


EXACT FORMULAE AND TURÁN INEQUALITIES FOR Vafa–WITTEN INVARIANTS OF $K3$ SURFACES

DANIEL R. JOHNSTON¹ AND JOSHUA MALES² 

¹*School of Science, The University of New South Wales, Canberra, ACT, Australia*
(daniel.johnston@adfa.edu.au)

²*Department of Mathematics, Machray Hall, University of Manitoba, Winnipeg, Canada*
(joshua.males@umanitoba.ca)

(Received 12 April 2022)

Abstract We consider three different families of Vafa–Witten invariants of $K3$ surfaces. In each case, the partition function that encodes the Vafa–Witten invariants is given by combinations of twisted Dedekind η -functions. By utilizing known properties of these η -functions, we obtain exact formulae for each of the invariants and prove that they asymptotically satisfy all higher-order Turán inequalities.

Keywords: Vafa–Witten invariants; exact formulae; asymptotics; Turán inequalities

2020 Mathematics subject classification: 11F12; 11F23

1. Introduction and statement of results

In 1994, Vafa and Witten [24] introduced invariants that count the number of solutions to the $N = 4$ supersymmetric Yang–Mills equations over four-dimensional spaces, now called Vafa–Witten invariants. Given a surface and group action, many results in the literature focus on providing the generating function Z of the Vafa–Witten invariants (also known as the partition function). These generating functions are often linear combinations of modular objects. Thus, with the tools of modular forms at our disposal, we are in a position to obtain precise information on the Vafa–Witten invariants in question. One of the most natural families of surfaces, which we consider throughout, are the $K3$ surfaces, and there have been several recent results for the generating function of Vafa–Witten invariants in this case [15, 22, 23].

In what follows, we let

$$\eta(q) := q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n) \tag{1.1}$$

be the usual Dedekind η -function, where $q := e^{2\pi i\tau}$ with $\tau \in \mathbb{H}$.



1.1. Formulae for Vafa–Witten invariants

Tanaka and Thomas proved the following formula for the generating function of $SU(r)$ Vafa–Witten invariants on $K3$ surfaces, thereby proving a conjecture of Vafa and Witten (here and throughout, $r \in \mathbb{N}$).

Theorem 1.1 (Theorem 5.24 of [22]). *For a $K3$ surface, the partition function of rank r trivial determinant $SU(r)$ Vafa–Witten invariants is*

$$Z^{SU(r)}(q) := \sum_{n \in \mathbb{Z}} \alpha_{1,r}(n)q^n = \sum_{d|r} \frac{d}{r^2} q^r \sum_{j=0}^{d-1} \eta \left(\zeta_d^j q^{\frac{r}{d^2}} \right)^{-24},$$

where $\zeta_d = e^{\frac{2\pi i}{d}}$.

Moreover, in [22, Theorem 5.24], it is shown that the same generating function also gives the rank r , trivial determinant, weighted Euler characteristic Vafa–Witten invariants, and so any results for $Z^{SU(r)}$ also apply to these invariants.

In [15], Jiang and Kool determined the generating function for $SU(p)/\mathbb{Z}_p$ Vafa–Witten invariants for prime p . To describe the result, fix a Chern class $c_1 \in H^2(S, \mathbb{Z})$, let μ_p be the cyclic group of order p and let

$$\delta_{a,b} := \begin{cases} 1 & \text{if } a - b \in pH^2(S, \mathbb{Z}), \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 1.2 (Theorem 1.4 of [15]). *For a $K3$ surface S , prime p , generic polarization and $c_1 \in H^2(S, \mathbb{Z})$ algebraic, we have the partition function*

$$Z_{c_1}^{SU(p)/\mathbb{Z}_p}(q) = \sum_{w \in H^2(S, \mu_p)} e^{\frac{2\pi i(w \cdot c_1)}{p}} Z_w(q),$$

where $w \cdot c_1$ is the natural inner product on $H^2(S, \mathbb{Z})$ (see [24, pages 45 and 46]) and

$$Z_w(q) := \sum_{n \in \mathbb{Z}} \alpha_{2,p}(w; n)q^{\frac{n}{p}} = \frac{\delta_{w,0}}{p^2} \eta(q^p)^{-24} + \frac{1}{p} \sum_{j=0}^{p-1} e^{-\frac{\pi i j w^2}{p}} \eta \left(\zeta_p^j q^{\frac{1}{p}} \right)^{-24}. \tag{1.2}$$

We also consider the twisted Vafa–Witten invariants for smooth projective $K3$ surfaces as described by Jiang (see also [14, Definition 5.13]).

Theorem 1.3 (Theorem 6.36 of [14]). *Let S be a smooth projective $K3$ surface with Picard number $\rho(S)$ and p prime. Then we have the partition function*

$$Z_{twist}^{SU(p)/\mathbb{Z}_p}(q) := \sum_{n \in \mathbb{Z}} \alpha_{3,p}(n)q^{\frac{n}{p}} = \frac{q^p}{p^2} \eta(q^p)^{-24} + q^p \left(p^{21} \eta \left(q^{\frac{1}{p}} \right)^{-24} + p^{\rho(S)-1} \sum_{j=1}^{p-1} \eta \left(\zeta_p^j q^{\frac{1}{p}} \right)^{-24} \right). \tag{1.3}$$

Remark. Note that the generating functions in Theorems 1.2 and 1.3 consist of n/p th powers of q unlike the generating function in Theorem 1.1, which only contains integer powers of q . However, in special cases, the non-integral powers of q in Equations (1.2) and (1.3) vanish. See Corollaries 3.1 and 3.2 for more information.

To obtain an exact formula for each of the Vafa–Witten invariants discussed above, we first obtain an exact formula for the coefficients of $\eta(q)^{-24}$. This is achieved using a result of Zuckerman (see Section 2.1), which relates the coefficients of a weakly holomorphic form to the principal part of the form at each cusp.

Lemma 1.4. *Let $a(n)$ denote the n th Fourier coefficient of $\eta(q)^{-24}$. Then, $a(n) = 0$ for $n < -1$, $a(-1) = 1$ and $a(0) = 24$. For $n > 0$, we have*

$$a(n) = \frac{2\pi}{n^{\frac{13}{2}}} \sum_{k=1}^{\infty} \frac{1}{k} \sum_{\substack{0 \leq h < k \\ \gcd(h,k)=1}} \omega_{h,k}^{24} e^{-\frac{2\pi i(n+1)h}{k}} I_{13} \left(\frac{4\pi}{k} \sqrt{n} \right).$$

Moreover,

$$a(n) = \frac{2\pi}{n^{\frac{13}{2}}} I_{13} (4\pi\sqrt{n}) \left(1 + O \left(\frac{1}{e^{2\pi\sqrt{n}}} \right) \right) \sim \frac{2\pi}{n^{\frac{13}{2}}} \frac{e^{4\pi\sqrt{n}}}{\sqrt{8\pi^2\sqrt{n}}}, \tag{1.4}$$

where I_{13} is the usual I -Bessel function.

We then prove the following theorem, which combined with Lemma 1.4 gives exact formulae for the Vafa–Witten invariants.

Theorem 1.5. *For all $n \in \mathbb{Z}$, we have*

$$\alpha_{1,r}(n) = \sum_{s|(n-r,r)} \frac{1}{s^2} a \left(\frac{(n-r)r}{s^2} \right), \tag{1.5}$$

along with

$$\alpha_{2,p}(w; n) = \frac{1}{p} \sum_{j=0}^{p-1} e^{\frac{\pi i j(2n-w^2)}{p}} a(n) + \begin{cases} \frac{\delta_{w,0}}{p^2} a \left(\frac{n}{p^2} \right) & (p^2 | n) \\ 0 & (\text{otherwise}) \end{cases}, \tag{1.6}$$

and

$$\alpha_{3,p}(n) = p^{21} a(n - p^2) + \begin{cases} \frac{1}{p^2} a \left(\frac{n-p^2}{p^2} \right) & \text{if } (p^2 | n) \\ 0 & (\text{otherwise}) \end{cases} + \begin{cases} p^{\rho(S)-1} (r-1) a(n - p^2), & \text{if } (p | n) \\ -p^{\rho(S)-1} a(n - p^2), & \text{if } (p \nmid n) \end{cases}. \tag{1.7}$$

From these exact formulae, we also deduce the asymptotic behaviours of the invariants using Equation (1.4).

Theorem 1.6. *As $n \rightarrow \infty$, we have*

$$\alpha_{1,r}(n) = \frac{2\pi}{(rn)^{\frac{13}{2}}} I_{13}(4\pi\sqrt{rn}) \left(1 + O_r \left(\frac{1}{e^{2\pi\sqrt{rn}}} \right) \right). \tag{1.8}$$

For $\alpha_{2,p}(n)$, we have

$$\alpha_{2,p}(n) = \begin{cases} \frac{2\pi}{n^{\frac{13}{2}}} I_{13}(4\pi\sqrt{n}) \left(1 + O_r \left(\frac{1}{e^{2\pi\sqrt{n}}} \right) \right) & \text{if } 2p \mid 2n - w^2, \\ \frac{2\pi\delta_{w,0}p^{\frac{9}{2}}}{n^{\frac{13}{2}}} I_{13} \left(\frac{4\pi}{p} \sqrt{n} \right) \left(1 + O_p \left(\frac{1}{e^{\frac{2\pi}{p}\sqrt{n}}} \right) \right) & \text{if } 2p \nmid 2n - w^2 \text{ and } p^2 \mid n, \\ 0 & \text{otherwise.} \end{cases} \tag{1.9}$$

Finally,

$$\alpha_{3,p}(n) = \frac{2\pi c_p(n)}{n^{\frac{13}{2}}} I_{13}(4\pi\sqrt{n}) \left(1 + O \left(\frac{1}{e^{2\pi\sqrt{n}}} \right) \right), \tag{1.10}$$

where

$$c_p(n) = \begin{cases} p^{21} + p^{\rho(S)-1}(p-1), & (p \mid n), \\ p^{21} - p^{\rho(S)-1}, & (p \nmid n). \end{cases}$$

The case of $\alpha_{2,p}$ is a little complicated in the statement of the theorem, which is explained by the fact that the sum over j can vanish or not as n varies, yielding a non-uniform asymptotic.

1.2. Turán inequalities

The Turán inequalities for functions in the real entire Laguerre–Pólya class are intricately linked to the Riemann hypothesis [9, 21] and have seen renewed interest in recent years. In fact, the Riemann hypothesis is true if and only if the Riemann Xi function lies in the Laguerre–Pólya class, and a necessary condition is that the Maclaurin coefficients satisfy Turán inequalities of all orders [9, 20].

The second-order Turán inequality for a sequence $\{a_n\}_{n \geq 0}$ is known as log-concavity and is satisfied if

$$a_n^2 \geq a_{n-1}a_{n+1}$$

for all $n \geq 1$. Log-concavity for modular objects is well-studied in the literature, for example [4, 6–8, 17]. The higher-order Turán inequalities are linked to the Jensen polynomial

associated to a sequence of real numbers $\{\alpha(n)\}$, given by

$$J_\alpha^{d,n}(X) := \sum_{j=0}^{d-1} \binom{d}{j} \alpha(n+j) X^j.$$

Consider a polynomial with real coefficients, $f(x) = x^n + a_{n-1}x^{n-1} + \dots + a_1x + a_0$. Then a classical result of Hermite states that the polynomial f is hyperbolic if and only if the Hankel matrix of f is positive definite. In turn, this gives a set of inequalities on the coefficients of f as all $k \times k$ minors of the Hankel matrix must be positive definite (see, e.g., the introduction of [7] for an explicit description). When applied to $J_\alpha^{d,n}$, Hermite’s theorem gives a set of inequalities associated to the sequence $\{\alpha(n)\}_{n \geq 0}$ and are known as the order k Turán inequalities¹. Thus, the higher order (strict) Turán inequalities for $1 \leq k \leq d$ are satisfied for $\{\alpha(n+j)\}_{n \geq 0}$ if and only if $J_\alpha^{d,n}$ is hyperbolic.

In [10], the authors proved new results on the (asymptotic) hyperbolicity of Jensen polynomials attached to a vast array of sequences, including those that are coefficients of weakly holomorphic modular forms. Following this, several other recent papers have made progress in this area, for example [16], [11] and [7].

Our result in this direction readily follows from results of [10], and we obtain the following which, to the best of the authors’ knowledge, is the first description of higher-order Turán inequalities for Vafa–Witten invariants. The case for $\alpha_{2,p}$ is slightly more complicated than the others, as it depends inherently on the choice of w .

Theorem 1.7. *For fixed r and p , as $n \rightarrow \infty$, the Vafa–Witten invariants $\alpha_{1,r}(n)$ and $\alpha_{3,p}(pn)$ each satisfy all order k Turán inequalities. If $\gcd(2p, w^2) = 1$, then $\alpha_{2,p}(w; n)$ satisfies all order k Turán inequalities. If $w^2 = 2pm$ for some $m \in \mathbb{Z}$, then $\alpha_{2,r}(w; pn)$ satisfies all order k Turán inequalities. Moreover, $\alpha_{2,p}(w; p^2n)$ satisfies all order k Turán inequalities.*

Since the Jensen polynomial attached to each of the sequences $\alpha_{1,r}(n)$, $\alpha_{2,p}(pn)$ and $\alpha_{3,p}(pn)$ (replacing $\alpha_{2,p}(n)$ by $\alpha_{2,p}(pn)$ or $\alpha_{2,p}(p^2n)$ as necessary) for fixed r and p are all eventually hyperbolic, in Corollary 4.1, we obtain several examples of generating functions, which can be regarded as new examples of functions in the shifted Laguerre–Pólya class, as in recent work of Wagner [25]. To the best of the authors’ knowledge, such functions have so far only previously been studied in the context of number theory and combinatorics. That our functions lie in the shifted Laguerre–Pólya class is a direct consequence of the generating function of $\alpha_{j,r}(n)$ arising from specializations of (twists of) Dedekind η -functions. A natural question to pose is whether further examples of topological invariants define objects in the same class of functions, and therefore what are the implications of general properties of this class of function to the topological spaces at hand, motivating further investigation of the situation in [25].

2. Preliminaries

In this section, we collect several preliminary results required for the rest of the paper.

¹ One also allows the equality case.

2.1. An exact formula of Zuckerman

We require a powerful result of Zuckerman [26] that builds on the work of Rademacher [19]. Zuckerman obtained exact formulae for Fourier coefficients of weakly holomorphic modular forms of arbitrary non-positive weight on finite index subgroups of $SL_2(\mathbb{Z})$ in terms of the cusps of the underlying subgroup and the principal parts of the form at each cusp. Let $\tau \in \mathbb{H}$ and let F be a weakly holomorphic modular form of weight $\kappa \leq 0$ with multiplier χ satisfying the transformation law

$$F(\gamma\tau) = \chi(\gamma)(-i(c\tau + d))^\kappa F(\tau) \tag{2.1}$$

for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in some finite index subgroup of $SL_2(\mathbb{Z})$. The transformation law (2.1) can be expressed equivalently in terms of the cusp $\frac{h}{k} \in \mathbb{Q}$. Let h' be defined by the congruence $hh' \equiv -1 \pmod{k}$. Letting $\tau = \frac{h'}{k} + \frac{i}{kz}$ and choosing $\gamma = \gamma_{h,k} := \begin{pmatrix} h & \beta \\ k & -h' \end{pmatrix} \in SL_2(\mathbb{Z})$, we obtain the equivalent transformation law

$$F\left(\frac{h}{k} + \frac{iz}{k}\right) = \chi(\gamma_{h,k})z^{-\kappa} F\left(\frac{h'}{k} + \frac{i}{kz}\right). \tag{2.2}$$

Let F have the Fourier expansion at $i\infty$ given by

$$F(\tau) = \sum_{n \gg -\infty} a(n)q^{n+\alpha}$$

and Fourier expansions at each rational number $0 \leq \frac{h}{k} < 1$ given by

$$F|_\kappa \gamma_{h,k}(\tau) = \sum_{n \gg -\infty} a_{h,k}(n)q^{\frac{n+\alpha_{h,k}}{c_k}}.$$

In this framework, Zuckerman’s result [26, Theorem 1] (see also [3]) reads as follows.

Theorem 2.1. *Assume the notation and hypotheses above. If $n + \alpha > 0$, then*

$$a(n) = 2\pi(n + \alpha)^{\frac{\kappa-1}{2}} \sum_{k=1}^{\infty} \frac{1}{k} \sum_{\substack{0 \leq h < k \\ \gcd(h,k)=1}} \chi(\gamma_{h,k}) e^{-\frac{2\pi i(n+\alpha)h}{k}}$$

$$\times \sum_{m+\alpha_{h,k} \leq 0} a_{h,k}(m) e^{\frac{2\pi i}{kc_k}(m+\alpha_{h,k})h'} \left(\frac{|m + \alpha_{h,k}|}{c_k}\right)^{\frac{1-\kappa}{2}} I_{-\kappa+1}\left(\frac{4\pi}{k} \sqrt{\frac{(n + \alpha)|m + \alpha_{h,k}|}{c_k}}\right),$$

where $I_{-\kappa+1}$ is the usual I -Bessel function.

2.2. Transformation laws of q -Pochhammer symbols

For coprime $h, k \in \mathbb{N}$, let

$$\omega_{h,k} := \exp(\pi i \cdot s(h, k)),$$

with the Dedekind sum

$$s(h, k) := \sum_{\mu \pmod{k}} \left(\left(\frac{\mu}{k} \right) \right) \left(\left(\frac{h\mu}{k} \right) \right).$$

Here,

$$((x)) := \begin{cases} x - [x] - \frac{1}{2} & \text{if } x \in \mathbb{R} \setminus \mathbb{Z}, \\ 0 & \text{if } x \in \mathbb{Z}. \end{cases}$$

Let $(a; q)_\infty$ be the usual q -Pochhammer symbol and $q := e^{\frac{2\pi i}{k}(h+iz)}$ for $z \in \mathbb{C}$ with $\text{Re}(z) > 0$. The classical modular transformation law for the Dedekind η -function (see e.g. [2, Theorem 3.4]) is

$$\eta\left(\frac{a\tau + b}{c\tau + d}\right) = \epsilon(a, b, c, d) (-i(c\tau + d))^{\frac{1}{2}} \eta(\tau),$$

where

$$\epsilon(a, b, c, d) = e^{i\pi\left(\frac{a+d}{12c} + s(-d, c)\right)}$$

if $c > 0$. Thus, if $(a, b, c, d) = (h, \beta, k, -h')$, we get

$$\eta\left(\frac{h'}{k} + \frac{i}{kz}\right) = e^{i\pi\left(\frac{h-h'}{12k} + s(h', k)\right)} z^{-\frac{1}{2}} \eta\left(\frac{h}{k} + \frac{iz}{k}\right)$$

so that

$$\frac{1}{\eta\left(\frac{h'}{k} + \frac{i}{kz}\right)} = e^{i\pi\left(\frac{h'-h}{12k} + s(h, k)\right)} z^{\frac{1}{2}} \frac{1}{\eta\left(\frac{h}{k} + \frac{iz}{k}\right)}. \tag{2.3}$$

Here we have used a standard result regarding the Dedekind sum ([2, Theorem 3.6b]), which implies that $s(h, k) = -s(h', k)$.

2.3. Hyperbolicity of Jensen polynomials

In [10], it was shown that for certain classes of functions, the Jensen polynomials of the coefficients tend to the order d Hermite polynomials $H_d(X)$, in turn implying asymptotic hyperbolicity.

Theorem 2.2 (Theorem 3 of [10]). *Let $\{\alpha(n)\}$, $\{A(n)\}$ and $\{\delta(n)\}$ be three sequences of positive real numbers with $\delta(n)$ tending to 0 and satisfying*

$$\log \left(\frac{\alpha(n+j)}{\alpha(n)} \right) = A(n)j - \delta(n)^2 j^2 + \sum_{i=3}^d g_i(n)j^i + o(\delta(n)^d)$$

as $n \rightarrow \infty$, where each $g_i(n) = o(\delta(n)^i)$ for each $3 \leq i \leq d$. Then the renormalized Jensen polynomials

$$\widehat{J}_\alpha^{d,n}(X) := \frac{\delta(n)^{-d}}{\alpha(n)} J_\alpha^{d,n} \left(\frac{\delta(n)X - 1}{\exp(A(n))} \right)$$

satisfy

$$\lim_{n \rightarrow \infty} \widehat{J}_\alpha^{d,n}(X) = H_d(X)$$

uniformly for X in any compact subset of \mathbb{R} . Moreover, this implies that the Jensen polynomials $J_\alpha^{d,n}$ are each hyperbolic for all but finitely many n .

3. Vafa–Witten invariants

The goal of this section is to obtain the claimed exact formulae and asymptotics for $\alpha_{1,r}(n)$, $\alpha_{2,p}(w;n)$ and $\alpha_{3,p}(n)$. We begin with a short subsection computing an exact formula for the coefficients $a(n)$.

3.1. Formula for the coefficients of $\eta(q)^{-24}$

We begin by proving Lemma 1.4, which gives an exact formula for the n th coefficient of $\eta(q)^{-24} = \sum_{n \in \mathbb{Z}} a(n)q^n$. Note that in [12], a longer and more general argument, not using Zuckerman’s formula, is used to compute exact formulae for fractional powers of η .

Proof of Lemma 1.4. By the definition (1.1) of $\eta(q)$, we have $a(n) = 0$ for $n < -1$, $a(-1) = 1$ and $a(0) = 24$. For $n > 0$, we apply Zuckerman’s formula. So, let $F(q) = \eta(q)^{-24}$. For the setup of Zuckerman’s formula, we write

$$\tau = \frac{h'}{k} + \frac{i}{kz}, \quad q = e^{2\pi i \tau}, \quad \text{and} \quad q_1 = e^{2\pi i(\gamma_{h,k}(\tau))},$$

where $\gamma_{h,k} = \begin{pmatrix} h & \beta \\ k & -h' \end{pmatrix}$ as in § 2.1.

Now, using Equation (2.3), we have

$$F(q_1) = e^{24\pi i \left(\frac{h'-h}{12k} + s(h,k) \right)} z^{12} F(q). \tag{3.1}$$

In terms of Zuckerman’s formula, $\alpha = 0$, and from Equation (3.1), we see that $\kappa = -12$ and

$$\chi(\gamma_{h,k}) = \omega_{h,k}^{24} e^{2\pi i \left(\frac{h'-h}{k}\right)}.$$

Next, by Equation (3.1),

$$F|_{\kappa}\gamma_{h,k}(\tau) = F(\tau) = \sum_{n \in \mathbb{Z}} a(n)q^n.$$

Thus, we can set $c_k = 1$, $\alpha_{h,k} = \alpha = 0$ and $a_{h,k}(n) = a(n)$. Moreover, the only term in the principal part of $F(\tau)$ is q^{-1} with coefficient $a(-1) = 1$. Putting all of this information into Theorem 2.1 gives, for $n > 0$,

$$\begin{aligned} a(n) &= \frac{2\pi}{n^{\frac{13}{2}}} \sum_{k=1}^{\infty} \frac{1}{k} \sum_{\substack{0 \leq h < k \\ \gcd(h,k)=1}} \omega_{h,k}^{24} e^{2\pi i \left(\frac{h'-h}{k}\right)} e^{-\frac{2\pi i n h}{k}} e^{-\frac{2\pi i h'}{k}} I_{13} \left(\frac{4\pi}{k} \sqrt{n}\right) \\ &= \frac{2\pi}{n^{\frac{13}{2}}} \sum_{k=1}^{\infty} \frac{1}{k} \sum_{\substack{0 \leq h < k \\ \gcd(h,k)=1}} \omega_{h,k}^{24} e^{-\frac{2\pi i (n+1)h}{k}} I_{13} \left(\frac{4\pi}{k} \sqrt{n}\right), \end{aligned} \tag{3.2}$$

as desired.

For the asymptotic expression, we note that for any $\nu \geq 0$ (see, e.g., [1, (4.12.7)])

$$I_{\nu}(x) = \frac{e^x}{\sqrt{2\pi x}} (1 + O(x^{-1})), \tag{3.3}$$

meaning the $k = 1$ term dominates in Equation (3.2). □

We are now in a position to compute the exact formulae and asymptotics for $\alpha_{1,r}(n)$, $\alpha_{2,p}(w; n)$ and $\alpha_{3,p}(n)$ (Theorems 1.5 and 1.6).

3.2. Exact formula for $\alpha_{1,r}(n)$

In this subsection, we prove the exact formula (1.5) for $\alpha_{1,r}(n)$ and the corresponding asymptotic Equation (1.8). So, recall that

$$Z^{\text{SU}(r)}(q) = \sum_{n \in \mathbb{Z}} \alpha_{1,r}(n)q^n = q^r \sum_{d|r} \frac{d}{r^2} \sum_{j=0}^{d-1} \eta \left(\zeta_d^j q^{\frac{r}{d^2}}\right)^{-24}. \tag{3.4}$$

Using the substitution $q \mapsto \zeta_d^j q^{\frac{r}{d^2}}$, we see that for any $k \in \mathbb{Z}$, the $k \frac{r}{d^2}$ th coefficient of

$$\eta \left(\zeta_d^j q^{\frac{r}{d^2}} \right)^{-24}$$

is equal to $\zeta_d^{jk} a(k)$. Therefore, the $k \frac{r}{d^2}$ th coefficient of

$$\frac{d}{r^2} \sum_{j=0}^{d-1} \eta \left(\zeta_d^j q^{\frac{r}{d^2}} \right)^{-24} \tag{3.5}$$

is $\frac{d^2}{r^2} a(k)$ if $d \mid k$ and 0 if $d \nmid k$. In particular, the only non-zero coefficients of q in Equation (3.5) are the multiples of r/d . Hence, the n th coefficient of

$$\sum_{d|r} \frac{d}{r^2} \sum_{j=0}^{d-1} \eta \left(\zeta_d^j q^{\frac{r}{d^2}} \right)^{-24}$$

is given by

$$\sum_{d|r, \frac{r}{d}|n} \frac{d^2}{r^2} a \left(\frac{nd^2}{r} \right).$$

Incorporating the factor of q^r in Equation (3.4) and letting $s = r/d$ then gives

$$\alpha_{1,r}(n) = \sum_{s|(n-r,r)} \frac{1}{s^2} a \left(\frac{(n-r)r}{s^2} \right), \tag{3.6}$$

as required. Now, from the asymptotic expression (1.4) for $a(n)$, we see that the $s = 1$ term dominates in Equation (3.6) so that

$$\alpha_{1,r}(n) \sim a((n-r)r) \sim a(rn)$$

with each of the error terms in these asymptotics less than $O_r \left(\frac{I_{13}(4\pi\sqrt{n})}{e^{2\pi\sqrt{rn}} n^{\frac{13}{2}}} \right)$. This gives Equation (1.8).

3.3. Exact formula for $\alpha_{2,p}(w; n)$

We now turn to the second Vafa–Witten invariant. To begin with, let $\alpha_{2,p}(w; n) = \alpha_{2,p,1}(w; n) + \alpha_{2,p,2}(w; n)$, where

$$\sum_{n \in \mathbb{Z}} \alpha_{2,p,1}(w; n) q^{\frac{n}{p}} = \frac{\delta_{w,0}}{p^2} \eta(q^p)^{-24},$$

$$\sum_{n \in \mathbb{Z}} \alpha_{2,p,2}(w; n) q^{\frac{n}{p}} = \frac{1}{p} \sum_{j=0}^{p-1} e^{-\frac{\pi i j w^2}{p}} \eta(\zeta_p^j q^{\frac{1}{p}})^{-24}.$$

Now, the n/p th coefficient of $\eta(q^p)^{-24}$ is given by $a(n/p^2)$ if $n \mid p^2$ and 0 otherwise. Thus,

$$\alpha_{2,p,1}(w; n) = \begin{cases} \frac{\delta_{w,0}}{p^2} a\left(\frac{n}{p^2}\right) & (n \mid p^2), \\ 0 & (\text{otherwise}). \end{cases} \tag{3.7}$$

Then, the n/p th coefficient of $\eta(\zeta_p^j q^{\frac{1}{p}})^{-24}$ is given by $\zeta_p^{nj} a(n)$ so that

$$\alpha_{2,p,2}(w; n) = \frac{1}{p} \sum_{j=0}^{p-1} e^{\frac{\pi i j (2n-w^2)}{p}} a(n). \tag{3.8}$$

Combining Equations (3.7) and (3.8) then gives Equation (1.6) as desired. Similar to the asymptotics for $\alpha_{1,r}(n)$, the asymptotics for $\alpha_{2,p}(w; n)$ are obtained directly from the asymptotic expression (1.4) for $a(n)$. Some care must be taken, however, as the sum

$$\sum_{j=0}^{p-1} e^{\frac{\pi i j (2n-w^2)}{p}}$$

may vanish depending on the value of w , as indicated by Equation (1.9). We draw particular attention to the case when w^2 is an even integer. Namely, writing $w^2 = 2m$, we have

$$\alpha_{2,p,2}(w; n) = \begin{cases} a(n) & (p \mid n - m), \\ 0 & (\text{otherwise}). \end{cases} \tag{3.9}$$

In addition, if $p \mid m$, we can do even more. In particular, we have the following corollary.

Corollary 3.1. *Let $\alpha'_{2,p}(w; n) = \alpha_{2,p}(w; pn)$. If $w^2 = 2m$ is an even integer and $p \mid m$, then*

$$Z_w(q) = \sum_{n \in \mathbb{Z}} \alpha'_{2,p}(w; n) q^n.$$

That is, all of the non-integer powers of q vanish from the expansion of $Z_w(q)$. Moreover,

$$\alpha'_{2,p}(w; n) = a(pn) + \begin{cases} \frac{\delta_{w,0}}{p^2} a\left(\frac{n}{p}\right) & (p \mid n) \\ 0 & (\text{otherwise}) \end{cases}.$$

As a result, $\alpha'_{2,p}(w; n) \sim a(pn)$ and satisfies the same asymptotics (1.8) as $\alpha_1(n)$.

Proof. We simply apply Equation (3.9), noting that if $p \mid m$, then $p \mid n - m$ is equivalent to $p \mid n$. □

3.4. Exact formulae for $\alpha_{3,p}(n)$

Finally, we turn to proving an exact formula for the third Vafa–Witten invariant in question. Let

$$\sum_{n \geq 0} \alpha_{3,p,1}(n)q^{\frac{n}{p}} = \frac{1}{p^2}q^p\eta(q^p)^{-24}, \tag{3.10}$$

$$\sum_{n \geq 0} \alpha_{3,p,2}(n)q^{\frac{n}{p}} = p^{21}q^p\eta\left(q^{\frac{1}{p}}\right)^{-24}, \tag{3.11}$$

$$\sum_{n \geq 0} \alpha_{3,p,3}(n)q^{\frac{n}{p}} = p^{\rho(S)-1}q^p \sum_{j=1}^{p-1} \eta\left(e^{\frac{2\pi ij}{p}}q^{\frac{1}{p}}\right)^{-24} \tag{3.12}$$

so that $\alpha_{3,p}(n) = \alpha_{3,p,1}(n) + \alpha_{3,p,2}(n) + \alpha_{3,p,3}(n)$. Now, Equations (3.10) and (3.11) are simple transformations of the function $\eta(q)^{-24} = \sum a(n)q^n$. In particular,

$$\alpha_{3,p,1}(n) = \begin{cases} \frac{1}{p^2}a\left(\frac{n}{p^2} - 1\right) & (p^2 \mid n), \\ 0 & (\text{otherwise}) \end{cases}$$

and

$$\alpha_{3,p,2}(n) = p^{21}a(n - p^2).$$

As for Equation (3.12), we note similarly as in § 3.2 that the n/p th coefficient of

$$\eta\left(e^{\frac{2\pi ij}{p}}q^{\frac{1}{p}}\right)$$

is given by $\zeta_p^{jn}a(n)$. Thus, the n/p th coefficient of

$$\sum_{j=1}^{p-1} \eta\left(e^{\frac{2\pi ij}{p}}q^{\frac{1}{p}}\right)^{-24} = -\eta\left(q^{\frac{1}{p}}\right)^{-24} + \sum_{j=0}^{p-1} \eta\left(e^{\frac{2\pi ij}{p}}q^{\frac{1}{p}}\right)^{-24}$$

is $(p - 1)a(n)$ if $p \mid n$ or $-a(n)$ if $p \nmid n$. Therefore,

$$\alpha_{3,p,3}(n) = \begin{cases} p^{\rho(S)-1}(p - 1)a(n - p^2) & (p \mid n), \\ -p^{\rho(S)-1}a(n - p^2) & (p \nmid n). \end{cases}$$

Combining our expressions for $\alpha_{3,p,1}(n)$, $\alpha_{3,p,2}(n)$ and $\alpha_{3,p,3}(n)$ gives Equation (1.7) as required. Again, to obtain the asymptotics (1.10), we use the asymptotic expression (1.4) for $a(n)$.

We also note that the exact formula (1.7) for $\alpha_{3,p}(n)$ simplifies in the case when the surface S is supersingular (i.e., $\rho(S) = 22$). In particular, all non-integral powers of q vanish, leading to the following corollary.

Corollary 3.2. *Let $\alpha'_{3,p}(n) = \alpha_{3,p}(pn)$. Then, if the surface in question is supersingular, we have*

$$Z_{\text{twist}}^{\text{SU}(p)/\mathbb{Z}_p}(q) = \sum_{n \geq 0} \alpha'_{3,p}(n)q^n$$

and

$$\alpha'_{3,p}(n) = p^{22}a(p(n-p)) + \begin{cases} \frac{1}{p^2}a\left(\frac{n}{p}-1\right) & (p \mid n) \\ 0 & (\text{otherwise}) \end{cases}.$$

Furthermore,

$$\alpha'_{3,p}(n) = \frac{2\pi p^{\frac{31}{2}}}{n^{\frac{13}{2}}} I_{13}(4\pi\sqrt{pn}) \left(1 + O_p\left(\frac{1}{e^{2\pi\sqrt{pn}}}\right)\right) \sim \frac{p^{\frac{61}{4}} e^{4\pi\sqrt{pn}}}{\sqrt{2n}^{\frac{27}{4}}}.$$

Proof. Let $\rho(S) = 22$ in Equations (1.7) and (1.10). □

3.5. Higher-order Turán inequalities

In this section, we prove that all relevant Vafa–Witten invariants asymptotically satisfy the higher-order Turán inequalities.

Proof of Theorem 1.7. We have

$$\alpha_{1,r}(n) = \frac{2\pi}{(nr)^{\frac{13}{2}}} I_{13}(4\pi\sqrt{nr}) \left(1 + O_r\left(\frac{1}{e^{2\pi\sqrt{nr}}}\right)\right)$$

as $n \rightarrow \infty$. This implies the asymptotic formula

$$\begin{aligned} \log\left(\frac{\alpha_{1,r}(n+k)}{\alpha_{1,r}(n)}\right) &\sim \log\left(\frac{n^{\frac{13}{2}} I_{13}(4\pi\sqrt{(n+j)r})}{(n+j)^{\frac{13}{2}} I_{13}(4\pi\sqrt{nr})}\right) \\ &= 4\pi\sqrt{r} \sum_{i \geq 1} \binom{1/2}{i} \frac{k^i}{n^{\frac{1}{2}-i}} + \frac{21}{4} \sum_{i \geq 1} \frac{(-1)^{i-1} k^i}{in^i} + \sum_{s,t \geq 1} c_t \binom{-t}{s} \frac{k^s}{n^{s+t}} \end{aligned}$$

for some (computable) constants c_t arising from the asymptotic expansion for the I -Bessel function, valid to all orders of $n^{-\frac{1}{2}}$. We may then apply Theorem 2.2 with $A(n)$ and $\delta(n)$

given by

$$A(n) = 2\pi\sqrt{\frac{r}{n}} + O\left(\frac{1}{n}\right), \quad \delta(n)^2 = \frac{\pi}{2}\sqrt{rn}^{-\frac{3}{2}} + O\left(n^{-\frac{5}{4}}\right).$$

Noting that the asymptotic hyperbolicity of the relevant Jensen polynomials implies that all higher-order Turán inequalities are satisfied for large enough n , finishing the proof in this case. The arguments for $\alpha_{2,p}$ and $\alpha_{3,p}$ are very similar, where for $\alpha_{2,p}$, one accounts for the variations depending on w^2 . □

4. Further results

We end by indicating related results and their implications to $\alpha_{1,r}(n)$, $\alpha_{2,p}(n)$ and $\alpha_{3,p}(n)$.

4.1. Laguerre–Polyá classes of functions

A real entire function $\psi(x) = \sum_{k \geq 0} \frac{\gamma_k}{k!} x^k$ is said to lie in the Laguerre–Polyá class if it can be represented by

$$\psi(x) = Cx^m e^{bx-ax^2} \prod_{k=1}^r \left(1 + \frac{x}{x_k}\right) e^{-\frac{x}{x_k}}, \quad 0 \leq r \leq \infty,$$

with $b, C, x_k \in \mathbb{R}$, $m \in \mathbb{Z}_{\geq 0}$, $a \geq 0$ and $\sum_{k=1}^r x_k^{-2} < \infty$. In [25], Wagner introduced new classes of objects known as the shifted-Laguerre–Polyá classes, and we recall the definitions here.

A real entire function $\psi(x)$ lies in the shifted-Laguerre–Polyá class of degree d if it is the uniform limit of polynomials $\{\psi_k(x)\}_{k \geq 0}$ with the property that there exists $N(d)$ such that $\psi_n^{(dn-d)}(x)$ has all real roots for $n \geq N(d)$. The function ψ is then said to lie in the shifted Laguerre–Polyá class if it satisfies the above for all $d \in \mathbb{N}$.

Most importantly for the present paper is [25, Theorem 1.1], which states that if $\{\gamma_k\}_{k \geq 0}$ is a sequence of eventually non-negative reals, then the following are equivalent²;

- (i) The sequence $\{\gamma_k\}$ is a shifted multiplier sequence of type 2 (in the sense of [25]).
- (ii) For every $d \in \mathbb{N}$, there exist $N(d)$ such that the Jensen polynomial $J_{\gamma}^{d,n}(X)$ has all real roots for $n \geq N(d)$.
- (iii) The formal power series $\sum_{k \geq 0} \frac{\gamma_k}{k!} x^k$ defines a function in the shifted Laguerre–Polyá class.

In § 3.5, we proved that the Jensen polynomial attached to each of the Vafa–Witten invariants $\alpha_{1,r}(n)$, $\alpha_{2,p}(n)$ and $\alpha_{3,p}(pn)$ (throughout replacing $\alpha_{2,p}(n)$ by $\alpha_{2,p}(pn)$ or $\alpha_{2,p}(p^2n)$ as necessary) for fixed r and p tend to the Hermite polynomials as $n \rightarrow \infty$, and so have all real roots. Therefore, we immediately obtain the following corollary.

² Here we use the second remark under [25, Theorem 1.1] to rewrite the theorem in the general setting where we do not assume that the real roots have the same sign.

Corollary 4.1. *Let r be a fixed positive integer and p be a fixed prime. Then each sequence $\alpha_{1,r}$, $\alpha_{2,p}$ and $\alpha_{3,p}$ defines a shifted multiplier sequence of type 2, and the formal power series*

$$\sum_{n \geq 0} \frac{\alpha_{1,r}(n)}{n!} x^n, \quad \sum_{n \geq 0} \frac{\alpha_{3,p}(pn)}{(pn)!} x^{pn},$$

along with

$$\sum_{n \geq 0} \frac{\alpha_{2,p}(w; n)}{(n)!} x^n \quad \text{if } \gcd(w^2, 2p) = 1, \quad \sum_{n \geq 0} \frac{\alpha_{2,p}(w; pn)}{(pn)!} x^{pn} \quad \text{if } w^2 = 2pm$$

and

$$\sum_{n \geq 0} \frac{\alpha_{2,p}(w; p^2n)}{(p^2n)!} x^{p^2n}$$

each define a function in the shifted Laguerre–Polyá class.

4.2. Representations as Poincaré series and algebraic formulae

Let Γ_∞ be the stabilizer of $i\infty$ in $\Gamma := \text{SL}_2(\mathbb{Z})$ and $|_{k,\nu}$ the usual Petersson slash operator with weight k and multiplier ν (see, e.g., [13]). For $\kappa \in \frac{1}{2}\mathbb{Z}$ with $\kappa \geq \frac{5}{2}$, and $m \in \frac{1}{24}\mathbb{Z}$ such that $\kappa - 12m \in 2\mathbb{Z}$, the Poincaré series of weight κ and index m that we consider is defined by

$$P_{k,m}(\tau) := \sum_{M \in \Gamma_\infty \backslash \Gamma} q^m |_{\kappa,\nu_\eta^{24m}} M,$$

with ν_η the multiplier of η . In [18], the authors showed that $a(n)$ is given by the coefficient of q in $P_{14,1-n}$, and thus we can build linear combinations of weight 14 Poincaré series with varying index whose coefficients are precisely $\alpha_{1,r}(n)$, $\alpha_{2,p}(n)$ or $\alpha_{3,p}(n)$. For example, for fixed r , we have

$$\alpha_{1,r}(n) = [\text{coefficient } q] \sum_{s|(n-r,r)} \frac{1}{s^2} P_{14,1-\frac{(n-r)r}{s^2}}(\tau).$$

Although this formula looks a little messy, it simplifies nicely in explicit examples; taking $r = 2$ for instance gives

$$\alpha_{1,2}(n) = [\text{coefficient } q] \left(P_{14,3-4m}(\tau) + \frac{\delta_{\text{even}}}{4} P_{14,2-m}(\tau) \right),$$

where $\delta_{\text{even}} = 0$ if $n = 2m + 1$ and 1 if $n = 2m$. Formulae for the remaining Vafa–Witten invariants as coefficients of linear combinations of Poincaré series follow similarly.

In their famous paper [5], Bruinier and Ono proved a formula for the coefficients of $\frac{1}{\eta(q)}$ as a sum of certain algebraic integers. Since each of the Vafa–Witten invariants in this paper have a generating function that is (essentially) a linear combination of $\frac{1}{\eta(q)^{24}}$, one could obtain exact algebraic formulae for each of $\alpha_{1,r}(n)$, $\alpha_{2,p}(n)$ and $\alpha_{3,p}(n)$ in a similar fashion.

Funding Statement. The research conducted by the second author of this paper is supported by the Pacific Institute for the Mathematical Sciences (PIMS). The research and findings may not reflect those of the Institute.

Competing Interests. The authors declare none.

References

- (1) G. E. Andrews, R. Askey and R. Roy, *Special functions*, Volume 1 (Cambridge University Press, 1999).
- (2) T. M. Apostol, *Modular functions and Dirichlet series in number theory* (Springer-Verlag, New York, 1990).
- (3) K. Bringmann, W. Craig, J. Males and K. Ono, Distributions on partitions arising from Hilbert schemes and hook lengths *Forum Math. Sigma*, **10** (2022), e49, 30.
- (4) K. Bringmann, C. Jennings-Shaffer, K. Mahlburg and R. Rhoades, Peak positions of strongly unimodal sequences, *Trans. Amer. Math. Soc.* **372**(10) (2019), 7087–7109.
- (5) J. Bruinier and K. Ono, Algebraic formulas for the coefficients of half-integral weight harmonic weak Maass forms, *Adv. Math.* **246** (2013), 198–219.
- (6) G. Cesana, W. Craig and J. Males, Asymptotic equidistribution for partition statistics and topological invariants, preprint.
- (7) W. Craig and A. Pun, A note on the higher order Turán inequalities for k -regular partitions, *Res. Number Theory* **7**(1) (2021), 5, 7.
- (8) S. DeSalvo and I. Pak, Log-concavity of the partition function, *Ramanujan J.* **38**(1) (2015), 61–73.
- (9) D. Dimitrov, Higher order Turán inequalities, *Proc. Amer. Math. Soc.* **126**(7) (1998), 2033–2037.
- (10) M. Griffin, K. Ono, K. L. Rolen and D. Zagier, Jensen polynomials for the Riemann zeta function and other sequences, *Proc. Natl. Acad. Sci. USA* **116**(23) (2019), 11103–11110.
- (11) M. Griffin, K. Ono, L. Rolen, J. Thorner, Z. Tripp and I. Wagner, Jensen polynomials for the Riemann xi-function, *Adv. Math.* **397** (2022), 108186.
- (12) J. Iskander, V. Jain and V. Talvola, Exact formulae for the fractional partition functions, *Res. Number Theory* **6**(2) (2020), 20, 17.
- (13) H. Iwaniec, *Topics in classical automorphic forms*. Graduate Studies in Mathematics, Volume 17, p. xii+259 (American Mathematical Society, Providence, RI, 1997).
- (14) Y. Jiang, Counting twisted sheaves and S-duality, *Adv. Math.* **400** (2022), 108332, 75.
- (15) Y. Jiang and M. Kool, Twisted sheaves and $SU(r)/\mathbb{Z}_r$ Vafa–Witten theory, *Math. Ann.* **382**(1–2) (2021), 719–743.
- (16) H. Larson and I. Wagner, Hyperbolicity of the partition Jensen polynomials, *Res. Number Theory* **5**(2) (2019), 19, 12.
- (17) J. Males, Asymptotic equidistribution and convexity for partition ranks, *Ramanujan J.* **54**(2) (2021), 397–413.

- (18) W. Pribitkin and B. Williams, Short proof of Rademacher’s formula for partitions, *Res. Number Theory* **5**(2) (2019), 17, 6.
- (19) H. Rademacher, A convergent series for the partition function $p(n)$, *Proc. Natl. Acad. Sci. USA* **23**(2) (1937), 78–84.
- (20) J. Schur and G. Pólya, Über zwei Arten von Faktorenfolgen in der Theorie der algebraischen Gleichungen, *J. Reine Angew. Math.* **144** (1914), 89–113.
- (21) G. Szegő, On an inequality of P. Turán concerning Legendre polynomials, *Bull. Amer. Math. Soc.* **54** (1948), 401–405.
- (22) Y. Tanaka and R. Thomas, Vafa-Witten invariants for projective surfaces II: semistable case, *Pure Appl. Math. Q.* **13**(3) (2017), 517–562.
- (23) R. Thomas, Equivariant K -theory and refined Vafa-Witten invariants, *Comm. Math. Phys.* **378** (2020), 2 1451–1500.
- (24) C. Vafa and E. Witten, A strong coupling test of S -duality, *Nuclear Phys. B* **431**(1–2) (1994), 3–77.
- (25) I. Wagner, On a new class of Laguerre-Pólya type functions with applications in number theory, *Pacific J. Math.* **320**(1) (2022), 177–192.
- (26) H. Zuckerman, On the coefficients of certain modular forms belonging to subgroups of the modular group, *Trans. Amer. Math. Soc.* **45** (1939), 298–321.