*Bull. Aust. Math. Soc.* **99** (2019), 10–14
doi:10.1017/S0004972718000783

# REPRESENTATION FUNCTIONS ON ABELIAN GROUPS

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(Received 21 May 2018; accepted 15 June 2018; first published online 15 August 2018)

**Abstract**

Let $G$ be a finite abelian group, $A$ a nonempty subset of $G$ and $h \geq 2$ an integer. For $g \in G$, let $R_{A,h}(g)$ denote the number of solutions of the equation $x_1 + \cdots + x_h = g$ with $x_i \in A$ for $1 \leq i \leq h$. Kiss *et al*. ['Groups, partitions and representation functions', *Publ. Math. Debrecen* **85**(3) (2014), 425–433] proved that (a) if $R_{A,h}(g) = R_{G\setminus A,h}(g)$ for all $g \in G$, then $|G| = 2|A|$, and (b) if $h$ is even and $|G| = 2|A|$, then $R_{A,h}(g) = R_{G\setminus A,h}(g)$ for all $g \in G$. We prove that $R_{G\setminus A,h}(g) - (-1)^h R_{A,h}(g)$ does not depend on $g$. In particular, if $h$ is even and $R_{A,h}(g) = R_{G\setminus A,h}(g)$ for some $g \in G$, then $|G| = 2|A|$. If $h > 1$ is odd and $R_{A,h}(g) = R_{G\setminus A,h}(g)$ for all $g \in G$, then $R_{A,h}(g) = \frac{1}{2}|A|^{h-1}$ for all $g \in G$. If $h > 1$ is odd and $|G|$ is even, then there exists a subset $A$ of $G$ with $|A| = \frac{1}{2}|G|$ such that $R_{A,h}(g) \neq R_{G\setminus A,h}(g)$ for all $g \in G$.

2010 *Mathematics subject classification*: primary 11B34.

*Keywords and phrases*: representation functions, abelian groups.

## 1. Introduction

Let $G$ be a finite abelian group and $A$ a nonempty subset of $G$. For $g \in G$ and an integer $h \geq 2$, let $R_{A,h}(g)$ denote the number of solutions of the equation $x_1 + \cdots + x_h = g$, with $x_i \in A$ for $1 \leq i \leq h$.

Recently, Kiss *et al.* [5] proved the following results.

THEOREM A (Kiss *et al.* [5]). *Let $h \geq 2$ be a fixed integer, $G$ a finite abelian group and $A$ a nonempty subset of $G$.*

(i) *If $R_{A,h}(g) = R_{G\setminus A,h}(g)$ for all $g \in G$, then $|G| = 2|A|$.*
(ii) *If $h$ is even and $|G| = 2|A|$, then $R_{A,h}(g) = R_{G\setminus A,h}(g)$ for all $g \in G$.*

THEOREM B (Kiss *et al.* [5]). *If $h > 2$ is a fixed odd integer and $A \subseteq \mathbb{Z}_m$ with $|A| = \frac{1}{2}m$, then there exists $g \in \mathbb{Z}_m$ such that $R_{A,h}(g) \neq R_{\mathbb{Z}_m\setminus A,h}(g)$.*

For related results, one may refer to [1–4] and [6–11]. In this paper, we prove further results about these representation functions.

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This work was supported by the National Natural Science Foundation of China, No. 11771211.

THEOREM 1.1. *Let $h \geq 2$ be an integer, $G$ a finite abelian group and $A$ a nonempty subset of $G$. Then, for every $g \in G$,*

$$R_{G\setminus A,h}(g) - (-1)^h R_{A,h}(g) = \frac{(|G| - |A|)^h - (-1)^h |A|^h}{|G|}.$$

COROLLARY 1.2. *Let $h \geq 2$ be an even integer, $G$ a finite abelian group and $A$ a nonempty subset of $G$. Then $R_{A,h}(g) = R_{G\setminus A,h}(g)$ for some $g \in G$ if and only if $|G| = 2|A|$. Moreover, $R_{A,h}(g) = R_{G\setminus A,h}(g)$ for some $g \in G$ if and only if $R_{A,h}(g) = R_{G\setminus A,h}(g)$ for all $g \in G$.*

Corollary 1.2 follows immediately from Theorem 1.1.

COROLLARY 1.3. *Let $h \geq 3$ be an odd integer, $G$ a finite abelian group and $A$ a nonempty subset of $G$. Suppose that $R_{A,h}(g) = R_{G\setminus A,h}(g)$ for all $g \in G$. Then*

$$R_{A,h}(g) = R_{G\setminus A,h}(g) = \tfrac{1}{2}|A|^{h-1} \quad \text{for all } g \in G.$$

Corollary 1.3 follows immediately from Theorems 1.1 and A(i).

COROLLARY 1.4. *Let $h \geq 3$ be an odd integer, $G$ a finite abelian group and $A$ a nonempty subset of $G$.*

(a) *If $R_{A,h}(g) = R_{G\setminus A,h}(g)$ for some $g \in G$, then $|G|$ is even.*
(b) *If $R_{A,h}(g) = R_{G\setminus A,h}(g)$ for all $g \in G$, then $|G| = 2|A|$ is divisible by 4.*

THEOREM 1.5. *Let $h \geq 3$ be an odd integer and $G$ be a finite abelian group of even order. Then, there exists a subset $A \subseteq G$ with $|A| = \frac{1}{2}|G|$ such that*

$$R_{G\setminus A,h}(g) \neq R_{A,h}(g) \quad \text{for all } g \in G.$$

In Section 2, we will give proofs of Theorem 1.1, Corollary 1.4 and Theorem 1.5.

## 2. Proofs

PROOF OF THEOREM 1.1. Let $\chi_A(x)$ be the characteristic function of the set $A \subseteq G$, that is,

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \in G \setminus A. \end{cases}$$

Observe that

$$R_{A,h}(g) = \sum_{a_1,a_2,\ldots,a_{h-1}\in G} \chi_A(a_1)\cdots\chi_A(a_{h-1})\chi_A(g - a_1 - \cdots - a_{h-1}),$$
$$R_{G\setminus A,h}(g) = \sum_{a_1,a_2,\ldots,a_{h-1}\in G} (1 - \chi_A(a_1))\cdots(1 - \chi_A(a_{h-1}))(1 - \chi_A(g - a_1 - \cdots - a_{h-1})).$$

https://doi.org/10.1017/S0004972718000783 Published online by Cambridge University Press

Consequently, the terms in $R_{A,h}(g)$ cancel in $R_{G\setminus A,h}(g)-(-1)^h R_{A,h}(g)$ and the remaining terms of $R_{G\setminus A,h}(g)$ are given by

$$\begin{aligned}
R_{G\setminus A,h}(g)-(-1)^h R_{A,h}(g) &= \sum_{a_1,a_2,\dots,a_{h-1}\in G}\ \sum_{I\subseteq\{1,2,\dots,h-1\}}(-1)^{|I|}\prod_{i\in I}\chi_A(a_i)\\
&\quad-\sum_{a_1,a_2,\dots,a_{h-1}\in G}\chi_A(g-a_1-\cdots-a_{h-1})\sum_{\substack{I\subseteq\{1,2,\dots,h-1\}\\|I|\le h-2}}(-1)^{|I|}\prod_{i\in I}\chi_A(a_i)\\
&=\sum_{I\subseteq\{1,2,\dots,h-1\}}(-1)^{|I|}\sum_{a_1,a_2,\dots,a_{h-1}\in G}\ \prod_{i\in I}\chi_A(a_i)\\
&\quad-\sum_{\substack{I\subseteq\{1,2,\dots,h-1\}\\|I|\le h-2}}(-1)^{|I|}\sum_{a_1,a_2,\dots,a_{h-1}\in G}\chi_A(g-a_1-\cdots-a_{h-1})\prod_{i\in I}\chi_A(a_i).
\end{aligned}$$

For any $I\subseteq\{1,2,\dots,h-1\}$,

$$\begin{aligned}
\sum_{a_1,a_2,\dots,a_{h-1}\in G}\ \prod_{i\in I}\chi_A(a_i) &= \sum_{a_1,a_2,\dots,a_{h-1}\in G}\ \prod_{i\in I}\chi_A(a_i)\prod_{j\in\{1,2,\dots,h-1\}\setminus I}\chi_G(a_j)\\
&=\prod_{i\in I}\Big(\sum_{a_i\in G}\chi_A(a_i)\Big)\cdot\prod_{j\in\{1,2,\dots,h-1\}\setminus I}\Big(\sum_{a_j\in G}\chi_G(a_j)\Big)\\
&=|A|^{|I|}|G|^{h-1-|I|}. \qquad (2.1)
\end{aligned}$$

Suppose $I\subseteq\{1,2,\dots,h-1\}$ with $|I|\le h-2$ and let $k\in\{1,2,\dots,h-1\}\setminus I$. Then

$$\begin{aligned}
&\sum_{a_1,a_2,\dots,a_{h-1}\in G}\chi_A(g-a_1-\cdots-a_{h-1})\prod_{i\in I}\chi_A(a_i)\\
&\quad=\sum_{\substack{a_t\in G\\1\le t\le h-1,\,t\ne k}}\ \sum_{a_k\in G}\chi_A(g-a_1-\cdots-a_{h-1})\prod_{i\in I}\chi_A(a_i)\\
&\quad=\sum_{\substack{a_t\in G\\1\le t\le h-1,\,t\ne k}}\ \sum_{a\in G}\chi_A(a)\prod_{i\in I}\chi_A(a_i)=|A|\sum_{\substack{a_t\in G\\1\le t\le h-1,\,t\ne k}}\ \prod_{i\in I}\chi_A(a_i).
\end{aligned}$$

Noting that $I\subseteq\{t:1\le t\le h-1,t\ne k\}$, by the argument used previously for (2.1),

$$\sum_{\substack{a_t\in G\\1\le t\le h-1,\,t\ne k}}\ \prod_{i\in I}\chi_A(a_i)=|A|^{|I|}|G|^{h-2-|I|}.$$

Thus

$$\sum_{a_1,a_2,\dots,a_{h-1}\in G}\chi_A(g-a_1-\cdots-a_{h-1})\prod_{i\in I}\chi_A(a_i)=|A|^{|I|+1}|G|^{h-2-|I|}.$$

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Hence

$$\begin{aligned}
&R_{G\setminus A,h}(g) - (-1)^h R_{A,h}(g)\\
&\quad = \sum_{I\subseteq\{1,2,\ldots,h-1\}} (-1)^{|I|}|A|^{|I|}|G|^{h-1-|I|} - \sum_{\substack{I\subseteq\{1,2,\ldots,h-1\}\\ |I|\le h-2}} (-1)^{|I|}|A|^{|I|+1}|G|^{h-2-|I|}\\
&\quad = \sum_{k=0}^{h-1}\binom{h-1}{k}(-1)^k|A|^k|G|^{h-1-k} + \sum_{k=0}^{h-2}\binom{h-1}{k}(-1)^{k+1}|A|^{k+1}|G|^{h-2-k}\\
&\quad = \sum_{k=0}^{h-1}\binom{h-1}{k}(-1)^k|A|^k|G|^{h-1-k} + \sum_{k=1}^{h-1}\binom{h-1}{k-1}(-1)^k|A|^k|G|^{h-1-k}\\
&\quad = |G|^{h-1} + \sum_{k=1}^{h-1}\left(\binom{h-1}{k} + \binom{h-1}{k-1}\right)(-1)^k|A|^k|G|^{h-1-k}\\
&\quad = |G|^{h-1} + \sum_{k=1}^{h-1}\binom{h}{k}(-1)^k|A|^k|G|^{h-1-k} = \sum_{k=0}^{h-1}\binom{h}{k}(-1)^k|A|^k|G|^{h-1-k}\\
&\quad = \frac{(|G|-|A|)^h - (-1)^h|A|^h}{|G|}.
\end{aligned}$$

This completes the proof of Theorem 1.1. □

Proof of Corollary 1.4. (a) Suppose that $R_{A,h}(g) = R_{G\setminus A,h}(g)$ for some $g \in G$. By Theorem 1.1, $2|G|R_{A,h}(g) = (|G| - |A|)^h + |A|^h$. If $|G|$ is odd, then $(|G| - |A|)^h + |A|^h$ is odd, a contradiction. Hence $|G|$ is even.

(b) Suppose that $R_{A,h}(g) = R_{G\setminus A,h}(g)$ for all $g \in G$. By Theorem A(i), $|G| = 2|A|$. In view of Corollary 1.3,

$$R_{A,h}(g) = R_{G\setminus A,h}(g) = \tfrac{1}{2}|A|^{h-1}$$

for all $g \in G$. Therefore $|A|$ is even and $|G| = 2|A|$ is divisible by 4. This completes the proof of Corollary 1.4. □

Proof of Theorem 1.5. Since $|G|$ is even, it follows from the fundamental theorem of finite abelian groups that

$$G \cong \mathbb{Z}_{2^{\alpha_0}} \oplus \mathbb{Z}_{p_1^{\alpha_1}} \oplus \cdots \oplus \mathbb{Z}_{p_t^{\alpha_t}} := \mathbb{Z}_{2^{\alpha_0}} \oplus H,$$

where $p_1, \ldots, p_t$ are primes (not necessary distinct and not necessary odd), $\alpha_0 \ge 1$ and

$$H = \mathbb{Z}_{p_1^{\alpha_1}} \oplus \cdots \oplus \mathbb{Z}_{p_t^{\alpha_t}}.$$

Without loss of generality, we may assume that $G = \mathbb{Z}_{2^{\alpha_0}} \oplus H$. Let

$$A = \{(a_1, a_2) : a_1 \equiv 0 \pmod 2, a_1 \in \mathbb{Z}_{2^{\alpha_0}}, a_2 \in H\}.$$

Then $|A| = |G|/2$ and

$$G \setminus A = \{(b_1, b_2) : b_1 \equiv 1 \pmod 2, b_1 \in \mathbb{Z}_{2^{\alpha_0}}, b_2 \in H\}.$$

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For any $g = (g_1, g_2) \in G$ with $g_1 \in \mathbb{Z}_{2^{\alpha_0}}$ and $g_2 \in H$,

$$g = (g_1, g_2) = (h-1)(g_1, g_2) + (-(h-2)g_1, -(h-2)g_2).$$

If $g_1$ is even, then $R_{G\setminus A,h}(g) = 0$ since $h$ is odd and

$$(g_1, g_2), (-(h-2)g_1, -(h-2)g_2) \in A.$$

Thus

$$R_{A,h}(g) \geq 1 > 0 = R_{G\setminus A,h}(g).$$

If $g_1$ is odd, then $R_{A,h}(g) = 0$ and

$$(g_1, g_2), (-(h-2)g_1, -(h-2)g_2) \in G \setminus A$$

since $h$ is odd. Thus

$$R_{G\setminus A,h}(g) \geq 1 > 0 = R_{A,h}(g).$$

This completes the proof of Theorem 1.5. □

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