hat{K}$ torsion free and $\widehat{G/K}$ torsion.

The fibres, H and Σ, of these fibre bundles are totally disconnected and so have no nonconstant paths. Since fibre bundles (over a paracompact Hausdorff space) are also fibrations, it follows that the bundles $\pi: G \longrightarrow \mathbb{T}^d$ and $p: K \longrightarrow \mathbb{T}^d$ have unique path lifting.

The next proposition shows that K is a Bohr compactification of $\mathbb{R}^d$.

PROPOSITION 2.4. $\overline{\operatorname{Im}\theta} = K$.

PROOF. It is sufficient to show $K \subseteq \overline{\operatorname{Im}\theta}$. We first show that if $\sigma: I \longrightarrow G$ is a path with initial point $\tilde{e}$, the unit of G, then $\operatorname{Im}\sigma \subseteq \operatorname{Im}\theta$. Since $q: \mathbb{R}^d \longrightarrow \mathbb{T}^d$ is a covering projection with unique path lifting, there is a (unique) path w in $\mathbb{R}^d$ with initial point 0 covering the path $\pi \circ \sigma$. However, $\theta \circ w$ is also a path in G with initial point $\tilde{e}$ and $\pi(\theta \circ w) = q \circ w = \pi \circ \sigma$. By uniqueness of path lifting in the fibration $\pi: G \longrightarrow \mathbb{T}^d$, $\sigma = \theta \circ w$.

The subgroup $\operatorname{Im}\theta$ thus contains any one-parameter subgroup of K. The smallest closed subgroup of K containing all one-parameter subgroups of K, say L, is thus contained in $\overline{\operatorname{Im}\theta}$. However, since K is connected, $L = K$ ([9]). ∎

REMARK 2.5. Since the map $\theta: \mathbb{R}^d \longrightarrow (\mathbb{T}^d)^{\mathbb{Z}}$ only depends on the element $Q = M^{-1}F$ of $\mathrm{GL}(d, \mathbb{Q})$, we see immediately that the description of (K, σ) only depends on Q. It is also clear that given $Q = M^{-1}F$ one can always find $\tilde{F} \in \mathcal{M}_d(\mathbb{Z})$ and $a \in \mathbb{N}_+$ so that $Q = a^{-1}\tilde{F}$. Of course, the group G will vary with the M and F chosen to represent Q.

REMARK 2.6. (a) The computation of $\hat{K}$ is now straightforward (*cf.*, [11]). Since $K \leq (\mathbb{T}^d)^{\mathbb{Z}}$ we have $n = (n_k)$, an element of $\oplus_{\mathbb{Z}} \mathbb{Z}^d$, is in $K^{\perp}$ if and only if $\langle n, \theta(r)\rangle = 0$, $(r \in \mathbb{R}^d)$, *i.e.*, $\Sigma\langle n_k, qQ^k(r)\rangle = 0$, $(r \in \mathbb{R}^d)$. Thus $\Sigma\langle (Q_t)^k n_k, r\rangle_{\mathbb{R}^d} \in \mathbb{Z}$, $(r \in \mathbb{R}^d)$ where

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$\langle\ ,\ \rangle_{\mathbb{R}^d}$ is the usual inner product in $\mathbb{R}^d$. Since 0 is a value of this sum (when $r = 0$), one must have $\Sigma(Q_t)^k n_k = 0$. Thus $\hat{K} \cong \oplus\mathbb{Z}^d/K^\perp \cong \mathbb{Z}^d[Q_t, (Q_t)^{-1}]$, the subgroup of $\mathbb{Q}^d$ (endowed with the discrete topology) generated by $\cup\{(Q_t)^k n_k \mid n_k \in \mathbb{Z}^d, k \in \mathbb{Z}\}$. The dual automorphism $\hat{\sigma}$ of the shift on K is multiplication by Q_t.

(b) The family of dynamical systems $(\hat{\Lambda}_a, \alpha)$ for $a \in \mathbb{R}_+$ considered in [5] can be extended (see Proposition 6.3 also) to the following setting. For $A \in \mathcal{M}_d(\mathbb{C})$ with $\det A \neq 0$ consider the subgroup $\Lambda_A = \mathbb{Z}^d[A, A^{-1}]$ (generated by $\cup\{A^k\mathbb{Z}^d \mid k \in \mathbb{Z}\}$) of $\mathbb{C}^d$ endowed with the discrete topology. The automorphism σ dual to multiplication by A defines a dynamical system $(\hat{\Lambda}_A, \sigma)$ where $\hat{\Lambda}_A$ is compact, abelian and connected. It is clear that Λ_A is generated as a group by a finite union of orbits under multiplication by A. Also, since $\mathbb{Z}^d \subseteq \Lambda_A$, $d \leq \operatorname{rank} \Lambda_A$. If the minimal polynomial m of A divides an element of $\mathbb{Q}[\lambda]$, or equivalently, if the spectrum of A consists of algebraic numbers in $\mathbb{C}$, then $\operatorname{rank} \Lambda_A \leq d \cdot s$ where s is the degree of the generator P_A of the ideal $\{h \in \mathbb{Q}[\lambda] \mid m|h \in \mathbb{R}[\lambda]\}$ in $\mathbb{Q}[\lambda]$. In this case $(\hat{\Lambda}_A, \sigma)$ is a σ-f.g. solenoid and one can conclude that there is a $r \in \mathbb{N}_+$ and a $Q \in \mathrm{GL}(r, \mathbb{Q})$ with $\mathbb{Z}^d[A, A^{-1}] \cong \mathbb{Z}^r[Q, Q^{-1}]$ such that multiplication by A corresponds to multiplication by Q.

The homomorphism $\theta: \mathbb{R}^d \longrightarrow K$ restricts to a group homomorphism χ from $\mathbb{Z}^d = q^{-1}(e)$ to $\Sigma = p^{-1}(e)$. The following proof makes implicit use of the local triviality of the fibration $p: K \longrightarrow \mathbb{T}^d$.

PROPOSITION 2.7. *The group homomorphism $\chi: \mathbb{Z}^d \longrightarrow \Sigma$ has dense image.*

PROOF. Since $\operatorname{Im}\chi = \operatorname{Im}\theta \cap p^{-1}(e)$ one needs to show $\overline{\operatorname{Im}\theta \cap p^{-1}(e)} = \Sigma$. Now $K = \overline{\operatorname{Im}\theta}$, so given $x \in \Sigma$ there is a convergent sequence x_n in $\operatorname{Im}\theta$ with $\lim x_n = x$. Thus $\pi(x_n) \longrightarrow e$ and $\pi(x_n)$ eventually lies in any neighbourhood V of e in $\mathbb{T}^d$. If s is the local cross section for $p: K \longrightarrow \mathbb{T}^d$ defined above, then $\operatorname{Im} s \subseteq \operatorname{Im}\theta$, so (for n large enough) $y_n = x_n - s\big(\pi(x_n)\big) \in \operatorname{Im}\theta$. It is straightforward to check that $y_n \in p^{-1}(e)$ and $y_n \longrightarrow x - \tilde{e} = x$. ∎

REMARK 2.8. Although the specific cross section s defined in Proposition 2.2 was used in the proof of Proposition 2.7 this was not necessary, as it is the case that $\operatorname{Im}(s) \subseteq \operatorname{Im}\theta$ for any local cross section s of $p: K \longrightarrow \mathbb{T}^d$ defined on a path connected neighbourhood V of e in $\mathbb{T}^d$. To see this, let $v \in V$, let σ be a path in K from $\tilde{e}$ to $s(v)$ and let w be the unique path in $\mathbb{R}^d$ with $w(0) = 0$ and $q \circ w = p \circ \sigma$. By unique path lifting in the bundle $p: K \longrightarrow \mathbb{T}^d$, it follows that $\theta \circ w = \sigma$ and thus $\sigma(v) \in \operatorname{Im}\theta$.

The compact group Σ is thus a Bohr compactification of $\mathbb{Z}^d$ ([9]). The map χ may also be viewed in another context, one which (since Σ is abelian and so has only trivial inner automorphisms) shows that it serves to classify the equivalence class of the principle Σ-bundle over $\mathbb{T}^d$ given by p ([21]).

PROPOSITION 2.9. *The map $\chi: \mathbb{Z}^d \longrightarrow \Sigma$ is the characteristic class of the bundle $p: K \longrightarrow \mathbb{T}^d$.*

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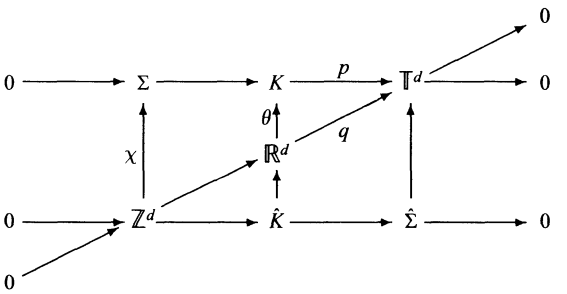
PROOF. The characteristic class of $p: K \longrightarrow \mathbb{T}^d$ is the group homomorphism defined by mapping the class $[w]$ in $\pi_1(\mathbb{T}^d, e)$ of a loop w at e in $\mathbb{T}^d$ to $w^\#$, where $w^\#$ is a self map of Σ. By definition ([21]), $w^\#(\xi) = W(1)$ where W is the unique lifting in K of w with $W(0) = \xi, \xi \in \Sigma$. (The definition actually involves W^{-1}, however, since Σ is abelian we can ignore this convention.) Note that $w^\#(\xi) = w^\#(\tilde{e}) + \xi$, $(\xi \in \Sigma)$, since $W + \xi$ is the lifting of w with initial point ξ if W is the lifting of w with initial point $\tilde{e}$. We may therefore identify $w^\#$ with the action of the group element $w^\#(\tilde{e})$ on Σ.

The fundamental group $\pi_1(\mathbb{T}^d, e)$ is isomorphic to $\mathbb{Z}^d$ by mapping $[w]$ to $\tilde{w}(1)$ where $\tilde{w}$ is the unique lifting in $\mathbb{R}^d$ of w with initial point 0 (in $\mathbb{R}^d$). Using this identification we have $\chi([w]) = \chi\big(\tilde{w}(1)\big) = \theta\big(\tilde{w}(1)\big) = (\theta \circ \tilde{w})(1) = w^\#(e)$. ∎

REMARK 2.10. In the following commutative self-dual diagram with exact rows (and diagonal) the maps χ and θ have dense ranges.



Using this (along with Remark 2.6(a)) the discrete abelian group $\hat{\Sigma}$ may be identified with the subgroup $q(\mathbb{Z}^d[Q_t, Q_t^{-1}])$ of $\mathbb{T}^d$ (with the discrete topology).

3. **Cantor fibres and connected bundles.** Since the fibre Σ of $p: K \longrightarrow \mathbb{T}^d$ is a group, it has an isolated point if and only if it is discrete, so Σ is either perfect, so a Cantor group, or it is discrete and hence, by compactness, finite. By the Proposition 2.7 the latter occurs if and only if $\operatorname{Im}\chi$ is finite. However, $\operatorname{Im}\chi \simeq \mathbb{Z}^d / \ker\chi$, so this is determined by the invariants of the submodule $\ker\chi = \cap\{Q^n\mathbb{Z}^d \mid n \in \mathbb{Z}\}$ in $\mathbb{Z}^d$. Denoting the submodule $\ker\chi$ by S, we have that Σ is finite if and only if S is a rank d submodule of $\mathbb{Z}^d$, otherwise Σ is a Cantor group.

REMARK 3.1. A fibration with unique path lifting whose base space is locally path connected and semilocally 1-connected must be a covering projection if the total space is locally path connected ([22]). Thus K is locally path connected exactly when the fibre Σ is discrete, *i.e.*, when Σ is finite.

We determine some conditions relating to whether or not Σ is a Cantor group. This also has bearing on the homomorphism θ, since $\ker\theta = \ker\chi$ $(= S)$. If $S = \bigcap_{n\in\mathbb{Z}} Q^n\mathbb{Z}^d$ has rank $r \neq 0$, there is a basis $\{e_i \mid 1 \leq i \leq d\}$ of $\mathbb{Z}^d$ so that $\{a_i e_i \mid 1 \leq i \leq r\}$ is a basis for S where the $a_i \in \mathbb{N}$, $a_1 \geq 1$, and $a_i \mid a_{i+1}$, $(1 \leq i \leq r-1)$. Since Q restricts to an

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automorphism of S, there is an $N = [n_{ij}] \in \mathrm{GL}(r, \mathbb{Z})$ with $Q|_S = N$. Thus, for $1 \le i \le r$, $Q(a_i e_i) = \sum_{j=1}^{r} n_{ji} a_j e_j$ and so (in $S \otimes \mathbb{Q}$, a submodule of $\mathbb{Q}^d$), $Q(e_i) = \sum_{j=1}^{r} n_{ji} a_i^{-1} a_j e_j$. However, with M denoting multiplication by a, $Q(e_i) = \sum_{j=1}^{d} a^{-1} f_{ji} e_j$ for $1 \le i \le r$ where $[f_{ij}]$ is the matrix of F relative to the basis $\{e_i \mid 1 \le i \le d\}$ of $\mathbb{Z}^d$. (It is similar over $\mathcal{M}_d(\mathbb{Z})$ to the given matrix F.) Since $\{e_i \mid 1 \le i \le d\}$ may also be viewed as a basis for $\mathbb{Q}^d$, it follows that $a^{-1} f_{ji} = a_j n_{ji} a_i^{-1}$ for $1 \le i, j \le r$ and $f_{ji} = 0$ for i, j with $1 \le i \le r$ and $r+1 \le j \le d$. We conclude that F is similar (via a unit in $\mathcal{M}_d(\mathbb{Z})$) to a block upper triangular matrix $\begin{bmatrix} A & B \\ 0 & C \end{bmatrix}$ with A a square $r \times r$ matrix, $A = a \cdot DND^{-1}$, $N \in \mathrm{GL}(r, \mathbb{Z})$ and D a diagonal matrix in $\mathcal{M}_r(\mathbb{Z})$,

$$D = \begin{bmatrix} a_1 & & \\ & \ddots & \\ & & a_r \end{bmatrix} \quad \text{with } 1 \le a_1 | a_2 | \cdots | a_r.$$

Notice that $\mathrm{char}\, F = \mathrm{char}\, A \cdot \mathrm{char}\, C$, a product of (monic) elements of $\mathbb{Z}[\lambda]$ of degree r and $d - r$ respectively, and also that $|\det F| = a^r \cdot |\det C|$.

We summarize these results and some of their consequences.

PROPOSITION 3.2. *For $a \in \mathbb{N}$, $F \in \mathcal{M}_d(\mathbb{Z})$ with $\det F \neq 0$ set $Q = a^{-1}F$ and $S = \cap\{Q^n \mathbb{Z}^d \mid n \in \mathbb{Z}\}$. If r is the nonnegative integer determined by the condition that $r \le d$ and a^r is the maximal power of a dividing $\det F$ then $\mathrm{rank}\, S \le r$. In particular, if $a > 1$ and $(a, \det F) = 1$, then $S = 0$. If $a = 1$, then $\cap\{F^n \mathbb{Z}^d \mid n \in \mathbb{Z}\}$ has $\mathrm{rank}\, d$ if, and only if, $F \in \mathrm{GL}(d, \mathbb{Z})$ (in which case S is $\mathbb{Z}^d$).*

PROPOSITION 3.3. *With the notation of Proposition 3.2, we have*

(a) If $\mathrm{rank}\, S = d$ then $\mathrm{char}\, Q \in \mathbb{Z}[\lambda]$.

(b) If $\mathrm{char}\, F$ is irreducible then $S = 0$ or S has $\mathrm{rank}\, d$, in which case $|\det F| = a^d$.

We give some examples.

EXAMPLE 3.4. (a) Set $a = 2$ and $F = \begin{bmatrix} 5 & 1 \\ 1 & 1 \end{bmatrix} \in \mathcal{M}_2(\mathbb{Z})$, so a^2 divides $\det F$ and $Q = 2^{-1}F \in \mathrm{GL}(2, \mathbb{Q})$. Then $e_1 = (1, 1)$ and $e_2 = (1, 0)$ form a basis for $\mathbb{Z}^2$ with e_1 and $2e_2$ a basis for the rank 2 submodule $S = \cap\{Q^n \mathbb{Z}^2 \mid n \in \mathbb{Z}\}$. The matrix of Q relative to this basis of S is $N = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$. The matrix F relative to the basis $\{e_1, e_2\}$ of $\mathbb{Z}^2$ is $\begin{bmatrix} 2 & 1 \\ 4 & 4 \end{bmatrix} = 2DND^{-1}$ where $D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$. We have $\mathrm{char}\, Q$ is an irreducible element of $\mathbb{Z}[\lambda]$. It is interesting to notice that $Q^3 \in SL(2, \mathbb{Z})$. Since $\mathrm{rank}\, S = 2$, the fibre for the bundle $p: K \to \mathbb{T}^2$ corresponding to the triple $(2, F, 2)$ is a finite group, not a Cantor group.

(b) Set $a = 2$ and $F = \begin{bmatrix} 2 & 1 \\ -2 & 1 \end{bmatrix}$, so $\det F = a^2$. Since $\mathrm{char}\, F = \lambda^2 - 3\lambda + 4$ is irreducible but $\mathrm{char}\, Q \notin \mathbb{Z}[\lambda]$, S must be 0 and so has rank strictly less than the highest power of a dividing $\det F$. The fibre of the fibration $p: K \to \mathbb{T}^2$ determined by the triple $(2, F, 2)$ is a Cantor group.

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After listing some preparatory results (to be mainly used in the next section) we discuss when the group G described by a triple (d, F, M) is connected.

The translation invariant metric on $\mathbb{R}^d$ defined by a norm on $\mathbb{R}^d$ yields a translation invariant metric δ_0 on the quotient group $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$. The metric $\delta = \sum_{n\in\mathbb{Z}} \delta_n$ on $(\mathbb{T}^d)^{\mathbb{Z}}$ with $\delta_n = 2^{-|n|}\alpha\delta_0$, $n \neq 0$, then defines a translation invariant metric on G and K, which we still denote by δ. Here α is a strictly positive real number (to be further determined in Section 4 if $Q = M^{-1}F$ is a hyperbolic map of $\mathbb{R}^d$).

For H a nonzero discrete (and therefore closed) subgroup of $\mathbb{R}^d$, let $s(H) = \inf\{\|h\| \mid h \in H, h \neq 0\}$, a strictly positive number. Letting U be the domain of the local section h of $q\colon \mathbb{R}^d \to \mathbb{T}^d$ defined previously, we see that $\|h(x)\| = \delta_0(q(x), e)$ for $x \in U_0$, where U_0 is the ball of radius $s(\mathbb{Z}^d)/2$ about e.

LEMMA 3.5. *Let $\eta > 0$ be less than $s(M^{-1}\mathbb{Z}^d)/2\|Q\|$ (or $s(M^{-1}\mathbb{Z}^d)/2$ if $\|Q\| < 1$). Let $y, z \in \mathbb{T}^d$ with $\delta_0(y, e) < \eta$, $\delta_0(z, e) < \eta$ and $Mz = Fy$. Then there are $x, w \in \mathbb{R}^d$ with $\|x\| < \eta$, $\|w\| < \eta$, $q(x) = z$, $q(w) = y$ and $x = Qw$.*

PROOF. Since $s(M^{-1}\mathbb{Z}^d) \leq s(\mathbb{Z}^d)$, we have $\|x\| = \delta_0(z, e) < \eta$ and $\|w\| = \delta_0(y, e) < \eta$ if $x = h(z)$ and $w = h(y)$. Since $Mz = Fy$ we have $Mx - Fw \in \mathbb{Z}^d$, so $x - Qw \in M^{-1}(\mathbb{Z}^d)$. However, $\|x - Qw\| < s\big(M^{-1}(\mathbb{Z}^d)\big)$, so $x = Qw$. ∎

Note that there can be only one element w with $q(w) = y$ and $\|w\| < \eta$, since $2\eta < s(\mathbb{Z}^d)$. Similarly, x is uniquely determined.

PROPOSITION 3.6. *If z_n, $(n \in \mathbb{N})$, is a sequence in $\mathbb{T}^d$ with $\delta_0(z_n, e) < \eta$, η as in Lemma 3.5, and $Mz_{n+1} = Fz_n$ then there is a unique $x \in \mathbb{R}^d$ with $\|x\| < \eta$ and $qQ^n x = z_n$, $(n \in \mathbb{N})$. In this case, $\|Q^n x\| < \eta$, $(n \in \mathbb{N})$.*

PROOF. Apply induction with Lemma 3.5 and the fact that Q is an isomorphism. ∎

Replacing n with $-n$ yields the next statement.

PROPOSITION 3.7. *If z_{-n}, $(n \in \mathbb{N})$ is a sequence in $\mathbb{T}^d$ with $\delta_0(z_{-n}, e) < \eta$ and $Mz_{-n} = Fz_{-(n+1)}$ then there is a unique $x \in \mathbb{R}^d$ with $\|x\| < \eta$ and $qQ^{-n}x = z_{-n}$, $(n \in \mathbb{N})$. In this case, $\|Q^{-n}x\| < \eta$, $(n \in \mathbb{N})$.*

For $g \in \Sigma$, define g_- by $(g_-)_n = \begin{cases} g_n, & n \geq 0 \\ e, & n \leq 0 \end{cases}$. The map $g \to g_-$ of Σ to G is continuous.

LEMMA 3.8. *If $g \in \Sigma$ then $g_- \in \Sigma$.*

PROOF. Since $\theta(\mathbb{Z}^d)$ is dense in Σ, it is sufficient to show $\theta(r)_- \in \Sigma$ for $r \in \mathbb{Z}^d$. However $\theta(r)_- = \begin{cases} q\big(Q^n(r)\big), & n \geq 0 \\ e, & n \leq 0 \end{cases}$ is path connected to $\tilde{e}$ in G, so lies in K. ∎

For $g \in \Sigma$, define $g_+ = g - g_-$, also an element of Σ. Denote by Σ_+ the compact subgroup $\{g_+ \mid g \in \Sigma\}$. We have $\Sigma = \Sigma_+ \oplus \Sigma_-$ where Σ_- is defined similarly. Note, for example, that if $a = 1$ in the defining triple (d, F, a) for G then $\Sigma = \Sigma_+$.

We will now describe conditions ensuring that $G = K$, *i.e.*, that G, which is described by the triple (d, F, M), is connected. Let $\mathcal{R}$, $\mathcal{R}_0$ be the relations (on $\mathbb{T}^d \times \mathbb{T}^d$) defined by $\{(s,t) \mid Mt = Fs\}$ and $\{(qr, qQr) \mid r \in \mathbb{R}^d\}$ respectively, so $G = \{x \in (\mathbb{T}^d)^{\mathbb{Z}} \mid (x_n, x_{n+1}) \in \mathcal{R}, n \in \mathbb{Z}\}$.

PROPOSITION 3.9. $K = \{x \in G \mid (x_n, x_{n+1}) \in \mathcal{R}_0, n \in \mathbb{Z}\}$.

PROOF. Since $\theta(\mathbb{R}^d) \subseteq K$ it is enough to show K is contained in the stated set. Choose $x \in K$, $n \in \mathbb{Z}$. Since $\theta(\mathbb{R}^d)$ is dense in K there is an $r \in \mathbb{R}^d$ with both $\delta_0(qQ^n r - x_n, e)$ and $\delta_0(qQ^{n+1}r - x_{n+1}, e)$ small enough to satisfy the hypothesis of Lemma 3.5. Thus, there are $v, w \in \mathbb{R}^d$ with $qv = qQ^n r - x_n$, $qw = qQ^{n+1}r - x_{n+1}$ and $Qv = w$. In particular, $x_{n+1} = q(Q^{n+1}r - w) = qQ(Q^n r - v)$ and since $x_n = q(Q^n r - v)$ we have $(x_n, x_{n+1}) \in \mathcal{R}_0$. ∎

PROPOSITION 3.10. *G is connected if and only if $F\mathbb{Z}^d + M\mathbb{Z}^d = \mathbb{Z}^d$.*

PROOF. Clearly G is connected if and only if $\mathcal{R} = \mathcal{R}_0$. Since $\mathcal{R}_0 \subseteq \mathcal{R}$ and $F\mathbb{Z}^d + M\mathbb{Z}^d \subseteq \mathbb{Z}^d$ it is enough to show $\mathcal{R} \subseteq \mathcal{R}_0$ if and only if $F\mathbb{Z}^d + M\mathbb{Z}^d \supseteq \mathbb{Z}^d$. An arbitrary $(qu, qv) \in \mathcal{R}$ with $u, v \in \mathbb{R}^d$ is an element of $\mathcal{R}_0$ if and only if there is an $r \in \mathbb{R}^d$ with $q(u - r) = q(v - Qr) = 0$. This is the same as requiring a $z\ (= u - r) \in \mathbb{Z}^d$ with $v - Qu - Qz \in \mathbb{Z}^d$. However $(qu, qv) \in \mathcal{R}$ is equivalent to $v - Qu \in M^{-1}(\mathbb{Z}^d)$. Thus $\mathcal{R} \subseteq \mathcal{R}_0$ if and only if given $w \in \mathbb{Z}^d$ there is $z \in \mathbb{Z}^d$ with $M^{-1}(w) - Qz \in \mathbb{Z}^d$. Applying the isomorphism M shows this is equivalent to $\mathbb{Z}^d \subseteq M\mathbb{Z}^d + F\mathbb{Z}^d$. ∎

PROPOSITION 3.11. *If $(\det M, \det F) = 1$ then G is connected.*

PROOF. Since $M\mathbb{Z}^d + F\mathbb{Z}^d$ is a rank d submodule of $\mathbb{Z}^d$, there is $C \in \mathcal{M}_d(\mathbb{Z})$ with $\det C \neq 0$ and $M\mathbb{Z}^d + F\mathbb{Z}^d = C\mathbb{Z}^d$. Since both $M\mathbb{Z}^d$ and $F\mathbb{Z}^d$ are contained in $C\mathbb{Z}^d$ it follows that C (left) divides both M and F. Thus $\det C$ divides $\det M$ and $\det F$, so $|\det C| = 1$ and $C\mathbb{Z}^d = \mathbb{Z}^d$. By Proposition 3.10, G is connected. ∎

Since G is connected if and only if $\mathcal{R}$ is connected ([11]) it follows that $\{(s,t) \in \mathbb{T}^d \times \mathbb{T}^d \mid Mt = Fs\}$ is connected if $(\det M, \det F) = 1$. The converse if not true. For example, if $M = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ and $F = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ then $M\mathbb{Z}^2 + F\mathbb{Z}^2 = \mathbb{Z}^2$, so G is connected, but $(\det M, \det F) \neq 1$. If, however, M is a multiplication by a, then the converse holds.

PROPOSITION 3.12. *Let $G = \{x \in (\mathbb{T}^d)^{\mathbb{Z}} \mid ax_{n+1} = Fx_n, n \in \mathbb{Z}\}$. Then G is connected if and only if $(a, \det F) = 1$.*

PROOF. Since G is connected, $a\mathbb{Z}^d + F\mathbb{Z}^d = \mathbb{Z}^d$. If $D \in \mathcal{M}_d(\mathbb{Z})$ is a common (left) divisor of aI and F then $\mathbb{Z}^d = a\mathbb{Z}^d + F\mathbb{Z}^d \subseteq D\mathbb{Z}^d + D\mathbb{Z}^d = D\mathbb{Z}^d \subseteq \mathbb{Z}^d$: so $D\mathbb{Z}^d = \mathbb{Z}^d$ and $|\det D| = 1$. Thus I is the greatest common (left) divisor of aI and F. If $c = (a, \det F)$ then c divides the dth invariant factor of the $d \times 2d$ matrix $[F, aI]$ and thus the dth invariant factor of the greatest common (left) divisor I. Thus $c = 1$. ∎

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4. A local product structure. We show that the dynamical system (K, σ) has a Smale space structure for an appropriate metric δ on K assuming that the map $Q = M^{-1}F \in \mathrm{GL}(d, \mathbb{Q})$ is a hyperbolic map of $\mathbb{R}^d$.

The map Q is hyperbolic if $\mathrm{sp}(Q) \cap \mathbb{T} = \phi$. This is equivalent to requiring that the homeomorphism σ of K be expansive ([13]). In this case there is a decomposition of $\mathbb{R}^d$, $\mathbb{R}^d = E_+ \oplus E_-$, into two Q-invariant subspaces and a norm $\| \ \|$ on $\mathbb{R}^d$ such that $\|Q|_{E_+}\| < 1$, $Q|_{E_-}$ is an isomorphism with $\|Q|_{E_-}^{-1}\| < 1$ and $\|(a, b)\| = \max\{\|a\|, \|b\|\}$ with $a \in E_+$, $b \in E_-$. Such a norm is called a norm adapted to Q. Since, in our situation, Q is an isomorphism of $\mathbb{R}^d$, $Q(E_\pm) = E_\pm$ and $Q|_{E_-}^{-1} = Q^{-1}|_{E_-}$. Also note that if $E_- \neq 0$, that $\|Q|_{E_-}\| \geq \|Q|_{E_-}\|^{-1} > 1$ and, similarly, if $E_+ \neq 0$, $\|Q^{-1}|_{E^+}\| > 1$. We have that $\|Q\| = \max\{\|Q|_{E_+}\|, \|Q|_{E_-}\|\}$ with respect to such a norm. Write $x = x_+ + x_-$ for $x \in \mathbb{R}^d$ where $x_+ \in E_+$ and $x_- \in E_-$. For Q hyperbolic we will automatically assume that a norm adopted to Q has been chosen. In particular the metric δ on G and K discussed in Section 3 will arise from such a norm.

A linear isomorphism, say T, of a normed space with $\|T^{-1}\| < 1$ must be (positively) expansive; in other words if $\|T^n z\|$ is a bounded set for $n \in \mathbb{N}$ then $z = 0$. This easily follows from $\|z\| = \|T^{-n}(T^n z)\| \leq \|T^{-1}\|^n \|T^n z\|$ which tends to zero as n approaches infinity if $\|T^n z\|$ is bounded, $(n \in \mathbb{N})$. Thus $Q^{-1}|_{E_+}$ and $Q|_{E_-}$ are (positively) expansive.

As in [6] define, for $x \in K$ and $\varepsilon > 0$, local stable and unstable sets by

$$V^s(x, \varepsilon) = \{y \in K \mid \delta(\sigma^n x, \sigma^n y) < \varepsilon, (n \in \mathbb{N})\}$$
$$V^u(x, \varepsilon) = \{y \in K \mid \delta(\sigma^{-n} x, \sigma^{-n} y) < \varepsilon, (n \in \mathbb{N})\}$$

respectively. Observe that $V^s(\tilde{e}, \varepsilon) + x = V^s(x, \varepsilon)$, $x \in K$ and similarly for V^u.

THEOREM 4.1. *Assume $Q = M^{-1}F$ is a hyperbolic map of $\mathbb{R}^d$. Let $z \in p^{-1}(U)$ where $U \subseteq \mathbb{T}^d$ denotes the neighbourhood of e in $\mathbb{T}^d$ described in Proposition 2.2. Let $\eta' > 0$ be less than $2\eta/3$ with η chosen as in Lemma 3.5. If $\delta(z, \tilde{e}) < \eta'$ there is a unique element w of K with $w \in V^s(z, \eta') \cap V^u(\tilde{e}, \eta')$.*

PROOF. For $w \in V^u(\tilde{e}, \eta')$ we have $\delta(\sigma^{-n} w, \tilde{e}) < \eta'$, $(n \in \mathbb{N})$, so $\delta_0(w_{-n}, e) < \eta'$, $(n \in \mathbb{N})$, and thus (Proposition 3.6) there is a unique $x \in \mathbb{R}^d$ with $qQ^{-n}x = w_{-n}$ and $\|Q^{-n}x\| < \eta'$, $(n \in \mathbb{N})$. In particular, $\|Q^{-n}|_{E_+}x_+\| < \eta'$, $(n \in \mathbb{N})$, and since $Q^{-1}|_{E_+}$ is (positively) expansive, we have $x_+ = 0$ and $x = x_- \in E_-$. Similarly if $w \in V^s(z, \eta')$ we have $\delta_0(w_n - z_n, e) < \eta'$, $(n \in \mathbb{N})$, and there is a unique $y \in \mathbb{R}^d$ with $qQ^n(y) = w_n - z_n$ and $\|Q^n y\| < \eta'$, $(n \in \mathbb{N})$. Using that $Q|_{E_-}$ is positively expansive, it follows that $y_- = 0$ and $y = y_+ \in E_+$. Thus, for $n \in \mathbb{N}$, $w_{-n} = qQ^{-n}x_-$ and $w_n = qQ^n y_+ + z_n$.

Let ϕ denote the local chart of the fibre bundle $p: K \to \mathbb{T}^d$ described in Section 2. Then $z = \phi(u, g)$ for some $u \in U$ with $\delta_0(u, e) < \eta'$ and $g \in \Sigma$. Thus $z_n = qQ^n h(u) + g_n$, $(n \in \mathbb{N})$. For the two above conditions on w_0 to agree we need $qx_- = qy_+ + q\big(h(u)\big)$. Since $\|x_- - y_+ - h(u)\| < 3\eta' < 2\eta < s(\mathbb{Z}^d)$, it follows that $h(u) = x_- - y_+$, so $h(u)_+ = -y_+$ and $h(u)_- = x_-$. Note that $y_+ + h(u) = h(u)_- = x_-$; this shows $qQ^n y_+ + z_n = qQ^n h(u)_- + g_n$, so

$$w_{-n} = qQ^{-n} h(u)_-$$

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and

$$w_n = qQ^n h(u)_- + g_n, \quad (n \in \mathbb{N}).$$

Thus w is in K and is unique. ∎

REMARK 4.2. It is straightforward to see that $V^s(x,\varepsilon) \cap V^u(y,\varepsilon)$ contains at most one point for any expansive homeomorphism of a compact metric space (of course, ε must be less than a certain constant).

Thus, if $x, y \in K$ are within η' of each other (in fact, one only requires $p(x)$ to be within η' of py in $\mathbb{T}^d$), we have $V^s(x,\eta') \cap V^u(y,\eta') = V^s(x,\eta') \cap \left(V^u(\tilde{e},\eta') + y\right) = \left(V^s(x-y,\eta') \cap V^u(\tilde{e},\eta')\right) + y$ is a unique element of K, denoted $[x,y]$, ([17, 20]). The usual algebraic properties (described in [17, 20] for example) of the locally defined map $[\,,\,]$ hold: $[x,x] = x$, $\left[[x,y],z\right] = [x,z]$ and $\left[x,[y,z]\right] = [x,z]$. Notice that $[\tilde{e},\tilde{e}] = \tilde{e}$ is a restatement of the fact that σ is expansive. The underlying group structure of K yields additional identities.

PROPOSITION 4.3. *The locally defined map* $[\,,\,]$ *on pairs* $(x,y) \in K \times K$ *with* x *sufficiently close to* y *satisfy the following:*

(a) $[x,y] = [x-y,\tilde{e}] + y = [\tilde{e}, y-x] + x$.

(b) $[-x,-y] = -[x,y]$.

(c) $[x,y] = [x,\tilde{e}] + [\tilde{e},y]$ *for* x, y *sufficiently close to* $\tilde{e}$.

(d) $\left[x - [\tilde{e},y], \tilde{e}\right] = [x,\tilde{e}]$ *and* $\left[\tilde{e}, y - [x,\tilde{e}]\right] = [\tilde{e},y]$ *for* x, y *sufficiently close to* $\tilde{e}$.

PROOF. Part (a) follows from the definition. The translation invariance of δ implies $\delta(x,y) = \delta(-y,-x)$ and since σ is a group homomorphism, part (b) follows. For part (c) note that $[x,\tilde{e}] + [\tilde{e},x] = [x,\tilde{e}] + ([-x,\tilde{e}] + x) = x$, so $[x,y] = \left[[x,y],\tilde{e}\right] + \left[\tilde{e},[x,y]\right] = [x,\tilde{e}] + [\tilde{e},y]$. Also $\left[x - [\tilde{e},y], \tilde{e}\right] = \left[x, [\tilde{e},y]\right] - [\tilde{e},y] = [x,y] - [\tilde{e},y] = [x,\tilde{e}]$ (by part (c)) and (d) follows. ∎

REMARK 4.4. For $a \in K$, $a = \phi(u,g)$ with $g \in \Sigma$ and u in $\mathbb{T}^d$ within distance η' of e (where η' is as in Theorem 4.1) we have

$$[a,\tilde{e}] = \phi(u_-, g_-)$$

and

$$[\tilde{e},a] = \phi(u_+, g_+),$$

where $u_\pm$ has the obvious meaning; $u_\pm = q\left(h(u)_\pm\right)$. The first equality follows from Theorem 4.1 where it was shown that $V^s\left(\phi(u,g),\eta'\right) \cap V^u(\tilde{e},\eta') = \theta\left(h(u)_-\right) + g_-$. Using $[\tilde{e},a] = a - [a,\tilde{e}]$ one obtains the second. We thus have a description of the local stable and unstable sets about $\tilde{e}$, since ([20])

$$V^s(\tilde{e},\varepsilon) = \{a \in K \mid [\tilde{e},a] = a \text{ and } \delta(\tilde{e},a) < \varepsilon\}$$

and

$$V^u(\tilde{e},\varepsilon) = \{a \in K \mid [a,\tilde{e}] = a \text{ and } \delta(a,\tilde{e}) < \varepsilon\}.$$

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To finish showing that (K, δ, σ) is a Smale space we need that $[\sigma x, \sigma y] = \sigma[x, y]$ locally and that σ (respectively σ^{-1}) is contractive on V^s (respectively V^u).

For the first property we may assume that $y = \tilde{e}$. For $w \in V^s(x, \eta'/2) \cap V^u(\tilde{e}, \eta'/2)$ with x close enough to $\tilde{e}$ such that $\delta\big(\sigma(x), \tilde{e}\big) < \eta'/2$ we have (since $\delta\big(\sigma(w), \sigma(x)\big) < \eta'/2$) that $\delta(\sigma(w), \tilde{e}) < \eta'$ and so $\sigma(w) \in V^s(\sigma(x), \eta') \cap V^u(\tilde{e}, \eta')$. Thus $[\sigma x, \tilde{e}] = \sigma[x, \tilde{e}]$ for x close enough to $\tilde{e}$.

PROPOSITION 4.5. *There is a constant α defining the metric δ on K and a constant $\lambda \in (0, 1)$ so that*

$$\delta(\sigma x, \sigma y) \leq \lambda \delta(x, y) \quad \textit{for } x, y \in V^s(\tilde{e}, \eta'/2)$$

and

$$\delta(\sigma^{-1}x, \sigma^{-1}y) \leq \lambda \delta(x, y) \quad \textit{for } x, y \in V^u(\tilde{e}, \eta'/2).$$

PROOF. Since $Q|_{E_+}$ and $Q^{-1}|_{E_-}$ are contractive there is $\lambda_0 \in (0, 1)$ with $\|Q|_{E_+}\| < \lambda_0$ and $\|Q^{-1}|_{E_-}\| < \lambda_0$. Choose $\alpha \in (0, 1)$ so that $\alpha/2 + \lambda_0 = \lambda \in (1/2, 1)$. If $a \in V^s(e, \eta')$ there is a unique $u \in E_+$ with $\|u\| < \delta'$ and $qQ^n u = a_n$, $(n \in \mathbb{N})$, so $a = \phi(u, g_+)$ for some $g \in \Sigma$. We have $\delta(\sigma(a), \tilde{e}) = \Sigma\delta_n(a_{n+1}, e) = \sum_{n\geq 1} \delta_n(a_{-n+1}, e) + \delta_0(a_1, e) + \sum_{n\geq 1} \delta_n(a_{n+1}, e) = \sum_{n\geq 0} \delta_{n+1}(a_{-n}, e) + \delta_0(a_1, e) + \sum_{n\geq 2} \delta_{n-1}(a_n, e) = \sum_{n\geq 0} 2^{-(n+1)}\alpha\delta_0(u_{-n}, e) + \delta_0(a_1, e) + \sum_{n\geq 2} 2^{-(n-1)}\alpha\delta_0(a_n, e)$. Since $\|Q|_{E_+}\| < \lambda_0$ and $\|u_+\| < \eta'$ we have $\delta_0(qQ^n u_+, e) \leq \lambda_0 \delta_0(qQ^{n-1}u_+, e)$ for $n \geq 1$, in other words, $\delta_0(a_n, e) \leq \lambda_0 \delta_0(a_{n-1}, e)$ for $n \geq 1$. Thus $\delta(\sigma(a), \tilde{e}) \leq \sum_{n\geq 0} 2^{-(n+1)}\alpha\delta_0(a_{-n}, e) + \lambda_0\delta_0(a_0, e) + \lambda_0 \sum_{n\geq 2} 2^{-(n-1)}\alpha\delta_0(a_{n-1}, e) = \frac{1}{2}\big(\sum_{n\geq 1} 2^{-n}\alpha\delta_0(a_{-n}, e)\big) + (\frac{\alpha}{2} + \lambda_0)\delta_0(a_0, e) + \lambda_0\big(\sum_{n\geq 1} 2^{-n}\alpha\delta_0(u_n, e)\big) \leq \lambda\delta(a, e)$.

A similar calculation shows $\delta\big(\sigma^{-1}(a), \tilde{e}\big) \leq \lambda\delta(a, \tilde{e})$ for $a \in V^u(\tilde{e}, \eta')$. If $x, y \in V^s(\tilde{e}, \eta'/2)$ then $a = x - y \in V^s(\tilde{e}, \eta')$, and similarly for V^u. The translation invariance of δ implies the result. ∎

5. **Isomorphism invariants.** We note some further properties of (K, σ). If (K, σ) is σ-finitely generated then the periodic points are dense in K ([14]). This also follows from [11] since then (K, σ) satisfies the descending chain condition. As noted previously, this is ensured, for example, if σ is expansive. In terms of the element $Q(= a^{-1}F)$ of $\mathrm{GL}(d, \mathbb{Q})$ defining K, σ expansive is equivalent to $\mathrm{sp}(Q) \cap \mathbb{T} = \phi$ ([13]). This condition also ensures that (K, σ) is ergodic with respect to μ, the (normalized) Haar measure on K, since the later is equivalent to $\mathrm{sp}(Q)$ not containing a n-th root of unity ([11], *cf.* [5]).

The density of the periodic points implies that the nonwandering set for σ is K. Coupled with the existence of hyperbolic coordinates (for σ expansive), the Smale and Bowen spectral decomposition results apply to (K, σ) ([6]). We remark that ergodicity of σ with respect to μ implies that σ is (one-sided) topologically transitive, *i.e.*, has a dense (forward) orbit ([23]). More is true. Since σ is an ergodic endomorphism of a compact metrizable group, σ is strong mixing with respect to u and so also topologically strong mixing ([23]). Thus (K, σ) must occur as an elementary part in the Bowen decomposition ([6]). It also follows (from σ being an ergodic endomorphism of K) that μ is a measure of maximum entropy ([23]). That μ is actually the unique such measure follows from

the fact that the entropy of σ is finite ([23]). An alternate route to this conclusion is by noting that since (K, σ) is an elementary part of the Bowen decomposition, it has the shadowing (tracing) property and the specification property ([6]). It is therefore, since σ is expansive, intrinsically ergodic, *i.e.*, there is a unique measure (so the Bowen measure) of maximal entropy ([6]).

Expansivity and the specification property imply that the (topological) entropy of (K, σ) is given by the growth rate of the number of periodic points, namely ([6])

$$h(\sigma) = \lim_{n\to\infty} \frac{1}{n} \log |\{x \in K \mid \sigma^n(x) = x\}|.$$

Since $|\{x \in K \mid \sigma^n(x) = x\}|$ is an invariant of the crossed product C^*-algebra $C(K) \rtimes_\sigma \mathbb{Z}$ ([10]) the entropy of σ is also (*cf.*, [5]).

THEOREM 5.1. *Let σ be an expansive automorphism of a solenoid K. There is a sequence of isomorphism invariants including the entropy of σ for the C^*-algebra $C(K) \rtimes_\sigma \mathbb{Z}$.*

Since (K, σ) is "irreducible" (an elementary part of the Bowen decomposition) in the strong sense described in [17], the considerations of that paper apply to the Smale space (K, σ). Given $x, y \in K$ then x is stably (respectively unstably, asymptotically) equivalent to y, written $x \overset{s}{\sim} y$ (respectively $x \overset{u}{\sim} y, x \overset{a}{\sim} y$) if $\delta(\sigma^n x, \sigma^n y) \to 0$ as $n \to \infty$ (respectively $-n \to \infty, |n| \to \infty$) ([17], [20]). Since these relations are defined in terms of the translation invariant metric δ, they are determined by the respective equivalence classes of the unit, $\tilde{e}$, of K, which are subgroups of K. Let $K_s = \{x \in K \mid x \overset{s}{\sim} \tilde{e}\}$, with K_u, K_a defined analogously. Note $K_a = K_s \cap K_u$ is dense in K ([20]) so these subgroups are not locally compact in the subspace topology. However, the inductive limit topology on these subgroups yields locally compact groups acting continuously by translation on K. For example, $K_s = \bigcup_{n\geq 0} \sigma^{-n}\left(V^s(\tilde{e}, \delta')\right)$ where $V^s(\tilde{e}, \delta')$ is homeomorphic (via ϕ) with the locally compact space $\{u \in E_+ \mid \|u\| < \delta'\} \times \{g_+ \mid g \in \Sigma\} = (E_+)_{\delta'} \times \Sigma_+$. With this topology, the C^*-algebras of the equivalence relations $C^*(\overset{s}{\sim})$, $C^*(\overset{u}{\sim})$, $C^*(\overset{a}{\sim})$ are the crossed product algebras obtained from the action of the groups K_s, K_u and K_a respectively on K. As observed in [17] these algebras are simple. In this situation one may see this by noting that the groups K_s, K_u and K_a each act freely on K with dense orbits. Note also that $C^*(\overset{s}{\sim}, Q) \cong C^*(\overset{u}{\sim}, Q^{-1})$ where $Q \in \mathcal{M}_d(\mathbb{Q})$ is the hyperbolic map determining the hyperbolic system (K, σ).

There are enough invariants (though K-theoretical ones play no role) to essentially distinguish the isomorphism class of the usual crossed product algebra $C(K) \rtimes_\sigma \mathbb{Z}$ in some restricted cases ([4]). In fact, in these cases, one can recover the dynamical system (as a member of the class of expansive automorphisms of a solenoid). Thus, one can naturally associate with these crossed product algebras the simple C^*-algebras $C^*(\overset{s}{\sim})$, $C^*(\overset{u}{\sim})$ and $C^*(\overset{a}{\sim})$ along with the action of $\mathbb{Z}$ on them. According to the classification programme ([7]), K-theoretical invariants should be able to classify these simple algebras. In the other direction one could also ask how well the associated simple C^*-algebras, along with their $\mathbb{Z}$-actions, determine the crossed product algebra $C(K) \rtimes \mathbb{Z}$.

We briefly indicate some examples using this framework (*cf.* [17]).

LEMMA 5.2. *Given a triple* (d, F, M) *with* $\det F \det M \neq 0$, *assume* $(\det M, \det F) = 1$ *and that* M *and* F *commute. Then* $\{p(\sigma^n(g)) \mid g \in \Sigma_-\} = \{q(u) \mid u \in M^{-n}(\mathbb{Z}^d)\} = M^{-n}(e)$ *and* $\{p(\sigma^{-n}(g)) \mid g \in \Sigma_+\} = \{q(u) \mid u \in F^{-n}(\mathbb{Z}^d)\} = F^{-n}(e)$, *for* $n \in \mathbb{Z}$.

PROOF. This follows by using $A\mathbb{Z}^d + B\mathbb{Z}^d = \mathbb{Z}^d$ whenever $(\det A, \det B) = 1$ for $A, B \in \mathcal{M}_d(\mathbb{Z})$ (Proposition 3.11) and an induction argument. ∎

If $M = 1$ in a given triple (d, F, M), (or if $Q \in \mathcal{M}_d(\mathbb{Z})$, consider the triple $(d, Q, 1)$) then $G = K = \{x \in (\mathbb{T}^d)^{\mathbb{Z}} \mid x_{n+1} = Fx_n, n \in \mathbb{Z}\}$ and $H = \Sigma = \Sigma_+$. It follows that $x \in K_u$ if and only if $x \in \theta(E_-)$. If we further assume that $\mathrm{sp}(F)$ and the closed unit disk are disjoint, so that $\mathbb{R}^d = E_-$, we have that $x \in K_s$ if and only if $x \in \bigcup_{n\geq 0} \sigma^{-n}(\Sigma_+)$ (with the inductive limit topology). The preceding lemma shows $p(K_s) = \bigcup F^{-n}(e)$ and in fact $p^{-1}\left(\bigcup F^{-n}(e)\right) = K_s$ since $\Sigma = \Sigma_+$. Thus, for $M = 1$ and $\mathrm{sp}(F)$ disjoint from the closed unit disk, we have (*cf.* [17]) that $C^*(\overset{u}{\sim}) = C(K) \rtimes K_u = C(K) \rtimes_\theta \mathbb{R}^d \cong C(\Sigma) \rtimes_\chi \mathbb{Z}^d \otimes \mathcal{K}\left(L^2(\mathbb{T}^d)\right)$. Also $C^*(\overset{s}{\sim}) = C(K) \rtimes K_s = C(K) \rtimes (\bigcup \sigma^{-n}(\Sigma)) \cong C(\mathbb{T}^d) \rtimes [\bigcup F^{-n}(e)] \otimes \mathcal{K}$, where the subgroup $\bigcup F^{-n}(e)$ of $\mathbb{T}^d$ is given the discrete topology.

REMARK 5.3. In this preceding example the discrete group $\bigcup F^{-n}(e)$ is the group $q\mathbb{Z}^d[F, F^{-1}] \cong \hat{\Sigma}_t$ (Remark 2.10) where Σ_t is the fibre of $p: K_t \to \mathbb{T}^d$ for the σ-f.g. solenoid obtained from the transposed matrix Q_t $(= F_t)$. Notice also that $C^*(\overset{s}{\sim})$ (and $C^*(\overset{u}{\sim})$) are limits of finite dimensional algebras over $C(\mathbb{T}^d)$.

In general, we can also associate with a given σ-f.g. solenoid (K, σ) (described by a nonsingular element Q of $\mathcal{M}_d(\mathbb{Q})$) the (isomorphic) pair of crossed product algebras $C(\Sigma) \rtimes_\chi \mathbb{Z}^d$ and $C(\mathbb{T}^d) \rtimes_{\hat{\chi}} \hat{\Sigma}$. These may also be viewed as algebras associated to the bundle K with characteristic class χ. Note that these algebras are, in general, not simple. For example, let $Q = 1$, so that $\Sigma \cong \hat{\Sigma} = \{e\}$. To ensure that these algebras are simple it is enough (since $\mathrm{Im}\,\chi$ is dense in Σ) that the group acts freely on the compact space, equivalent here to the homomorphism χ being injective. Propositions 3.2 and 3.3 give some sufficient conditions for the characteristic class χ to be injective, thus associating another simple C^*-algebra with (K, σ). Since $\hat{\Sigma}$ is an increasing union of finite subgroups of $\mathbb{T}^d$ (Remark 2.10), this algebra is a limit of finite dimensional algebras over $C(\mathbb{T}^d)$.

We return to the example (K, σ) defined by F $(= Q) \in \mathcal{M}_d(\mathbb{Z})$, $\det F \neq 0$ and $\mathrm{sp}(F)$ disjoint from the closed unit disk. A. Kumjian has suggested that by using results of M. Rørdam, for example, one possibly could show that the crossed products of the stable and unstable algebras with the automorphism defined by σ are purely infinite algebras (in the general Smale space setting). This is the case for the examples under consideration.

The automorphism σ of K yields an automorphism α of $C^*(\overset{s}{\sim})$ ([17]). In our context the automorphism of $C^*(\overset{s}{\sim}) = C(K) \rtimes \left(\bigcup \sigma^{-n}(\Sigma)\right)$ is defined by $\alpha(f)(g, k) = \lambda f\left(\sigma^{-1}(g), \sigma^{-1}(k)\right)$ for $g \in K_s$, $k \in K$, $f: K_s \to C(K)$ continuous and compactly supported and $\lambda = |\det F|$. Note that $\ln \lambda$ is the entropy of σ. Restricting α^{-1} to the unital, simple and nuclear C^*-algebra $\mathcal{A}_s = C(\mathbb{T}^d) \rtimes \left(\bigcup F^{-n}(e)\right)$ defines an endomorphism ρ of $\mathcal{A}_s$ given by $(\rho f)(g, k) = \lambda^{-1} f(Fg, Fk)$ where $g \in \bigcup F^{-n}(e)$, $k \in \mathbb{T}^d$ and

$f: \bigcup F^{-n}(e) \longrightarrow C(\mathbb{T}^d)$ with finite support. Recall $\mathcal{A}_s = \lim\big(C(\mathbb{T}^d) \rtimes F^{-n}(e)\big)$ with unital embeddings. For $h \in C(\mathbb{T}^d) \subseteq \mathcal{A}_s$, $\rho(h) = (h \circ F)\lambda^{-1} \sum_{g \in F^{-1}(e)} \delta_g$ where δ_g is the unitary in $C(\mathbb{T}^d) \rtimes F^{-1}(e)$ implementing translation by g on $\mathbb{T}^d$. Thus $\rho(h) = (h \circ F) \otimes p_1$ in the identification of $C(\mathbb{T}^d) \rtimes F^{-1}(e)$ with $C(\mathbb{T}^d) \otimes \mathcal{M}_\lambda(\mathbb{C})$, where $p_1 = \lambda^{-1}(\sum_{g \in F^{-1}(e)} \delta_g)$ is a proper projection of $\mathcal{M}_\lambda(\mathbb{C})$ (the order of the group $F^{-1}(e)$ is $|\det F| = \lambda$ which is larger than 1 by the condition on $\mathrm{sp}(F)$). Recall here that $h \circ F \in C(\mathbb{T}^d)^{F^{-1}(e)} \cong C(\mathbb{T}^d)$. Thus $\rho(1)$ is a proper projection in $\mathcal{A}_s$ and ρ is a corner endomorphism of $\mathcal{A}_s$.

Now observe that $\mathcal{A}_s$ has a unique trace and so has real rank zero ([2]). Also, since it is a limit of matrix algebras over $C(\mathbb{T}^d)$, it has the comparability property. Thus the crossed product of $\mathcal{A}_s$ with the corner endomorphism ρ (that scales the trace by λ^{-1}) yields a purely infinite simple unital nuclear algebra ([18]). These types of algebras have recently been classified.

6. **A universal property.** We briefly describe a universal property for the algebras $C(K) \rtimes_\sigma \mathbb{Z}$ associated to (K, σ).

THEOREM 6.1. *Let $F = [f_{ij}] \in \mathcal{M}_d(\mathbb{Z})$ with $\det F \neq 0$ and $a \in \mathbb{N}_+$. Let $\mathcal{A}$ be a C^*-algebra generated by $d+1$ unitaries $V_0, \ldots, V_{d-1}, U$ such that*

(i) $\{\mathrm{ad}\, U^k(V_i) \mid 0 \leq i \leq d-1, k \in \mathbb{Z}\}$ *is a commuting family of unitaries*

(ii) $\mathrm{ad}\, U(V_i^a) = \prod_{k=0}^{d-1} V_k^{f_{ki}}$ $(0 \leq i \leq d-1)$

(iii) for $W \in \mathcal{U}$, the group generated by the unitaries $V_0, \ldots, V_{d-1}, U$, we have $W = I$ if $W^a = I$.

*Then there is a *-homomorphism of $C(K) \rtimes_\sigma \mathbb{Z}$ onto $\mathcal{A}$ (intertwining the automorphism σ with $\mathrm{ad}\, U$) where $\hat{K} = \mathbb{Z}^d[Q, Q^{-1}]$ and $Q = a^{-1}F$. The C^*-algebra $C(K) \rtimes_\sigma \mathbb{Z}$ is such an $\mathcal{A}$. The *-homomorphism maps the corresponding unitaries in $C(K) \rtimes_\sigma \mathbb{Z}$ to those in $\mathcal{A}$.*

PROOF. We introduce notation. Choose a basis $\{e_i \mid 0 \leq i \leq d-1\}$ of $\mathbb{Z}^d$ and view $\mathbb{Z}^d$ as a subgroup of $\oplus_{\mathbb{Z}} \mathbb{Z}^d$ with $v \in \mathbb{Z}^d$ the element of $\oplus \mathbb{Z}^d$ mapping j in $\mathbb{Z}$ to $\delta_{oj} v$. The automorphism τ of $\oplus \mathbb{Z}^d$ is given by the shift; $\big(\tau(n)\big)_k = n_{k-1}$, $(n \in \oplus \mathbb{Z}^d)$ and $\tilde{F} \in \mathrm{Aut}(\mathbb{Z}^d)$ is the element $\oplus F$ which commutes with τ. For $v \in \mathbb{Z}^d$, v^k denotes the element $\tau^k(v)$ of $\oplus \mathbb{Z}^d$.

Define a homomorphism $\varphi: \oplus \mathbb{Z}^d \longrightarrow \mathcal{U}$ by

$$\varphi(be_i^k) = \mathrm{ad}\, U^k(V_i^b), \quad 0 \leq i \leq d-1;\ b, k \in \mathbb{Z}.$$

By definition, $\mathrm{ad}\, U^s \varphi(e_i^k) = \varphi(e_i^{k+s}) = \varphi\big(\tau^s(e_i^k)\big)$ for $0 \leq i \leq d-1$; $k, s \in \mathbb{Z}$. It follows that $\mathrm{ad}\, U^s \circ \varphi = \varphi \circ \tau^s$, $(s \in \mathbb{Z})$.

To recast condition (ii) using φ note that $\varphi\big(\tau(ae_i)\big) = \mathrm{ad}\, U \varphi(ae_i) = \varphi(\tilde{F} e_i)$. Thus, for $k \in \mathbb{Z}$, $\varphi(ae_i^k) = \mathrm{ad}\, U^{k-1} \varphi\big(\tau(ae_i)\big) = \mathrm{ad}\, U^{k-1} \varphi(\tilde{F} e_i) = \varphi(\tau^{k-1} \tilde{F} e_i) = \varphi(\tilde{F} e_i^{k-1})$, so

$$\varphi(a\tau n) = \varphi(\tilde{F} n), \quad \Big(n \in \bigoplus \mathbb{Z}^d\Big).$$

By induction, $\varphi(a^r \tau^r n) = \varphi(\tilde{F}^r n)$, $(r \in \mathbb{N}_+, n \in \oplus \mathbb{Z}^d)$.

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We show that if $n = (n_k) \in \oplus \mathbb{Z}^d$ satisfies $\Sigma Q^k n_k = 0$ (in $\mathbb{Q}^d$) then $\varphi(n) = I$. This yields a map of $\hat{K} \cong \oplus \mathbb{Z}^d / K^\perp \longrightarrow \mathcal{A}$ and so a homomorphism of $C^*(\hat{K}) \cong C(K) \longrightarrow \mathcal{A}$. Since multiplication by Q is the automorphism on the quotient group $\hat{K}$ corresponding to the automorphism τ on $\oplus \mathbb{Z}^d$, and since τ corresponds to ad U on $\mathcal{A}$, the result follows.

Choose $n = (n_k) \in \oplus \mathbb{Z}^d$ with $\Sigma Q^k n_k = 0$. Writing n as $(\dots 0, n_s, \dots, n_{r-s}, 0, \dots)$ with $r, s \in \mathbb{N}$ we see that there is an r and $s \in \mathbb{N}$ with $Q^s a^r(\Sigma Q^k n_k) = \sum_{j=0}^r F^j a^{r-j} n_{j-s} = 0$. The later is an equality in $\mathbb{Z}^d$, so $\varphi(\sum_{j=0}^r F^j a^{r-j} n_{j-s}) = I$. Since $\varphi(F^j a^{r-j} n_{j-s}) = \varphi(a^j \tau^j a^{r-j} n_{j-s}) = \varphi(a^r \tau^j n_{j-s})$ we have $\varphi(\sum_{j=0}^r \tau^j n_{j-s})^{a^r} = I$. Condition (iii) implies $\varphi(\sum_{j=0}^r \tau^j n_{j-s}) = I$ so $I = \text{ad } U^{-s}\left(\varphi(\sum_{j=0}^r \tau^j n_{j-s})\right) = \varphi(\sum_{j=0}^r \tau^{j-s} n_{j-s}) = \varphi(n)$. ∎

REMARK 6.2. We point out that the solenoid K in the preceding result is associated with the triple (d, F_t, a).

An immediate application is that $C(K) \rtimes_\sigma \mathbb{Z}$ is the universal C^*-algebra generated by two unitary operators D and U satisfying $UD = D^2U$ where K is the σ-f.g. solenoid associated with the element $2 = F \in M_1(\mathbb{Z})$. Unitary operators satisfying these relations occur in wavelet theory.

PROPOSITION 6.3. *Let* $\ell(x) = \sum_{i=0}^d a_i x^i \in \mathbb{Z}[x]$ $(a_d a_0 \neq 0)$ *with* $Q_t \in \mathcal{M}_d(\mathbb{Q})$ *the companion matrix of* $a_d^{-1}\ell$ *(so* $Q = a_d^{-1}F$ *with* $F \in \mathcal{M}_d(\mathbb{Z})$*).*

Then $\mathbb{Z}^d[Q, Q^{-1}] \cong \mathbb{Z}[x, x^{-1}]/(\ell)$ *(as rings) if and only if the content of* ℓ *(=* cont ℓ*) is* 1.

PROOF. Since $\mathbb{Z}[x, x^{-1}]/(\ell)$ is torsion free if and only if cont$(\ell) = 1$ and since $\mathbb{Z}^d[Q, Q^{-1}]$ is torsion free (it is a subgroup of $\mathbb{Q}^d$) we need only show that there is an isomorphism if $\mathbb{Z}[x, x^{-1}]/(\ell)$ is torsion free. Using the notation established in Theorem 6.1 define a group homomorphism $\gamma: \oplus \mathbb{Z}^d \longrightarrow \mathbb{Z}[x, x^{-1}]/(\ell)$ by $\gamma(e_i^k) = x^{k+i}$. We have $\gamma \circ \tau = M_x \circ \gamma$ where M_x is multiplication by x, so kerγ (along with $K^\perp$) is τ-invariant. It follows from $\ell(x) = 0$ that $\gamma(\text{ad } \tau e_i) = \gamma(\tilde{F} e_i)$ and so, as in Theorem 6.1, $\gamma(\text{ad}^r \tau^r n) = \gamma(\tilde{F}^r n)$ for $n \in \oplus \mathbb{Z}^d$, $r \in \mathbb{N}$. Since $\mathbb{Z}[x, x^{-1}]/(\ell)$ is torsion free, the argument in Theorem 6.1 shows $K^\perp \subseteq \ker\gamma$. Note further that $\mathbb{Z}^d \cap \ker\gamma = 0$. Also, for $v \in \mathbb{Z}^d$, $\text{ad } \tau^{k+1} v - \tau^k F v \in K^\perp$, $(k \in \mathbb{Z})$, since $\text{ad } Q^{k+1} v - Q^k F v = Q^k(\text{ad } Qv - Fv) = Q^k(0) = 0$. With these observations it now follows (Lemma 18 [13]) that $K^\perp = \ker\gamma$. ∎

The preceding proposition implies that the C^*-algebras considered in [4, 2, 3] fall within the framework of the crossed product algebras considered here. In particular, some of the results in those papers which where proved for irreducible elements $\ell \in \mathbb{Z}[x]$ are now true for (expansive systems defined by) elements $\ell \in \mathbb{Z}[x]$ with cont$(\ell) = 1$.

REFERENCES

1. E. Atkin, *The General Topology of Dynamical Systems*, Graduate Studies in Math. **1**, Amer. Math. Soc., 1991.
2. B. Blackadar, M. Dadarlat and M. Rørdam, *The real rank of inductive limit C*-algebras*, Math. Scand. **69**(1991), 211–216.
3. B. Brenken, *K-groups of solenoidal algebras* 1, Proc. Amer. Math. Soc. (5)**23**(1995), 1457–1464.
4. ———, *Isomorphism classes of solenoidal algebras* 1, Canad. Math. Bull. (4) **36**(1993), 414–418.

5. B. Brenken and P. Jorgensen, *A family of dilation crossed product algebras*, J. Operator Theory **25**(1991), 299–308.
6. M. Denker, C. Grillenberger and K. Sigmund, *Ergodic theory on compact spaces*, Springer Lecture Notes in Math. **527**, 1976.
7. G. Elliott, *Are amenable C^*-algebras classifiable?*, Contemp. Math. **145**(1993), 423–427.
8. D. Fried, *Finitely presented dynamical systems*, Ergodic Theory Dynamical Systems **7**(1987), 489–507.
9. E. Hewitt and K. A. Ross, *Abstract Harmonic Analysis, Vol. 1*, Springer-Verlag, 1963.
10. S. Kawamura, J. Tomiyama and Y. Watatani, *Finite dimensional irreducible representations of C^*-algebras associated with topological dynamical systems*, Math. Scand. **56**(1985), 241–248.
11. B. Kitchens and K. Schmidt, *Automorphisms of compact groups*, Ergodic Theory Dynamical Systems, **9**(1989), 691–735.
12. P. F. Lam, *On expansive automorphism transformation groups*, Trans. Amer. Math. Soc. **105**(1970), 131–138.
13. W. Lawton, *The structure of compact connected groups which admit an expansive automorphism*, Springer Lecture Notes in Math. **318**(1973), 182–196.
14. R. Laxton and W. Parry, *On the periodic points of certain automorphisms and a system of polynomial identities*, J. Algebra **6**(1967), 388–393.
15. D. Lind, K. Schmidt and T. Ward, *Mahler Measure and Entropy for Commuting Automorphisms of Compact Groups*, Invent. Math. (3) **101**(1990), 593–629.
16. D. Lind and T. Ward, *Automorphisms of solenoids and p-adic entropy*, Ergodic Theory Dynamical Systems **8**(1988), 411–419.
17. I. Putnam, *C^*-algebras from Smale spaces*, preprint.
18. M. Rørdam, *Classification of certain infinite simple C^*-algebras*, J. Funct. Anal., to appear.
19. D. Ruelle, *Noncommutative algebras for hyperbolic diffeomorphisms*, Invent. Math. **93**(1988), 1–13.
20. ———, *Thermodynamic Formalism*, Encyclopedia of Math. Appl. **5**, Addison-Wesley, Reading, 1978.
21. N. Steenrod, *The Topology of Fibre Bundles*, Princeton University Press, Princeton, 1951.
22. E. Spanier, *Algebraic Topology*, McGraw-Hill, New York, 1966.
23. P. Walters, *An Introduction to Ergodic Theory*, Graduate Texts in Math., Springer-Verlag, 1982.

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