Can. J. Math. Vol. XXXVI, No. 5, 1984, pp. 795-799

MODULAR AND ADMISSIBLE SEMILATTICES

C. S. HOO AND P. V. RAMANA MURTY

1. Introduction. We correct some errors in [**1**] and extend some of the results there. Generally, we shall follow the terminology and notation of [**1**]. There is an error in the proof of Lemma 3.13 there, and consequently the subsequent results which depend on it are incorrect as stated. However, they are correct if we replace the condition "a-admissible" by "strongly a-admissible" (see [**3**] where this notion was introduced). We also show that the results in [**1**] are correct if the semilattices are assumed to be modular.

2. Strongly admissible semilattices. We shall change the terminology in [**3**] slightly.

Definition 1 (see [**3**]). Let A be a Boolean algebra and let D be a meet semilattice with 1. An admissible map $f:A \times D \rightarrow D$ is called *strongly admissible* if

$$f(a, d)\, f(a', d) = d \quad \text{for each } a \in A,\ d \in D,$$

where a' is the complement of a in A.

The following result was proved in [**4**] (page 362).

THEOREM 2.1. *Let A be a Boolean algebra and let D be a meet semilattice with* 1. *Then an admissible map $f:A \times D \rightarrow D$ is strongly admissible if and only if*

$$f(a + b, d) = f(a, d)\, f(b, d) \quad \textit{for all } a, b \in A,\ d \in D.$$

Definition 2. An a-admissible semilattice is strongly a-admissible if the corresponding admissible map $f:C_a \times D_a \rightarrow D_a$ is strongly admissible.

For the rest of the paper, A will always denote a Boolean algebra and D a meet semilattice. The following are easily verified.

Let $f:A \times D \rightarrow D$ be an admissible map, where D is implicative. Then

(i) $f(a, d_1 * d_2) \leqq f(a, d_1) * f(a, d_2)$
(ii) $f(a, d_1) * f(a, d_2) = f(a, d_1) * f(a, d_1 * d_2)$
(iii) $f(a, d_1) * f(a, d_2) = d_1 * f(a, d_2)$.

Received November 22, 1982. The first author was supported by NSERC (Canada) Grant A3026.

LEMMA 2.2. *Let $f{:}A \times D \to D$ be strongly admissible, where D is implicative. Then f satisfies*

$$f(a, d) * f(a', d) = f(a, d) * d.$$

THEOREM 2.3. *Let $f{:}A \times D \to D$ be strongly admissible, where D is implicative. Then f satisfies*

$$f(a, d_1 * d_2) = [f(a, d_1) * f(a, d_2)]f(0, d_1 * d_2).$$

Proof. We have

$$\begin{aligned} d_1 * d_2 &= d_1 * \{f(a, d_2)f(a', d_2)\} \\ &= \{d_1 * f(a, d_2)\}\{d_1 * f(a', d_2)\} \\ &= \{f(a, d_1) * f(a, d_2)\}\{f(a'd_1) * f(a', d_2)\} \\ &\geqq \{f(a, d_1) * f(a, d_2)\}f(a', d_1 * d_2). \end{aligned}$$

Hence

$$f(a, d_1 * d_2) \geqq f(a, f(a, d_1) * f(a, d_2))f(a, f(a', d_1 * d_2)).$$

But

$$f(a, f(a, d_1) * f(a, d_2)) \geqq f(a, d_1) * f(a, d_2) \geqq f(a, d_1 * d_2)$$

and

$$f(a, f(a', d_1 * d_2)) = f(0, d_1 * d_2) \geqq f(a, d_1 * d_2).$$

Hence

$$\begin{aligned} f(a, d_1 * d_2) &\geqq [f(a, d_1) * f(a, d_2)]f(0, d_1 * d_2) \\ &\geqq f(a, d_1 * d_2). \end{aligned}$$

Thus Lemma 3.13 of [**1**] should be replaced by the following.

LEMMA 3.13′. *Let L be a strongly a-admissible semilattice and let*

$$f{:}C_a \times D_a \to D_a$$

be the corresponding strongly admissible map. If D_a is implicative, then f satisfies

$$f(b, d_1 * d_2) = \{f(b, d_1) * f(b, d_2)\}f(a, d_1 * d_2)$$

for all $b \in C_a$ and $d_1, d_2 \in D_a$.

Remark. In our definition of an admissible map $f{:}A \times D \to D$, we do not require that $f(0, d) = 1$ since this condition is not necessarily true in the case of a-admissible semilattices. In many results, this condition is not required.

https://doi.org/10.4153/CJM-1984-046-5 Published online by Cambridge University Press

THEOREM 2.4. *Let* $f:A \times D \to D$ *be strongly admissible where* D *is implicative. Then* $A \times D/f$ *is* $[1, d]$*-implicative for each* $d \in D$.

Proof. Let $d \in D$ and let $[a_1, d_1] \in A \times D/f$. We define

$$[a_1, d_1] * [1, d] = [1, f(a_1, d_1 * d)].$$

This makes $A \times D/f$ into an $[1, d]$-implicative semilattice. For, let

$$[a_2, d_2] \in A \times D/f.$$

Then

$$\begin{aligned} &[a_2, d_2] \leqq [1, f(a_1, d_1 * d)] \\ &\Leftrightarrow f(a_2, d_2) \leqq f(a_2a_1, d_1 * d) \\ &\Leftrightarrow f(a_2a_1, d_2) \leqq f(a_2a_1, d_1 * d) \\ &= [f(a_2a_1, d_1) * f(a_2a_1, d)]f(0, d_1 * d). \end{aligned}$$

But

$$f(a_2a_1, d_2) \leqq [f(a_2a_1, d_1) * f(a_2a_1, d)]f(0, d_1 * d)$$

implies that

$$\begin{aligned} f(a_2a_1, d_2)f(a_2a_1, d_1) &\leqq f(a_2a_1, d_1)f(a_2a_1, d)f(0, d_1 * d) \\ &\leqq f(a_2a_1, d). \end{aligned}$$

Conversely,

$$f(a_2a_1, d_2)f(a_2a_1, d_1) \leqq f(a_2a_1, d)$$

implies that

$$f(a_2a_1, d_2) \leqq f(a_2a_1, d_1) * f(a_2a_1, d).$$

But clearly

$$f(a_2a_1, d_2) \leqq f(a_2a_1, d_1 * d_2) \leqq f(0, d_1 * d_2).$$

Hence

$$f(a_2a_1, d_2) \leqq [f(a_2a_1, d_1) * f(a_2a_1, d)]f(0, d_1 * d_2).$$

Thus,

$$\begin{aligned} &[a_2, d_2] \leqq [1, f(a_1, d_1 * d)] \\ &\Leftrightarrow f(a_2a_1, d_2)f(a_2a_1, d_1) \leqq f(a_2a_1, d) \\ &\Leftrightarrow f(a_2a_1, d_2d_1) \leqq f(a_2a_1, d) \\ &\Leftrightarrow [a_2a_1, d_2d_1] \leqq [1, d] \\ &\Leftrightarrow [a_2, d_2][a_1, d_1] \leqq [1, d]. \end{aligned}$$

THEOREM 2.5. *Let $f:A \times D \to D$ be strongly admissible, where D has* 1. *Then $A \times D/f$ is implicative if and only if so is D.*

Proof. Suppose that D is implicative. Then by Theorem 2.4, $A \times D/f$ is $[1, d]$-implicative. We have seen by Theorem 3.1 of [**1**] that $A \times D/f$ is $[a, 1]$-admissible for all $a \in A$. Hence $A \times D/f$ is $[a, d]$-implicative for all $a \in D$. Conversely, if $A \times D/f$ is implicative, then since D is isomorphic to the $[0, 1]$-dense filter of $A \times D/f$, it is also implicative.

Thus Theorem 3.14, Theorem 3.15 and Lemma 3.18 of [**1**] hold if we replace "a-admissible" by "strongly a-admissible". Similarly, Corollary 3.16 and Theorem 3.17 of [**1**] also hold if we replace "0-admissible" by "strongly 0-admissible".

3. Modular a-admissible semilattices. We now show that Lemma 3.13 of [**1**] and the subsequent results there hold if the semilattices are required to be modular.

LEMMA 3.1. *Let L be a modular a-admissible semilattice, and suppose that D_a is implicative. Then the corresponding admissible map*

$$f:C_a \times D_a \to D_a$$

satisfies

$$d_1 * f(b, d_2) = f(b, d_1 * d_2) \quad \textit{for all } b \in C_a, \textit{ and all } d_1, d_2 \in D_a.$$

Proof. We have

$$d_1 f(b, d_1 * d_2) \leqq f(b, d_1) f(b, d_1 * d_2) = f(b, d_1(d_1 * d_2))$$
$$= f(b, d_1 d_2) \leqq f(b, d_2).$$

Hence

$$f(b, d_1 * d_2) \leqq d_1 * f(b, d_2).$$

Conversely, let $x \leqq d_1 * f(b, d_2)$. Then

$$x d_1 \leqq d_1[d_1 * f(b, d_2)] = d_1 f(b, d_2) \leqq f(b, d_2).$$

Hence $xbd_1 \leqq d_2$ and hence

$$xbd_1 \leqq d_1 d_2 \leqq d_1.$$

Since L is modular we can find $y \geqq xb$ such that $d_1 d_2 = y d_1$. Thus $d_1 d_2 \leqq y$, that is $y \in D_a$. We now have

$$y \leqq d_1 * (d_1 d_2) = d_1 * d_2.$$

Hence $xb \leqq y \leqq d_1 * d_2$, and hence

$$x \leqq f(b, d_1 * d_2).$$

https://doi.org/10.4153/CJM-1984-046-5 Published online by Cambridge University Press

This proves that

$$d_1 * f(b, d_2) \leqq f(b, d_1 * d_2)$$

and proves the lemma.

COROLLARY 3.2. *Let L be a modular a-admissible semilattice, where D_a is implicative. Then the corresponding admissible map*

$$f: C_a \times D_a \rightarrow D_a$$

satisfies

$$f(b, d_1 * d_2) = f(b, d_1) * f(b, d_2)$$

for all $b \in C_a$ and all $d_1, d_2 \in D_a$.

Proof. In general,

$$d_1 * f(b, d_2) = f(b, d_1) * f(b, d_2).$$

The corollary now follows from Lemma 3.1.

Remark. Because of Corollary 3.2, it follows that Lemma 3.13 and all subsequent results in [**1**] hold if the semilattices are required to be modular.

REFERENCES

1. C. S. Hoo, *Pseudocomplemented and implicative semilattices*, Can. J. Math. *35* (1982), 423-437.

2. P V. Ramana Murtv, *Prime and implicative semilattices*, Algebra Universalis *10* (1980), 31-35.

3. P. V. Ramana Murty and V. V. Rama Rao, *Characterization of certain classes of pseudocomplemented semilattices*, Algebra Universalis *4* (1974), 289-300.

4. P. V. Ramana Murty and M. Krishna Murty, *On admissible semilattices*, Algebra Universalis *6* (1976), 355-366.

University of Alberta,
Edmonton, Alberta;
Andhra University,
Waltair, India