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ON PLATE DECOMPOSITIONS OF CONE MULTIPLIERS

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Abstract An important inequality due to Wolff on plate decompositions of cone multipliers is known to have consequences for sharp L^p results on cone multipliers, local smoothing for the wave equation, convolutions with radial kernels, Bergman projections in tubes over cones, averages over finite-type curves in $\mathbb{R}^3$ and associated maximal functions. We observe that the range of p in Wolff's inequality, for the conic and the spherical versions, can be improved by using bilinear restriction results. We also use this inequality to give some improved estimates on square functions associated to decompositions of cone multipliers in low dimensions. This gives a new L^4 bound for the cone multiplier operator in $\mathbb{R}^3$.

Keywords: cone multipliers; Wolff's inequality; plate decompositions

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1. Introduction

Let $\Gamma = \{(\tau,\xi) \in \mathbb{R} \times \mathbb{R}^d : \tau = |\xi|\}$ denote the forward light cone in $\mathbb{R}^{d+1}$, $d \geqslant 2$. For fixed $c > 0$ and small $\delta > 0$, we consider δ-neighbourhoods of the truncated cone

$$\Gamma_\delta(c) = \{(\tau,\xi) \in \mathbb{R}^{d+1} : 1 \leqslant \tau \leqslant 2 \text{ and } |\tau - |\xi|| \leqslant c\delta\},$$

with the usual decomposition into plates subordinated to a $\sqrt{\delta}$-*separated* sequence in the sphere $\{\omega_k\} \subset S^{d-1}$:

$$\Pi_k^{(\delta)} = \left\{(\tau,\xi) \in \Gamma_\delta(c) \;\middle|\; \left|\frac{\xi}{|\xi|} - \omega_k\right| \leqslant c'\sqrt{\delta}\right\}, \quad \operatorname{dist}(\omega_k, \omega_{k'}) \geqslant \sqrt{\delta} \quad \text{if } k \neq k'. \tag{1.1}$$

Let

$$\alpha(p) := d\left(\frac{1}{2} - \frac{1}{p}\right) - \frac{1}{2}, \tag{1.2}$$

Table 1. *Range of exponents for the validity of (1.3) for light cones*

dimension	[**11**, **22**]	improvements	conjecture
$d = 2$	$p > 74$	$p > p_2 := 63 + \frac{1}{3}$	$p > 6$
$d = 3$	$p > 18$	$p > p_3 := 15$	$p > 4$
$d = 4$	$p > 8.4$	$p > p_4 := 7.28$	$p > \frac{10}{3}$
$d \geqslant 5$	$p > 2 + \dfrac{8}{d-3}$	$p > p_d := 2 + \dfrac{8}{d-3}\left(1 - \dfrac{1}{d+1}\right)$	$p > 2 + \dfrac{4}{d-1}$

the standard Bochner–Riesz critical index in d dimensions. Then Wolff's inequality is the assertion that, for all $\varepsilon > 0$,

$$\Big\| \sum_k f_k \Big\|_p \leqslant C_\varepsilon \delta^{-\alpha(p)-\varepsilon} \Big(\sum_k \|f_k\|_p^p \Big)^{1/p} \tag{1.3}$$

provided that

$$\operatorname{supp} \widehat{f_k} \subset \varPi_k^{(\delta)}. \tag{1.4}$$

The power $\alpha(p)$ is optimal for each p (except perhaps for $\varepsilon > 0$), and the inequality is conjectured to hold for all $p > 2+4/(d-1)$. In his fundamental work [**22**], Wolff developed a method to prove such inequalities for large values of p, and obtained a positive answer for $d = 2$ and $p > 74$. Subsequently, the method has been extended by Łaba and Wolff [**11**] to higher dimensions. It is shown in [**11**] that (1.3) holds for $p > 2 + 32/(3d - 7)$ when $d \geqslant 3$ and $p > 2 + 8/(d - 3)$ when $d \geqslant 4$. In this paper we modify the weakest part of their proof to obtain a better range of exponents in all dimensions (see Table 1). The improvement relies on certain square function bounds which follow from Wolff's bilinear Fourier extension theorem [**23**].

Theorem 1.1. *Let $d \geqslant 2$ and let p_d be as in Table 1. Then, under the assumption (1.4), the inequality (1.3) holds for all $\varepsilon > 0$ and all $p \geqslant p_d$.*

Remark 1.2. Various further and more technical improvements on the range of Theorem 1.1 (and by implication on the results of Corollaries 1.5 and 1.6) have been obtained by the authors, and also by Wilhelm Schlag. After this paper was submitted for publication these improvements were combined and included in a joint paper [**9**].

A similar result can be proved for decompositions of spheres in $\mathbb{R}^d$. We now let

$$\mathcal{U}_\delta(c) = \{\xi \in \mathbb{R}^d : ||\xi| - 1| \leqslant c\delta\},$$

and consider the decomposition into plates subordinated to a $\sqrt{\delta}$-separated sequence in the sphere $\{\omega_k\} \subset S^{d-1}$,

$$B_k^{(\delta)} = \{\xi \in \mathcal{U}_\delta(c) : |\xi/|\xi| - \omega_k| \leqslant c'\sqrt{\delta}\}.$$

Theorem 1.3. *The analogue of Wolff's inequality for the sphere,*

$$\Big\| \sum_k f_k \Big\|_p \leqslant C_\varepsilon \delta^{-\alpha(p)-\varepsilon} \Big(\sum_k \|f_k\|_p^p \Big)^{1/p}, \quad \operatorname{supp} \widehat{f_k} \subset B_k^{(\delta)}, \tag{1.5}$$

holds for $p \geqslant 2 + 8/(d-1) - 4/((d-1)d)$ *and all* $\varepsilon > 0$.

Again (1.5) is conjectured to hold for the optimal range $p > 2 + 4/(d-1)$. It has been known to hold for $p > 2 + 8/(d-1)$; this follows from a modification of the argument in [**11**] (see also [**10**]). Note that in two dimensions the range is improved from the previous value of $p > 10$ to $p > 8$.

Remark 1.4. Theorem 1.3 may be extended to convex surfaces with non-vanishing Gaussian curvature and, similarly, Theorem 1.1 may be extended to cones with $d-1$ positive principal curvatures. This can be achieved by using scaling and induction on scales arguments such as in [**16**, § 2] (see also [**10**] for related results).

We proceed to list some of the known implications of Theorem 1.1.

Corollary 1.5. *Let* $d \geqslant 2$ *and* p_d *be as in Table 1. Then the following hold.*

(i) *For all* $p > p_d$, $\alpha > (d-1)/2 - d/p$, *we have*

$$\Big(\int_1^2 \|e^{it\sqrt{-\Delta}} f\|_{L^p(\mathbb{R}^d)}^p \, dt \Big)^{1/p} \lesssim \|f\|_{L^p_\alpha(\mathbb{R}^d)}. \tag{1.6}$$

(ii) *For all* $p \in (p_d, \infty)$, $\alpha > (d-1)/2 - d/p$ *the Fourier multiplier*

$$m_\alpha(\tau, \xi) = (1 - |\xi|^2/\tau^2)_+^\alpha \tag{1.7}$$

defines a bounded operator in $L^p(\mathbb{R}^{d+1})$.

(iii) *Let* $K \in \mathcal{S}'(\mathbb{R}^d)$ *be radial, let* $\varphi \in C_0^\infty(\mathbb{R}^d \setminus \{0\})$ *so that* φ *is radial and not identically zero and let* $\varepsilon > 0$. *Let* $K_t = \mathcal{F}^{-1}[\varphi \hat{K}(t\cdot)]$. *Then, for all Schwartz functions* f *and* $1 < r < p_d/(p_d - 1)$,

$$\|K * f\|_r \leqslant C_\varepsilon \sup_{t>0} \Big(\int |K_t(x)|^r (1+|x|)^\varepsilon \, dx \Big)^{1/r} \|f\|_r.$$

(iv) *Let* $\chi \in C_0^\infty(\mathbb{R})$ *and let* $s \mapsto \gamma(s) \in \mathbb{R}^3$ *be a smooth curve satisfying*

$$\sum_{j=1}^n |\langle \theta, \gamma^{(j)}(s) \rangle| \neq 0$$

for every unit vector θ *and every* $s \in \operatorname{supp} \chi$. *For* $t > 0$ *define the convolution operator* A_t *by*

$$A_t f(x) = \int f(x - t\gamma(s)) \chi(s) \, ds.$$

Suppose that $\max\{n, 32 + \frac{2}{3}\} < p < \infty$. *Then* A_t *maps* $L^p(\mathbb{R}^3)$ *into the* L^p-*Sobolev space* $L^p_{1/p}(\mathbb{R}^3)$. *Moreover, the maximal function* $Mf = \sup_t |A_t f|$ *defines a bounded operator on* $L^p(\mathbb{R}^3)$.

Parts (i)–(iii) are standard consequences of Theorem 1.1; see [**22**] for (i) and the local version of (ii). The global version follows by results on dyadic decompositions of multipliers and L^p Calderón–Zygmund theory (see [**6**, **17**]). The proof of [**15**, Theorem 1.6] together with these arguments can be used to deduce (iii) from Theorem 1.1. For (iv) see [**16**].

In addition to the connection to cone multipliers, a major motivation for this paper was the relevance of inequalities for plate decompositions for the boundedness properties of the Bergman projection in tube domains over full light cones (see [**2**, **3**]). Denote by $\Delta(Y) = y_0^2 - |y'|^2$ the Lorentz form and consider the forward light cone on which Δ is positive:

$$\Lambda^{d+1} = \{Y = (y_0, y') \in \mathbb{R} \times \mathbb{R}^d : y_0^2 - |y'|^2 > 0,\ y_0 > 0\}.$$

Let $\mathcal{T}^{d+1} \subset \mathbb{C}^{d+1}$ be the tube domain over Λ^{d+1}, i.e.

$$\mathcal{T}^{d+1} = \mathbb{R}^{d+1} + \mathrm{i}\Lambda^{d+1}.$$

Let $w_\gamma(Y) = \Delta(Y)^\gamma$ and consider the weighted space $L^p(\mathcal{T}^{d+1}, w_\gamma)$ with norm

$$\|F\|_{p,\gamma} = \biggl(\iint_{\mathcal{T}^{d+1}} |F(X + \mathrm{i}Y)|^p \Delta^\gamma(Y)\, \mathrm{d}Y\, \mathrm{d}X\biggr)^{1/p}.$$

Let $\mathcal{P}_\gamma$ be the orthogonal projection mapping the weighted space $L^2(\mathcal{T}^{d+1}, w_\gamma)$ to its subspace $\mathcal{A}^p_\gamma$ consisting of the holomorphic functions. Only the case when $\gamma > -1$ is interesting since $\mathcal{A}^p_\gamma = \{0\}$ for $\gamma \leqslant -1$. We are interested in the L^p boundedness properties of $\mathcal{P}_\gamma$. For $\gamma > -1$ the operator $\mathcal{P}_\gamma$ can only be bounded on $L^p(\mathcal{T}^{d+1}, w_\gamma)$ in the range

$$1 + \frac{d-1}{2(\gamma + d + 1)} < p < 1 + \frac{2(\gamma + d + 1)}{d - 1} \tag{1.8}$$

(see, for example, [**1**, Theorem 4.3]), and (1.8) is indeed the conjectured range for L^p boundedness (except for $d = 2$ and $\gamma \in (-1, -\frac{1}{2})$, in which case there are additional counter-examples for $p \geqslant 8 + 4\gamma$ [**2**]).

Corollary 1.6. *Let $d \geqslant 2$ and p_d as in Table 1. Then, for all*

$$\gamma \geqslant \frac{d-1}{2}\biggl(p_d - \frac{2(d+1)}{d-1}\biggr),$$

the Bergman projection $\mathcal{P}_\gamma$ is a bounded operator in $L^p(\mathcal{T}^{d+1}, w_\gamma)$ in the sharp range (1.8).

In addition to Corollary 1.6, both Theorem 1.1 and Theorem 4.4 have further implications for the range of boundedness of the Bergman projector $\mathcal{P}_\gamma$ in natural weighted mixed norm spaces. For the derivation of Corollary 1.6 and further discussion of mixed norm estimates, we refer the reader to [**2**] (cf. in particular Proposition 5.5 and Corollaries 5.12 and 5.17 therein).

Our approach to Theorem 1.1 is based on bilinear methods, for which we consider a closely related inequality:

$$\Big\|\sum_k f_k\Big\|_p \leqslant C_\alpha \delta^{-\alpha}\Big(\sum_k \|f_k\|_p^2\Big)^{1/2}. \tag{1.9}$$

One can conjecture the validity of (1.9) for all $\alpha > 0$ and all $2 < p < 2 + 4/(d-1)$, but for the moment no positive result for any such p seems to be known. The limiting point $p = 2(d+1)/(d-1)$ should be the hardest case, since by interpolation and Hölder's inequality it implies both (1.9) and (1.3) in all the conjectured ranges. This kind of inequality arises naturally in the study of weighted mixed norm inequalities for the Bergman projection operator $\mathcal{P}_\gamma$ [**2**].

We shall deduce Theorem 1.1 by using a stronger version of (1.9) for $p = 2(d+3)/(d+1)$, but with a power of $1/\delta$ which is (probably) not optimal. Namely, under the assumption (1.4) we have

$$\Big\|\sum_k f_k\Big\|_{2(d+3)/(d+1)} \leqslant C_\varepsilon \delta^{-((d-1)/4(d+3))-\varepsilon}\Big\|\Big(\sum_k |f_k|^2\Big)^{1/2}\Big\|_{2(d+3)/(d+1)} \tag{1.10}$$

for all $\varepsilon > 0$.

We prove this inequality in § 2 using the bilinear approach of Tao and Vargas [**20**, § 5] and the optimal bilinear cone extension inequality of Wolff [**23**] (see Proposition 2.3). By Minkowski's inequality and interpolation, (1.10) trivially implies non-optimal estimates for the inequality (1.9) for all $p \in (2, \infty)$ (see Corollary 2.4). In § 3 we use these to refine a part of Wolff's proof of (1.3) and obtain the new sharp estimates for large p given in Table 1. In § 4 we improve on some of the square function results in low dimensions; these yield, in particular, the following estimate for the cone multiplier in $\mathbb{R}^{2+1}$.

Theorem 1.7. *Suppose that*

$$\alpha > \frac{5}{44}\left(\frac{p_2 - 4}{p_2 - \frac{41}{11}}\right).$$

Then the cone Fourier multiplier m_α defines a bounded operator on $L^4(\mathbb{R}^3)$ and the local smoothing result (1.6) holds in two dimensions.

This is a small improvement over the known range $\alpha > \frac{5}{44}$, which follows from a combination of [**20**] and [**23**].

Notation

We shall use the notation $A \lesssim B$ if there is a constant (which may depend on d) so that $A \leqslant CB$. For families (A_δ, B_δ), $\delta \leqslant 1$, we use $A_\delta \lessapprox B_\delta$ if for every $\varepsilon \in (0, 1)$ there is a constant C_ε such that $A_\delta \leqslant C_\varepsilon \delta^{-\varepsilon} B_\delta$ for $\delta \leqslant 1$.

2. The bilinear estimate

Following the approach by Tao and Vargas, we first establish an equivalence between linear and bilinear versions of (1.10), which is a higher-dimensional analogue of [**20**, Lemma 5.2].

Lemma 2.1. *Let $d \geqslant 2$, and suppose that, for some $p \in [2, \infty)$ and $\alpha > \max\{0, (d-1)(1/4 - 1/p)\}$,*

$$\bigg\| \bigg(\sum_{\omega_k \in \Omega} f_k \bigg) \bigg(\sum_{\omega_{k'} \in \Omega'} f_{k'} \bigg) \bigg\|_{p/2} \leqslant C\delta^{-2\alpha} \bigg\| \bigg(\sum_{\omega_k \in \Omega} |f_k|^2 \bigg)^{1/2} \bigg\|_p \bigg\| \bigg(\sum_{\omega_{k'} \in \Omega'} |f_{k'}|^2 \bigg)^{1/2} \bigg\|_p \tag{2.1}$$

holds for all $f_k \in \mathcal{S}(\mathbb{R}^{d+1})$ with $\operatorname{supp} \widehat{f_k} \subset \Pi_k^{(\delta)}$, all pairs of 1-separated subsets $\Omega, \Omega' \subset S^{d-1}$ and all $\delta \ll 1$. Then we also have

$$\bigg\| \sum_k f_k \bigg\|_p \leqslant C'\delta^{-\alpha} \bigg\| \bigg(\sum_k |f_k|^2 \bigg)^{1/2} \bigg\|_p, \quad \operatorname{supp} \widehat{f_k} \subset \Pi_k^{(\delta)}. \tag{2.2}$$

We remark that the restriction on α for $p > 4$ is never severe. To see this we note that the condition $(d-1)(1/4-1/p) \leqslant \alpha(p)/2$ holds if and only if $d \geqslant 2$ and that (2.2) cannot hold with $\alpha < \alpha(p)/2$; this can be proved using Knapp examples.

Proof of Lemma 2.1. Let $\Phi : Q \equiv [0,1]^{d-1} \to S^{d-1}$ be a smooth parametrization of (a compact subset of) the sphere and let $\mathcal{D}$ denote the set of all dyadic intervals $I \subset Q$ with $|I| \geqslant \delta^{(d-1)/2}$. As in [**21**, p. 971], we may consider a Whitney decomposition $Q \times Q = \biguplus_{I \sim J} I \times J$, where $I \sim J$ means that

(i) $I, J \in \mathcal{D}$ and $|I| = |J|$,

(ii) if $|I| > \delta^{(d-1)/2}$, then I and J are not adjacent but their parents are,

(iii) if $|I| = \delta^{(d-1)/2}$, then I and J have adjacent or equal parents.

For simplicity, we assume (by splitting the sphere into finitely many pieces) that all $\omega_k \in \Phi(Q)$ and let $y_k = \Phi^{-1}(\omega_k) \in Q$. We also denote $\mathcal{D}_j = \{I \in \mathcal{D} : |I| = 2^{-j(d-1)}\}$. Then

$$\bigg(\sum_k f_k \bigg)^2 = \sum_{y_k, y_{k'} \in Q} f_k f_{k'} = \sum_{\sqrt{\delta} \leqslant 2^{-j} \leqslant 1} \sum_{\substack{I, J \in \mathcal{D}_j, \\ I \sim J}} \bigg(\sum_{y_k \in I} f_k \bigg) \bigg(\sum_{y_{k'} \in J} f_{k'} \bigg).$$

To establish (2.2) we take $L^{p/2}$-norms in the above expression and use Minkowski's inequality in j, so that we reduce the problem to showing that, for each j,

$$\bigg\| \sum_{\substack{I, J \in \mathcal{D}_j, \\ I \sim J}} \bigg(\sum_{y_k \in I} f_k \bigg) \bigg(\sum_{y_{k'} \in J} f_{k'} \bigg) \bigg\|_{p/2} \lesssim (2^{2j}\delta)^{-2\alpha} \max\{1, 2^{j(d-1)(1-4/p)}\} \bigg\| \bigg(\sum_k |f_k|^2 \bigg)^{1/2} \bigg\|_p^2. \tag{2.3}$$

Inequality (2.3) is trivial when $2^{-j} \approx \sqrt{\delta}$, since by assumption the number of y_ks in each I is approximately constant. We consider the general case $\sqrt{\delta} < 2^{-j} \leqslant 1$. By construction, we must have

$$\sum_{I\in\mathcal{D}_j} \sum_{J\sim I} \chi_{I+J} \lesssim 1. \tag{2.4}$$

Indeed, if c_I denotes the centre of I, then

$$I + J \subset (c_I + B_{c2^{-j}}) + (c_J + B_{c2^{-j}}) \subset 2c_I + B_{c'2^{-j}}.$$

Since for each I there are at most $O(1)$ cubes J with $J \sim I$, and since the centres c_I are 2^{-j} separated, (2.4) follows easily.

From (2.4) it follows that the functions $F_{I,J} = (\sum_{y_k\in I} f_k)(\sum_{y_{k'}\in J} f_{k'})$ have pairwise (almost) disjoint spectra when $I \sim J \in \mathcal{D}_j$. We may conclude by orthogonality and standard interpolation arguments that

$$\biggl\| \sum_{I\sim J\in\mathcal{D}_j} F_{I,J} \biggr\|_{p/2} \lesssim \max\{1, 2^{j(d-1)(1-4/p)}\} \biggl(\sum_{I\sim J\in\mathcal{D}_j} \|F_{I,J}\|_{p/2}^{p/2} \biggr)^{2/p}. \tag{2.5}$$

(The case $p/2 = 2$ follows by orthogonality and the cases $p/2 = 1$ and $p/2 = \infty$ are trivial; see, for example, [**20**, Lemma 7.1].) Next, we wish to use the bilinear assumption (2.1) to estimate $\|F_{I,J}\|_{p/2}$. This can only be used directly when $2^j \approx 1$, since $\operatorname{dist}(I, J) \sim 1$. For other js we must use Lorentz transformations to rescale the problem. To do this, let $\{\eta_1, \dots, \eta_d\}$ be an orthonormal basis of $\mathbb{R}^d$ with η_1 being the centre of $\Phi(I)$. Then we define $L \in \mathrm{SO}(1, d)$ acting on a basis of $\mathbb{R}^{d+1}$ by

$$L(1, \eta_1) = (1, \eta_1), \quad L(-1, \eta_1) = \frac{\sigma}{\delta}(-1, \eta_1) \quad \text{and} \quad L(0, \eta_\ell) = \sqrt{\frac{\sigma}{\delta}}(0, \eta_\ell), \quad \ell = 2, \dots, d,$$

where we choose $\sigma = 2^{2j}\delta$ (so that $\delta < \sigma < 1$). The spectrum of the function $f_k \circ L$ is contained in (perhaps a multiple of) the plates $\Pi_k^{(\sigma)}$ corresponding to the $\sqrt{\sigma}$-separated centres $\{L(1, \omega_k)\}$. Moreover, by the choice of σ, the plates corresponding to $y_k \in I$ and $y_{k'} \in J$ are c-separated, and therefore after a change of variables we can apply (2.1) at scale σ to obtain

$$\begin{aligned} \|F_{I,J}\|_{p/2} &= \biggl\| \biggl(\sum_{y_k\in I} f_k \biggr) \biggl(\sum_{y_{k'}\in J} f_{k'} \biggr) \biggr\|_{p/2} \\ &\lesssim (2^{2j}\delta)^{-2\alpha} \biggl\| \biggl(\sum_{y_k\in I} |f_k|^2 \biggr)^{1/2} \biggr\|_p \biggl\| \biggl(\sum_{y_{k'}\in J} |f_{k'}|^2 \biggr)^{1/2} \biggr\|_p, \end{aligned}$$

and then also

$$\begin{aligned} &\biggl(\sum_{I\sim J\in\mathcal{D}_j} \|F_{I,J}\|_{p/2}^{p/2} \biggr)^{2/p} \\ &\quad \lesssim (2^{2j}\delta)^{-2\alpha} \biggl[\sum_{I\sim J\in\mathcal{D}_j} \biggl\| \biggl(\sum_{y_k\in I} |f_k|^2 \biggr)^{1/2} \biggr\|_p^{p/2} \biggl\| \biggl(\sum_{y_{k'}\in J} |f_{k'}|^2 \biggr)^{1/2} \biggr\|_p^{p/2} \biggr]^{2/p} \end{aligned}$$

$$
\begin{aligned}
&\lesssim (2^{2j}\delta)^{-2\alpha}\biggl[\int\biggl(\sum_{I}\sum_{y_k\in I}|f_k|^2\biggr)^{p/2}\biggr]^{2/p}\\
&\leqslant (2^{2j}\delta)^{-2\alpha}\biggl\|\biggl(\sum_{k}|f_k|^2\biggr)^{1/2}\biggr\|_p^2,
\end{aligned}
$$

where in the second inequality we have used $2ab \leqslant a^2+b^2$ followed by the imbedding $\ell^1 \hookrightarrow \ell^{p/2}$. Combining this with (2.5), we obtain

$$
\biggl\|\sum_{I\sim J\in\mathcal{D}_j} F_{I,J}\biggr\|_{p/2} \lesssim (2^{2j}\delta)^{-2\alpha}\max\{1, 2^{j(d-1)(1-4/p)}\}\biggl\|\biggl(\sum_{k}|f_k|^2\biggr)^{1/2}\biggr\|_p^2. \tag{2.6}
$$

This proves (2.3). By our assumption on α we may sum in j and the lemma follows. □

We turn to the proof of (a generalization of) the square function estimate (1.10). We shall use the following statement of Wolff's Fourier extension theorem [**23**, p. 680].

Wolff's bilinear estimate

Let $p \geqslant (d+3)/(d+1)$, $\varepsilon > 0$ and let E, E' be 1-separated subsets of $\Gamma_{1/N}$. Then, for all smooth f and g supported in E and E', and all N-cubes Q, we have

$$
\|\hat{f}\hat{g}\|_{L^p(Q)} \leqslant C_\varepsilon N^{-1+\varepsilon}\|f\|_2\|g\|_2. \tag{2.7}
$$

Denote by $\mathcal{Q} \equiv \mathcal{Q}(\delta^{-1/2})$ a tiling of $\mathbb{R}^{d+1}$ with cubes Q of disjoint interior and side length $\delta^{-1/2}$, with centres c_Q in $\delta^{-1/2}\mathbb{Z}^{d+1}$.

Proposition 2.2. *Let $d \geqslant 2$ and suppose that $\operatorname{supp}\widehat{f_k} \subset \Pi_k^{(\delta)}$, $\operatorname{supp}\hat{g}_k \subset \Pi_k^{(\delta)}$ and let $\Omega, \Omega' \subset S^{d-1}$ be 1-separated subsets. Suppose $2(d+3)/(d+1) \leqslant q \leqslant p \leqslant \infty$ and let*

$$
\mu(p) = \frac{d}{4} - \frac{d+1}{2p}. \tag{2.8}
$$

Then, for all $\varepsilon > 0$,

$$
\begin{aligned}
&\biggl(\sum_{Q\in\mathcal{Q}(\delta^{-1/2})}\biggl\|\biggl(\sum_{\omega_k\in\Omega} f_k\biggr)\biggl(\sum_{\omega_{k'}\in\Omega'} g_{k'}\biggr)\biggr\|_{L^{q/2}(Q)}^{p/2}\biggr)^{2/p}\\
&\qquad\lesssim \delta^{-2\mu(p)-\varepsilon}\biggl\|\biggl(\sum_{\omega_k\in\Omega}|f_k|^2\biggr)^{1/2}\biggr\|_p\biggl\|\biggl(\sum_{\omega_{k'}\in\Omega'}|g_{k'}|^2\biggr)^{1/2}\biggr\|_p.
\end{aligned} \tag{2.9}
$$

Proof. Let $\psi \in \mathcal{S}(\mathbb{R}^{d+1})$ be such that $\operatorname{supp}\hat{\psi} \subset B_{1/10}$ and $\psi(x) > 1$ if $|x_i| \leqslant 2$, $i = 1, \dots, d+1$; then $\sum_{n\in\mathbb{Z}^{d+1}}\psi(\cdot+n)^2 \approx 1$. Let $\psi_Q = \psi(\sqrt{\delta}(\cdot - c_Q))$, so that $\sum_Q \psi_Q^2 \approx 1$. We write

$$
F^Q = \biggl(\sum_{\omega_k\in\Omega} f_k\biggr)\psi_Q \quad\text{and}\quad G^Q = \biggl(\sum_{\omega_{k'}\in\Omega'} g_{k'}\biggr)\psi_Q,
$$

so that the supports of $\hat{F}^Q$ and $\hat{G}^Q$ are 1-separated sets in $\varGamma_{\sqrt{\delta}}$. Thus, we can use Wolff's estimate (2.7) with $N = \delta^{-1/2}$ to obtain

$$\bigg\| \bigg(\sum_{\omega_k \in \varOmega} f_k \bigg) \bigg(\sum_{\omega_{k'} \in \varOmega'} g_{k'} \bigg) \bigg\|_{L^{q/2}(Q)} \lesssim \|F^Q G^Q\|_{L^{q/2}(Q)} \lessapprox \delta^{1/2} \|\hat{F}^Q\|_2 \|\hat{G}^Q\|_2. \tag{2.10}$$

Now, by almost orthogonality we can write

$$\|\hat{F}^Q\|_2^2 \approx \sum_k \|\widehat{f}_k * \hat{\psi}_Q\|_2^2 = \bigg\| \bigg(\sum_k |f_k|^2 \bigg)^{1/2} \psi_Q \bigg\|_2^2,$$

and similarly for G^Q. We write

$$S_\varOmega = \bigg(\sum_{\omega_k \in \varOmega} |f_k|^2 \bigg)^{1/2}, \qquad \tilde{S}_{\varOmega'} = \bigg(\sum_{\omega_k \in \varOmega'} |g_k|^2 \bigg)^{1/2},$$

raise (2.10) to the power $p/2$ and sum in Q. Thus,

$$\bigg(\sum_Q \bigg\| \bigg(\sum_{\omega_k \in \varOmega} f_k \bigg) \bigg(\sum_{\omega_{k'} \in \varOmega'} g_{k'} \bigg) \bigg\|_{L^{q/2}(Q)}^{p/2} \bigg)^{2/p} \lessapprox \sqrt{\delta} \bigg(\sum_Q \|S_\varOmega \psi_Q\|_2^{p/2} \|\tilde{S}_{\varOmega'} \psi_Q\|_2^{p/2} \bigg)^{2/p}$$

and by the Cauchy–Schwarz and Hölder inequalities the right-hand side is estimated as

$$\begin{aligned}
&\sqrt{\delta} \bigg(\sum_Q \|S_\varOmega \psi_Q\|_2^{p/2} \|\tilde{S}_{\varOmega'} \psi_Q\|_2^{p/2} \bigg)^{2/p} \\
&\qquad \lesssim \sqrt{\delta} \bigg(\sum_Q \|S_\varOmega \psi_Q\|_2^p \bigg)^{1/p} \bigg(\sum_Q \|\tilde{S}_{\varOmega'} \psi_Q\|_2^p \bigg)^{1/p} \\
&\qquad \lesssim \sqrt{\delta} \bigg(\sum_Q \|S_\varOmega \psi_Q\|_p^p |Q|^{-1+p/2} \bigg)^{1/p} \bigg(\sum_Q \|\tilde{S}_{\varOmega'} \psi_Q\|_p^p |Q|^{-1+p/2} \bigg)^{1/p} \\
&\qquad \lesssim \delta^{1/2-(d+1)(1/2-1/p)} \|S_\varOmega\|_p \|\tilde{S}_{\varOmega'}\|_p,
\end{aligned}$$

which yields the assertion. □

We combine Proposition 2.2 for $q = p$ and Lemma 2.1 to obtain the following.

Proposition 2.3. *Let $d \geqslant 2$ and let $\mu(p)$ be as in (2.8) and suppose that $p \geqslant 2(d+3)/(d+1)$.*

Then, for all $\varepsilon > 0$,

$$\bigg\| \sum_k f_k \bigg\|_p \leqslant C_\varepsilon \delta^{-\mu(p)-\varepsilon} \bigg\| \bigg(\sum_k |f_k|^2 \bigg)^{1/2} \bigg\|_p \quad \textit{if } \operatorname{supp} \widehat{f}_k \subset \varPi_k^{(\delta)}. \tag{2.11}$$

We may apply Minkowski's inequality on the right-hand side of (2.11) and obtain (1.9) for the limiting case $p = 2(d+3)/(d+1)$ and $\alpha > \mu(p) = d/4 - (d+1)/2p$.

It turns out this is all that is needed to obtain the claimed improvements in Theorem 1.1. The resulting inequality can also be interpolated with the trivial estimates for L^2 and L^∞ to give the following.

Corollary 2.4. *The inequality (1.9) holds for all*

$$\alpha > \frac{d-1}{4}\left(\frac{1}{2} - \frac{1}{p}\right) \qquad \text{when } 2 \leqslant p \leqslant \frac{2(d+3)}{d+1}$$

and for all

$$\alpha > \frac{d-1}{4}\left(1 - \frac{2(d+2)}{p(d+1)}\right) \quad \text{when } \frac{2(d+3)}{d+1} \leqslant p \leqslant \infty.$$

3. An improvement of Wolff's estimate

We turn to Theorem 1.1. The proof in [**11**, **22**] for inequality (1.3) is based on a subtle localization procedure, induction on scales and certain combinatorial arguments. Here we discuss only the modifications leading to the claimed improvements based on Proposition 2.3. A more self-contained exposition with further improvements can be found in [**9**].

For simplicity, when δ is fixed (and small) we use the notation $A \lessapprox B$ to indicate the inequality $A \leqslant C_\varepsilon \delta^{-\varepsilon} B$ for all $\varepsilon > 0$. Recall that the number of plates $\Pi_k^{(\delta)}$ covering Γ_δ is approximately $\delta^{-(d-1)/2}$. Also, throughout this section we fix $q(d) = 2(d+3)/(d+1)$.

Due to various reductions (see [**11**, § 3]), it is sufficient to show that, for all f_k with $\operatorname{supp} \widehat{f_k} \subset \Pi_k^{(\delta)}$ and $\|f_k\|_\infty \leqslant 1$, and for all $\lambda > 0$ we have

$$\left|\left\{\left|\sum_k f_k\right| > \lambda\right\}\right| \lessapprox \lambda^{-p} \delta^{d-(d-1)p/2} \|f\|_2^2, \tag{3.1}$$

where $f = \sum_k f_k$. In [**11**,**22**] it is observed that, by Chebyshev's inequality, this property trivially holds for small enough λ; namely, for all $\lambda \leqslant \delta^{-(d-1)/2+1/(p-2)}$. We use (1.9) to enlarge this range of λ.

Lemma 3.1. *Let $q = q(d) = 2(d+3)/(d+1)$. Then inequality (3.1) holds for all*

$$\lambda \leqslant \delta^{-((d-1)/2)+q/4(p-q)}. \tag{3.2}$$

Proof. Let $\beta = (d-1)/4(d+3)$. By Chebyshev's inequality and (1.9), we have

$$|\{|f| > \lambda\}| \leqslant \lambda^{-q} \|f\|_q^q \lessapprox \delta^{-q\beta} \lambda^{-q} \left(\sum_k \|f_k\|_q^2\right)^{q/2}$$

and estimate

$$\begin{aligned}\left(\sum_k \|f_k\|_q^2\right)^{q/2} &\lesssim \delta^{-((d-1)/2)((q/2)/(q/2)')} \sum_k \|f_k\|_q^q \\ &\lesssim \delta^{-((d-1)/2)(q/2-1)} \sum_k \|f_k\|_2^2 \sup_k \|f_k\|_\infty^{q-2}.\end{aligned}$$

Since by assumption $\|f_k\|_\infty \leqslant 1$ and by almost orthogonality $\sum_k \|f_k\|_2^2 \approx \|f\|_2^2$, it suffices to show that in the desired range of λ we have $\delta^{-q\beta-((d-1)/2)(q/2-1)}\lambda^{-q} \leqslant \delta^{-((d-1)p/2)-d}\lambda^{-p}$, which is equivalent to (3.2). $\square$

At this point one can proceed exactly as in the proof of [**11**, Proposition 3.2] (or [**22**, p. 1277], when $d = 2$). The desired gain comes from using $\lambda \geqslant \delta^{-((d-1)/2)+q(d)/4(p-q(d))}$ (rather than $\lambda \leqslant \delta^{-((d-1)/2)+1/(p-2)}$) in [**11**, Step (54)] (or [**22**, (68)]).

For completeness, we shall briefly sketch this procedure here, referring always to the notation in [**11**]. Localizing with $\sqrt{N}$-cubes Δ as in [**11**, Lemma 6.1], one can find a collection of functions $\{f_\Delta\}$ with spectrum in $\Gamma_{\sqrt{\delta}}$ and a number

$$\lambda_* \in (\lambda\delta^{(d-1)/4+\varepsilon}, c\delta^{-(d-1)/4}) \tag{3.3}$$

so that

$$|\{|f| > \lambda\}| \lessapprox \sum_\Delta |\{|f_\Delta| > \lambda_*\}|$$

and

$$\operatorname{card}(\mathcal{P}(f_\Delta)) \lessapprox \lambda_*^2\lambda^{-2}\delta^{-(3d-1)/4}. \tag{3.4}$$

Here $\mathcal{P}(f_\Delta)$ refers to the set of plates in the wave-packet decomposition of f_Δ. When the cardinality of this set is 'small', a further localization argument and induction on scales allows us to conclude the theorem (see [**11**, Lemmas 6.2 and 6.3]).

In [**11**, **22**], the size of $\operatorname{card}(\mathcal{P}(f_\Delta))$ which ensures the validity of these arguments is controlled in three different ways, each depending on a different combinatorial estimate:

$$\operatorname{card}(\mathcal{P}(f_\Delta)) \leqslant c_\varepsilon\delta^\varepsilon\lambda_*^2 \tag{3.5}$$

or

$$\operatorname{card}(\mathcal{P}(f_\Delta)) \leqslant c_\varepsilon\delta^{(3d-3)/8+\varepsilon}\lambda_*^4 \tag{3.6}$$

or, in three dimensions (i.e. $d = 2$) only,

$$\operatorname{card}(\mathcal{P}(f_\Delta)) \leqslant c_\varepsilon\delta^{(11/8)+\varepsilon}\lambda_*^9. \tag{3.7}$$

The last estimate is by far the most difficult (see [**11**, Lemmas 5.2 and 5.3] and [**22**, Lemma 3.2]).

Given the lower bound for λ_* in (3.3) and

$$\lambda \geqslant \delta^{-((d-1)/2)+q/4(p-q)} \tag{3.8}$$

and given (3.4), it remains to verify the estimates (3.5) in the claimed range $p > p_d$, $d \geqslant 5$, (3.6) for $p > p_d$, $d = 3, 4$ and (3.7) for $p > p_2$.

This is straightforward. By (3.4) and (3.8) we have

$$\operatorname{card}(\mathcal{P}(f_\Delta)) \lessapprox \delta^{-\varepsilon}\lambda_*^2\delta^{d-1-(q(d)/2(p-q(d)))}\delta^{-(3d-1)/4},$$

which, in the case when $d \geqslant 4$, gives the assertion (3.5) if

$$d - 1 - \frac{q(d)}{2(p - q(d))} - \frac{3d-1}{4} > 0$$

or, after a short computation,

$$p > q\biggl(1 + \frac{2}{d-3}\biggr) = 2 + \frac{8}{d-3}\frac{d}{d+1}.$$

This is the asserted range if $d \geqslant 5$.

Next we examine the validity of the inequality (3.6) under condition (3.8). We now have

$$\operatorname{card}(\mathcal{P}(f_\Delta)) \leqslant C_\varepsilon \frac{\lambda_*^4 \delta^{-((3d-1)/4)-\varepsilon}}{\lambda_*^2 \lambda^2} \leqslant \frac{\lambda_*^4 \delta^{-((3d-1)/4)-\varepsilon}}{\lambda^4 \delta^{((d-1)/2)+2\varepsilon}} \leqslant \frac{\delta^{-((5d-3)/4)-3\varepsilon}}{\delta^{-2(d-1)+(q(d)/p-q(d))}} \lambda_*^4.$$

This quantity is $\lesssim \delta^\varepsilon \delta^{(3d-3)/8} \lambda_*^4$ if and only if

$$\frac{5d-3}{4} - 2(d-1) + \frac{q(d)}{p - q(d)} + 4\varepsilon < -\frac{3d-3}{8},$$

which yields the range $p > q(d)(1 + 8/(3d-7))$. Notice that this inequality amounts to $p > 7.28$ if $d = 4$ and $p > 15$ if $d = 3$, which is the assertion in those cases.

Finally, we consider the case $d = 2$ when $q(2) = \frac{10}{3}$. By (3.4) we need to have

$$\lambda_*^2 \lambda^{-2} \delta^{-5/4-\varepsilon} \leqslant c_\varepsilon \delta^{11/8} \lambda_*^9,$$

i.e. $\lambda^{-2}\delta^{-21/8-\varepsilon} \leqslant c_\varepsilon \lambda_*^7$ provided that $\lambda_* > \lambda\delta^{1/4+\varepsilon}$. Thus, taking the smallest possible λ^* yields $\delta^{-35/8-10\varepsilon} \leqslant \lambda^9$ and this has to hold for all λ satisfying (3.8), i.e. $\lambda \geqslant \delta^{-1/2+q(2)/4(p-q(2))}$. Taking the minimal λ, this is achieved if $\frac{35}{8} - 10\varepsilon < \frac{9}{2} - 9q/(4p-4q)$ with $q = q(2) = \frac{10}{3}$. Solving in p and letting $\varepsilon \to 0$ yields the range $p > 19q(2) = 63 + \frac{1}{3}$. □

Sketch of proof of Theorem 1.3. The proof is similar to the proof of Theorem 1.1. Instead of (1.10) we use a square function inequality for the sphere

$$\biggl\| \sum_k f_k \biggr\|_q \leqslant C_\varepsilon \delta^{-\alpha(q)/2-\varepsilon} \biggl\| \biggl(\sum_k |f_k|^2 \biggr)^{1/2} \biggr\|_q, \quad \operatorname{supp} \widehat{f_k} \subset B_k^{(\delta)}, \tag{3.9}$$

with $\alpha(q) = d(1/2 - 1/q) - \frac{1}{2}$ and $q = 2(d+2)/d$. In two dimensions this is an old observation by Fefferman [**8**] and holds for $q = 4$ with $\varepsilon = 0$. In higher dimensions the inequality (3.9) was proved by Bourgain [**4**] for the range of the Stein–Tomas restriction theorem (i.e. $q \geqslant 2(d+1)/(d-1)$). For the larger range $q > 2(d+2)/d$ the proof of (3.9) is analogous to the proof of Proposition 2.3; one now uses Tao's bilinear Fourier extension inequality [**19**] (see also [**12**] for related results). Unlike (2.11) in the conic case, the inequality (3.9) in the spherical case is essentially optimal for the given range $q \geqslant 2(d+2)/d$. We omit further details. □

4. More on square functions

We shall now discuss some improvements of the square function estimate in Proposition 2.3 in low dimensions; thus, we seek estimates of the form

$$\Big\| \sum_k f_k \Big\|_p \leqslant C_\varepsilon \delta^{-\beta-\varepsilon} \Big\| \Big(\sum_k |f_k|^2 \Big)^{1/2} \Big\|_p, \quad \operatorname{supp} \widehat{f_k} \subset \Pi_k^{(\delta)} \tag{4.1}$$

for some $\beta < \mu(p) = d/4 - (d+1)/2p$.

We shall assume throughout this section the following Wolff hypothesis and aim to prove estimates of the form (4.1) conditional on this hypothesis.

Hypothesis $\mathcal{W}(w;d)$. For all $\delta \in (0,1)$ and all families $\{h_k\}$ of functions satisfying $\operatorname{supp} \hat{h}_k \subset \Pi_k^{(\delta)}$,

$$\Big\| \sum_k h_k \Big\|_w \leqslant C_\varepsilon \delta^{-\alpha(w)-\varepsilon} \Big(\sum_k \|h_k\|_w^w \Big)^{1/w}, \tag{4.2}$$

where

$$\alpha(w) = d\Big(\frac{1}{2} - \frac{1}{w}\Big) - \frac{1}{2}$$

(see Table 1).

We note that in view of the embedding $L^p(\ell^2) \subset L^p(\ell^p)$ the inequality (4.2) trivially implies (4.1) with $\beta = \alpha(p)$, for $w \leqslant p < \infty$. Another trivial observation is that (4.1) holds with $\beta \geqslant \frac{1}{4}(d-1)$ in view of the Cauchy–Schwarz inequality, as

$$\sum_k |f_k(x)| \lesssim \delta^{-(d-1)/4} \Big(\sum_k |f_k(x)|^2 \Big)^{1/2}$$

for every x.

The method for our improvement over the exponent $\min\{\mu(p), \frac{1}{4}(d-1)\}$ will be limited to the case where

$$\alpha(p) < \min\{\mu(p), \tfrac{1}{4}(d-1)\}, \tag{4.3}$$

which holds if and only if

$$p < \min\Big\{ \frac{2(d-1)}{d-2}, \frac{4d}{d-1} \Big\}.$$

We have the additional restriction $p > 2(d+3)/(d+1)$ in Proposition 2.3. Summarizing, we obtain an improvement which is limited to $d = 2, 3, 4$ and to the ranges

$$d = \begin{cases} 2, & \frac{10}{3} < p < \min\{8, w\}, \\ 3, & 3 < p < 4, \\ 4, & \frac{14}{5} < p < 3. \end{cases} \tag{4.4}$$

We emphasize that square function estimates such as (4.1) cannot *a priori* be interpolated when subject to the Fourier support condition (1.4). We shall, however, start with a preliminary result that is proved using an interpolation.

We let ϕ_k be a bump function adapted to the plate $\Pi_k^{(\delta)}$ satisfying the natural estimates, so that ϕ_k equals 1 on the plate, and is supported on the 'double plate'. Define the operator P_k by

$$\widehat{P_k f} = \phi_k \hat{f}. \tag{4.5}$$

Each P_k is bounded on $L^p(\mathbb{R}^{d+1})$, $1 \leqslant p \leqslant \infty$, with uniform bounds.

Lemma 4.1. *Let $d = 2$, and suppose that Hypothesis $\mathcal{W}(w;2)$ holds. Let*

$$\beta = \beta_*(p,w) = \frac{3w-13}{6w-20} - \frac{9w-40}{(6w-20)p} \tag{4.6}$$

and let $r = r(p,w)$ be defined by

$$\frac{1}{r(p,w)} = \frac{1}{2} - \frac{w-2}{6w-20}\left(3 - \frac{10}{p}\right). \tag{4.7}$$

Then, for $\frac{10}{3} \leqslant p \leqslant w$,

$$\Big\| \sum_k P_k g_k \Big\|_p \leqslant C_\varepsilon \delta^{-\beta-\varepsilon} \Big\| \Big(\sum_k |g_k|^r \Big)^{1/r} \Big\|_p$$

for all families $\{g_k\}$ with $g_k \in \mathcal{S}(\mathbb{R}^{d+1})$.

Proof. By $\mathcal{W}(w;2)$ and the embedding $L^p(\ell^2) \subset \ell^p(L^p)$ we have the inequality

$$\begin{aligned} \Big\| \sum_k P_k g_k \Big\|_w &\leqslant C_\varepsilon \delta^{-(\alpha(w)+\varepsilon)} \Big(\sum_k \|P_k g_k\|_w^w \Big)^{1/w} \\ &\lesssim C_\varepsilon \delta^{-(\alpha(w)+\varepsilon)} \Big\| \Big(\sum_k |g_k|^w \Big)^{1/w} \Big\|_w. \end{aligned} \tag{4.8}$$

We also observe that, for $2 \leqslant p \leqslant 4$,

$$\Big\| \Big(\sum_k |P_k g_k|^2 \Big)^{1/2} \Big\|_p \leqslant C(1 + \log \delta^{-1})^{1/2-1/p} \Big\| \Big(\sum_k |g_k|^2 \Big)^{1/2} \Big\|_p. \tag{4.9}$$

Indeed the left-hand side is estimated by using

$$\sup_{\omega \in L^{(p/2)'}} \Big(\sum_k \int |P_k g_k|^2 \omega \, \mathrm{d}x \Big)^{1/2} \lesssim \sup_{\omega \in L^{(p/2)'}} \Big(\sum_k \int |g_k|^2 M_\delta \omega \, \mathrm{d}x \Big)^{1/2}, \tag{4.10}$$

where M_δ is a Besicovitch-type maximal operator associated to the light cone which is bounded on L^2 with norm $O(\sqrt{\log(2+\delta^{-1})})$ if $\delta < \frac{1}{2}$ [**7**, **14**]. Thus, Hölder's inequality implies (4.9).

Now we can combine Proposition 2.3 with respect to the double plates, applied to $f_k = P_k g_k$, and (4.9) to obtain

$$\Big\| \sum_k P_k g_k \Big\|_{10/3} \leqslant C_\varepsilon \delta^{-1/20-\varepsilon} \Big\| \Big(\sum_k |g_k|^2 \Big)^{1/2} \Big\|_{10/3}. \tag{4.11}$$

After a little arithmetic the claimed bound follows by interpolation between (4.8) and (4.11). ☐

Since $r(p,w) \geqslant 2$ in Lemma 4.1, we immediately get the following result.

Corollary 4.2. *Let $d = 2$ and suppose that Hypothesis $\mathcal{W}(w;2)$ holds. Then for all families of functions $\{f_k\}$ with $\operatorname{supp}\widehat{f}_k \subset \mathit{\Pi}_k^{(\delta)}$ the estimate (4.1) holds for $\frac{10}{3} \leqslant p \leqslant w$ with $\beta = \beta_*(p,w)$.*

In particular note that $\beta_*(4,w) = (3w-12)/(24w-80)$, so that $\beta_*(4,6) = \frac{3}{32}$. If we use the exponent obtained in Theorem 1.1, i.e. $w = p_2 = \frac{190}{3}$, we get only $\beta_*(4,p_2) = \frac{89}{720}$, which is worse than the $\frac{5}{44}$ exponent that is already known from [**20**, **23**].

For large values of w one can improve on the result of Corollary 4.2. Our approach will be similar to the one by Tao and Vargas [**20**] in $2+1$ dimensions. By using $\mathcal{W}(w;2)$ in that approach one can slightly improve on the previously known exponents.

Theorem 4.3. *Let $2 \leqslant d \leqslant 4$ and let p be as in (4.4). If Hypothesis $\mathcal{W}(w;d)$ holds, then for all families of Schwartz functions $\{f_k\}$ with $\operatorname{supp}\widehat{f}_k \subset \mathit{\Pi}_k^{(\delta)}$ the estimate (4.1) holds with*

$$\beta = \mu(p) - \frac{d-1}{2}\left(\frac{((d+1)/2(d+3)) - 1/p}{((d+1)/2(d+3)) + (1/p) - 2(p-1)/(w-1)p}\right)\left(\frac{1}{p} - \frac{d-2}{2(d-1)}\right). \tag{4.12}$$

The proof (of a slightly more general result) will be given in § 5.

In $2+1$ dimensions, Theorem 4.3 yields inequality (4.1) for the range $\frac{10}{3} \leqslant p \leqslant w$ with β equal to

$$\beta_{**}(p,w) = \frac{1}{2p}\,\frac{(3p^2-2p-20)w - 23p^2 + 82p - 40}{(10+3p)w - 23p + 10}; \tag{4.13}$$

in particular, we have $\beta_{**}(4,w) = (5w-20)/(44w-164)$, which (with $p_2 \equiv w$) occurs in Theorem 1.7. We compare this result with (4.6). Notice that $\frac{3}{32} = \beta_*(4,6) < \beta_{**}(4,6) = \frac{1}{10}$. A straightforward computation shows the inequality $\beta_{**}(p,w) < \beta_*(p,w)$ holds if and only if $(9p-30)w^2 + (-9p^2 - 39p + 230)w + 23p(3p-10) > 0$ and after factoring we see that for $\frac{10}{3} < p < w$ we have $\beta_{**}(p,w) < \beta_*(p,w)$ if and only if $(p - \frac{10}{3})(w - \frac{23}{3})(w-p) > 0$. Thus, for any $p \in (\frac{10}{3}, w)$ we have

$$\beta_{**}(p,w) < \beta_*(p,w) \quad\iff\quad w > \tfrac{23}{3}, \tag{4.14}$$

so that the L^p result in Theorem 4.3 is better than the result of Corollary 4.2 in the range $w > \frac{23}{3}$. We obtain the following corollary, which yields Theorem 1.7.

Corollary 4.4. *Let $d = 2$ and suppose that $\mathcal{W}(w;2)$ holds for some $w > 6$. Let $\frac{10}{3} < p \leqslant 4$ and let $\alpha > \min\{\beta_*(p,w), \beta_{**}(p,w)\}$ (i.e. $\alpha > \beta_{**}(p,w)$ if $w > \frac{23}{3}$). Then*

(i) *the smoothing inequality (1.6) holds true and*

(ii) *the Fourier multiplier m_α in (1.7) defines a bounded operator on $L^p(\mathbb{R}^3)$.*

We also observe that by interpolation we obtain the analogous boundedness results for the range $4 \leqslant p \leqslant w$ under the assumption that

$$\alpha > \frac{1}{2} - \frac{2}{p} + \frac{4(w-p)}{p(w-4)} \min\{\beta_*(4,w), \beta_{**}(4,w)\}.$$

If we use the result of Theorem 1.1 in $2+1$ dimensions (i.e. Hypothesis $\mathcal{W}(w;2)$ with $w = p_2 = \frac{190}{3}$) we obtain this result for

$$\alpha > \beta_{**}(p, \tfrac{190}{3}) = \frac{501p^2 - 134p - 3920}{2p(501p + 1930)},$$

which equals $\frac{445}{3934}$ if $p = 4$. This represents a slight improvement over the Tao–Vargas result [**20**], which yields the L^4 boundedness for $\alpha > \frac{5}{44} = 0.11\overline{363}$; note that $\frac{445}{3934} \approx 0.11311642\ldots$. We also see from Corollary 4.2 that the validity of (1.3) for the optimal (conjectured) range $p \geqslant 6$ implies the L^4 boundedness for $\alpha > \frac{3}{32} = 0.09375$; however, it has been conjectured that it should hold for all $\alpha > 0$.

Proof of Corollary 4.4. It remains to estimate the L^p-norm of the square function. For part (ii) this is done as in [**13**]; namely, one first uses a weighted L^2 bound as in (4.10) together with the optimal $L^{(p/2)'}$ bound for a Besicovitch maximal function associated with the light cone. Now let S_k be the region in $\mathbb{R}^2$ obtained by projecting the plate $\Pi_k^{(\delta)}$ to the ξ_1–ξ_2-plane. Now define an operator $\mathfrak{S}_k$ by $\widehat{\mathfrak{S}_k g}(\tau,\xi) = \eta_k(\xi)\hat{g}(\tau,\xi)$, where $\xi = (\xi_1,\xi_2)$ and η_k is a function adapted to the double of S_k, with the property that $P_k\mathfrak{S}_k = P_k$. We then obtain

$$\bigg\|\bigg(\sum_k |P_k g|^2\bigg)^{1/2}\bigg\|_p \leqslant C(1+\log\delta^{-1})^{1/2-1/p}\bigg\|\bigg(\sum_k |\mathfrak{S}_k g|^2\bigg)^{1/2}\bigg\|_p \tag{4.15}$$

and by Córdoba's estimate for a sectorial square function [**7**] one dominates the latter L^p-norm by $C(\log\delta)^C\|g\|_p$.

For part (i) one argues similarly, except that now one has to use a result for a Besicovitch maximal function which sends functions on $\mathbb{R}^3$ to functions on $\mathbb{R}^2$; this variant and its application are discussed in [**14**]. □

5. Proof of Theorem 4.3

We work with the operators P_k in (4.5), which localize in Fourier space to the doubles of the plates $\Pi_k^{(\delta)}$. It will be convenient to consider the following mixed norm variant of the 'Wolff hypothesis'.

Hypothesis $\boldsymbol{\mathcal{W}(r,s;d)}$. Given $r \geqslant 2(d+1)/(d-1)$ and $1 \leqslant s \leqslant r$, we say that Hypothesis $\mathcal{W}(r,s;d)$ holds if, for all $\delta < 1$, $\varepsilon > 0$ and all families of Schwartz functions $\{h_k\}$, we have

$$\bigg\|\sum_k P_k h_k\bigg\|_r \leqslant C_\varepsilon \delta^{-\alpha(r,s)-\varepsilon}\bigg(\sum_k \|h_k\|_r^s\bigg)^{1/s}, \tag{5.1}$$

where $\alpha(r,s) = (d-1)/2s' - (d+1)/2r$.

We shall prove the following variant of Theorem 4.3.

Theorem 5.1. *Let $2 \leqslant d \leqslant 4$, and let p be as in (4.4). Let $q = 2(d+3)/(d+1)$. If Hypothesis $\mathcal{W}(r,p;d)$ holds, then for all families of Schwartz functions $\{f_k\}$ with $\operatorname{supp} \widehat{f}_k \subset \Pi_k^{(\delta)}$ the estimate (4.1) holds with*

$$\beta = \mu(p) - \frac{d-1}{2}\left[\frac{1/q - 1/p}{1/q + 1/p - 2/r}\right]\left(\frac{1}{p} - \frac{d-2}{2(d-1)}\right). \tag{5.2}$$

Theorem 4.3 is an immediate consequence of Theorem 5.1, by the following observation.

Lemma 5.2. *Let $w \geqslant 2(d+1)/(d-1)$ and fix $p \in [2,w]$. Then $\mathcal{W}(w;d)$ implies $\mathcal{W}(r,p;d)$ with $r = p'(w-1)$.*

Proof. This follows by interpolation between the Wolff inequality (i.e. (4.8) in d dimensions) and the trivial bound $\|\sum_k P_k h_k\|_\infty \lesssim \sum_k \|h_k\|_\infty$. □

Remark 5.3. In [**9**] we establish certain cases of the mixed norm inequality $\mathcal{W}(r,s;d)$ which do not simply follow by interpolation from the original Wolff inequality (as formulated in $\mathcal{W}(w;d)$). In such cases Theorem 5.1 leads to further improvements of Theorem 1.7.

To establish Theorem 5.1 we shall work with the following hypothesis.

Hypothesis $\mathcal{SQ}(\gamma,p)$. For all $\delta < 1$, $\varepsilon > 0$,

$$\left\|\sum_k h_k\right\|_p \leqslant C_\varepsilon \delta^{-\gamma-\varepsilon}\left\|\left(\sum_k |h_k|^2\right)^{1/2}\right\|_p, \tag{5.3}$$

provided that $\operatorname{supp} \hat{h}_k \subset \Pi_k^{(\delta)}$.

By Proposition 2.3 we know already that for $p > 2(d+3)/(d+1)$ this inequality holds true with the exponent $\gamma = \mu(p) = d/4 - (d+1)/2p$ and we seek an improvement in the ranges (4.4).

We use Hypothesis $\mathcal{W}(r,p;d)$ to prove the following proposition, which amounts to an improved version of [**20**, Proposition 5.4] (where the case $r = \infty$ was considered in the $(2+1)$-dimensional situation). As in § 2 we work with a covering $\mathcal{Q}(\delta^{-1/2})$ of $\sqrt{1/\delta}$ cubes.

Proposition 5.4. *Let $d \geqslant 2$, $2 < p < r$, and suppose that hypotheses $\mathcal{W}(r,p;d)$ and $\mathcal{SQ}(\gamma,p)$ hold. Then, for all functions h_k with $\operatorname{supp} \hat{h}_k \in \Pi_k^{(\delta)}$ we have*

$$\left(\sum_{Q \in \mathcal{Q}(\delta^{-1/2})} \left\|\sum_k h_k\right\|_{L^r(Q)}^p\right)^{1/p} \leqslant C_\varepsilon \delta^{-(\gamma+\alpha(p))/2-\varepsilon}\left\|\left(\sum_k |h_k|^2\right)^{1/2}\right\|_{L^p(\mathbb{R}^{d+1})}. \tag{5.4}$$

Proof. We group the indices k (and therefore the corresponding plates $\Pi_k^{(\delta)}$) into $O(\delta^{-(d-1)/4})$ disjoint families S_l so that $\operatorname{dist}(\omega_k, \omega_{k'}) \lesssim \delta^{1/4}$ for $k, k' \in S_l$. Define

$$G_l = \sum_{k \in S_l} g_k.$$

As in the proof of Proposition 2.2 we also work with the functions ψ_Q adapted to the cubes $Q \in \mathcal{Q}(\delta^{-1/2})$. By the support property of $\hat{\psi}_Q$ the Fourier transform of $\psi_Q G_l$ is supported in a $C\sqrt{\delta}$ plate and these plates form an essentially disjoint plate family. Therefore,

$$\Big\|\sum_l G_l\Big\|_{L^r(Q)} \lesssim \Big\|\psi_Q \sum_l G_l\Big\|_r \lessapprox \delta^{-\alpha(r,p)/2}\Big(\sum_l \|\psi_Q G_l\|_r^p\Big)^{1/p}, \tag{5.5}$$

by Hypothesis $\mathcal{W}(r,p;d)$ with δ replaced by $\sqrt{\delta}$. By the support property of $\widehat{\psi_Q G_l}$ and Young's inequality,

$$\|\psi_Q G_l\|_r \lesssim \delta^{((d+1)/4)(1/p-1/r)}\|\psi_Q G_l\|_p \tag{5.6}$$

and therefore

$$\Big(\sum_Q \Big\|\sum_l G_l\Big\|_{L^r(Q)}^p\Big)^{1/p} \lessapprox \delta^{-(\alpha(r,p)/2)+((d+1)/4)(1/p-1/r)}\Big(\sum_{Q,l}\|\psi_Q G_l\|_p^p\Big)^{1/p}.$$

A little algebra shows that

$$-\frac{\alpha(r,p)}{2} + \frac{d+1}{4}\Big(\frac{1}{p}-\frac{1}{r}\Big) = -\frac{\alpha(p)}{2}.$$

From some straightforward estimation using the decay of the ψ_Q we also obtain

$$\Big(\sum_Q \Big\|\sum_l G_l\Big\|_{L^r(Q)}^p\Big)^{1/p} \lessapprox \delta^{-\alpha(p)/2}\Big(\sum_l \|G_l\|_p^p\Big)^{1/p}. \tag{5.7}$$

As $\hat{G}_l$ is supported in a $C\sqrt{\delta}$ plate we may use rescaling arguments as in the proof of Lemma 2.1 to deduce from Hypothesis $\mathcal{SQ}(\gamma,p)$ applied with parameter $\sqrt{\delta}$ that

$$\|G_l\|_p \lessapprox \delta^{-\gamma/2}\Big\|\Big(\sum_{k\in S_l}|g_k|^2\Big)^{1/2}\Big\|_p$$

and hence

$$\begin{aligned}\Big(\sum_Q \Big\|\sum_l G_l\Big\|_{L^r(Q)}^p\Big)^{1/p} &\leqslant C_\varepsilon \delta^{-(\alpha(p)+\gamma)/2-\varepsilon}\Big(\sum_l \Big\|\Big(\sum_{k\in S_l}|g_k|^2\Big)^{1/2}\Big\|_p^p\Big)^{1/p}\\ &\lesssim C_\varepsilon \delta^{-(\alpha(p)+\gamma)/2-\varepsilon}\Big\|\Big(\sum_k |g_k|^2\Big)^{1/2}\Big\|_p,\end{aligned}$$

which is the assertion. □

In order to complete the Proof of Theorem 5.1, we begin by observing that Hypothesis $\mathcal{SQ}(\mu(p),p)$ holds by Proposition 2.3.

Assuming that $\mathcal{SQ}(\gamma, p)$ holds for some $\gamma \leqslant \mu(p)$, the following estimate for bilinear expressions is an immediate consequence of Proposition 5.4:

$$\Bigl(\sum_{Q\in\mathcal{Q}}\Bigl\|\Bigl(\sum_{\omega_k\in\Omega}f_k\Bigr)\Bigl(\sum_{\omega_{k'}\in\Omega'}g_{k'}\Bigr)\Bigr\|_{L^{r/2}(Q)}^{p/2}\Bigr)^{2/p} \lessapprox \delta^{-\alpha(p)-\gamma}\Bigl\|\Bigl(\sum_{\omega_k\in\Omega}|f_k|^2\Bigr)^{1/2}\Bigr\|_p\Bigl\|\Bigl(\sum_{\omega_{k'}\in\Omega'}|g_{k'}|^2\Bigr)^{1/2}\Bigr\|_p. \tag{5.8}$$

We now assume that Ω and Ω' are separated as in Proposition 2.2 and interpolate the inequalities (5.8) and (2.9) with $q = 2(d+3)/(d+1)$. As a result we obtain

$$\Bigl(\sum_{Q\in\mathcal{Q}}\Bigl\|\Bigl(\sum_{\omega_k\in\Omega}f_k\Bigr)\Bigl(\sum_{\omega_{k'}\in\Omega'}g_{k'}\Bigr)\Bigr\|_{L^{p/2}(Q)}^{p/2}\Bigr)^{2/p} \lessapprox \delta^{-2\Gamma(p,\gamma)}\Bigl\|\Bigl(\sum_{\omega_k\in\Omega}|f_k|^2\Bigr)^{1/2}\Bigr\|_p\Bigl\|\Bigl(\sum_{\omega_{k'}\in\Omega'}|g_{k'}|^2\Bigr)^{1/2}\Bigr\|_p,$$

where

$$\Gamma(p,\gamma) = (1-\vartheta)\mu(p) + \vartheta\frac{\alpha(p)+\gamma}{2} \quad \text{with } \vartheta = \Bigl(\frac{1}{q}-\frac{1}{p}\Bigr)\Bigl(\frac{1}{q}-\frac{1}{r}\Bigr)^{-1}.$$

By Lemma 2.1 we also obtain

$$\Bigl\|\sum_k f_k\Bigr\|_p \lessapprox \delta^{-\Gamma(p,\gamma)}\Bigl\|\Bigl(\sum_{\omega_k\in\Omega}|f_k|^2\Bigr)^{1/2}\Bigr\|_p. \tag{5.9}$$

The assumption $p < 2(d-1)/(d-2)$ in (4.4) implies that $\alpha(p) < \Gamma(p,\gamma) \leqslant \mu(p)$, provided that $\alpha(p) < \gamma \leqslant \mu(p)$. Moreover, $\gamma = \Gamma(p,\gamma)$ if and only if γ equals

$$\gamma_* = \frac{1}{1-\vartheta/2}\Bigl((1-\vartheta)\mu(p) + \vartheta\frac{\alpha(p)}{2}\Bigr) = \mu(p) - \frac{\vartheta}{2-\vartheta}(\mu(p)-\alpha(p)).$$

The fixed point is contained in the interval $(\alpha(p), \mu(p))$ and one observes that $\Gamma(p,\gamma) < \gamma$ for $\gamma^* < \gamma \leqslant \mu(p)$. Thus, if we define a sequence γ_n by setting $\gamma_0 = \mu(p)$ and $\gamma_{n+1} = \Gamma(p,\gamma_n)$ for $n \geqslant 0$, then γ_n is decreasing and bounded below and converges to γ^*. We compute that $\vartheta/(2-\vartheta) = (1/q - 1/p)/(1/q + 1/p - 2/r)$ and $\alpha(p) - \mu(p) = (d-2)/4 - (d-1)/2p$ and see that γ_* is equal to the right-hand side of (5.2). Thus, (5.9) and an iteration yield the assertion of the theorem.

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References

1. D. Békollé, A. Bonami, G. Garrigós, C. Nana, M. Peloso and F. Ricci, Lecture notes on Bergman projectors in tube domains over cones, An analytic and geometric viewpoint, *IMHOTEP J. Afric. Math. Pures Appl.* **5** (2004) (exposé I; available at www.univ-orleans.fr/mapmo/imhotep/archive/vol5/art1.pdf).
2. D. Békollé, A. Bonami, G. Garrigós and F. Ricci, Littlewood–Paley decompositions related to symmetric cones and Bergman projections in tube domains, *Proc. Lond. Math. Soc.* **89** (2004), 317–360.
3. D. Békollé, A. Bonami, M. Peloso and F. Ricci, Boundedness of weighted Bergman projections on tube domains over light cones, *Math. Z.* **237** (2001), 31–59.
4. J. Bourgain, Besicovitch type maximal operators and applications to Fourier analysis, *Geom. Funct. Analysis* **1** (1991), 147–187.
5. J. Bourgain, Estimates for cone multipliers, in *Geometric aspects of functional analysis* (ed. J. Lindenstrauss and V. Milman), Operator Theory, Advances and Applications, Volume 77 (Birkhäuser, 1995).
6. A. Carbery, Variants of the Calderón–Zygmund theory for L^p-spaces, *Rev. Mat. Ibero.* **2** (1986), 381–396.
7. A. Córdoba, Geometric Fourier analysis, *Annales Inst. Fourier* **32** (1982), 215–226.
8. C. Fefferman, A note on spherical summation multipliers, *Israel J. Math.* **15** (1973), 44–52.
9. G. Garrigós, W. Schlag and A. Seeger, Improvements in Wolff's inequality for the cone multiplier, preprint.
10. I. Łaba and M. Pramanik, Wolff's inequality for hypersurfaces, in *Proc. 7th Int. Conf. on Harmonic Analysis and Partial Differential Equations, El Escorial, 2004*, *Collectanea Math.* Extra Volume (2006), 293–326.
11. I. Łaba and T. Wolff, A local smoothing estimate in higher dimensions, *J. Analysis Math.* **88** (2002), 149–171.
12. S. Lee, Improved bounds for Bochner–Riesz and maximal Bochner–Riesz operators, *Duke Math. J.* **122** (2004), 205–232.
13. G. Mockenhaupt, A note on the cone multiplier, *Proc. Am. Math. Soc.* **117** (1993), 145–152.
14. G. Mockenhaupt, A. Seeger and C. D. Sogge, Wave front sets, local smoothing and Bourgain's circular maximal theorem, *Annals Math.* **136** (1992), 207–218.
15. D. Müller and A. Seeger, Regularity properties of wave propagation on conic manifolds and applications to spectral multipliers, *Adv. Math.* **161** (2001), 41–130.
16. M. Pramanik and A. Seeger, L^p regularity of averages over curves and bounds for associated maximal operators, *Am. J. Math.* **129** (2007), 61–103.
17. A. Seeger, Some inequalities for singular convolution operators in L^p-spaces, *Trans. Am. Math. Soc.* **308** (1988), 259–272.
18. T. Tao, Endpoint bilinear restriction theorems for the cone, and some sharp null form estimates, *Math. Z.* **238** (2001), 215–268.
19. T. Tao, A sharp bilinear restrictions estimate for paraboloids, *Geom. Funct. Analysis* **13** (2003), 1359–1384.

20. T. Tao and A. Vargas, A bilinear approach to cone multipliers, II, Applications, *Geom. Funct. Analysis* **10** (2000), 216–258.
21. T. Tao, A. Vargas and L. Vega, A bilinear approach to the restriction and Kakeya conjectures, *J. Am. Math. Soc.* **11** (1998), 967–1000.
22. T. Wolff, Local smoothing type estimates on L^p for large p, *Geom. Funct. Analysis* **10** (2000), 1237–1288.
23. T. Wolff, A sharp bilinear cone restriction estimate, *Annals Math. (2)* **153** (2001), 661–698.