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RESEARCH ARTICLE

On the structure of lower bounded HNN extensions

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Abstract

This paper studies the structure and preservational properties of lower bounded HNN extensions of inverse semigroups, as introduced by Jajcayová. We show that if $S^* = [S; U_1, U_2; \phi]$ is a lower bounded HNN extension then the maximal subgroups of S^* may be described using Bass-Serre theory, as the fundamental groups of certain graphs of groups defined from the $\mathscr{D}$-classes of S, U_1 and U_2. We then obtain a number of results concerning when inverse semigroup properties are preserved under the HNN extension construction. The properties considered are completely semisimpleness, having finite $\mathscr{R}$-classes, residual finiteness, being E-unitary, and 0-E-unitary. Examples are given, such as an HNN extension of a polycylic inverse monoid.

1. Introduction

Higman et al. [10] introduced the concept of an HNN extension of a group. In combinatorial group theory, HNN extensions play an important role in algorithmic problems.

Yamamura [20] showed the usefulness of HNN extensions in the variety of inverse semigroups by proving the undecidability of any Markov property and the undecidability of several non-Markov properties. Jajcayová introduced lower bounded HNN extensions in [12], mirroring the definition of lower bounded amalgams of inverse semigroups given in Bennett [3] and [4]. It was proved in Jajcayová [13] that an HNN extension of a free inverse semigroup with finitely generated subsemigroups has decidable word problem.

HNN extensions of a finite inverse semigroup have been considered by Cherubini and Rodaro [6], showing that an HNN extension of a finite inverse semigroup has decidable word problem. More recently, Ayyash and Cherubini [1] and [2] give necessary and sufficient conditions for an HNN extension of a finite inverse semigroup or a lower bounded HNN extension to be completely semisimple. Ayyash [1] also described the maximal subgroups in the finite case.

In the current paper, we use Bass-Serre theory to describe the maximal subgroups of a lower bounded HNN extension S^* containing the idempotents of S (Theorem 4.4). The maximal subgroups are the fundamental groups of graph of groups constructed from $\mathscr{D}$-classes and maximal subgroups of S. All other maximal subgroups of S^* are isomorphic to subgroups of S (Theorem 4.6). Conditions are given for S^* to have finite $\mathscr{R}$-classes (Theorem 4.16). Conditions are given for S^* to be E-unitary and 0-E-unitary (Theorem 4.19). We show that the HNN extension of a polycyclic inverse monoid can be 0-E-unitary, with group of units isomorphic to a free group and all other maximal subgroups are trivial.

2. Preliminaries

A semigroup S is an *inverse semigroup* if for all $s \in S$ there is a unique element s^{-1}, the *inverse of* s, such that $ss^{-1}s = s$ and $s^{-1}ss^{-1} = s^{-1}$. The *semilattice of idempotents* of S is the set $E(S) = \{e \in S : e^2 = e\}$. The *natural partial order* $\leq$ of S is defined by $a \leq b$ if and only if $a = eb$, for some $e \in E(S)$, for $a, b \in S$. A

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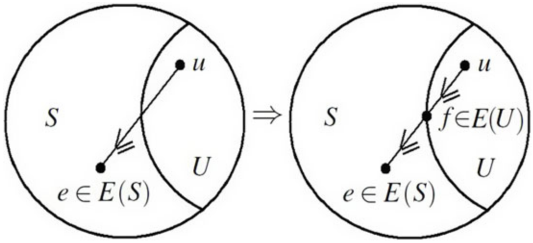

Figure 1. *The lower bounded subsemigroup condition.*

subsemigroup U of S is an *inverse subsemigroup* if $u^{-1} \in U$, for all $u \in U$. For inverse semigroups, see Howie [11], Petrich [18], and Lawson [16].

A *presentation* for an inverse semigroup S is a pair $\langle X \mid R\rangle$, where X is a non-empty set and R is a binary relation on $(X \cup X^{-1})^+$, with $S \cong (X \cup X^{-1})^+/\tau$, where τ is the congruence generated by R and the Vagner congruence ρ. We then say S is *presented* by the *generators* X and *relations* R, written $S = Inv\langle X \mid R\rangle$.

We study $\langle X \mid R\rangle$ by considering the *Schützenberger automaton* $\mathscr{A}(X, R, w)$ of w, for $w \in (X \cup X^{-1})^+$. The automaton $\mathscr{A}(X, R, w)$ has underlying graph $S\Gamma(X, R, w)$, with vertices $R_{w\tau}$, the $\mathscr{R}$-class of S containing $w\tau$, and an edge from s to t labeled by y, for $s, t \in R_{w\tau}$ and $y \in X \cup X^{-1}$ where $s \cdot y\tau = t$ in S. The initial state is $ww^{-1}\tau$ and the terminal state is $w\tau$. We also denote $\langle X \mid R\rangle$, $S\Gamma(X, R, w)$, $\mathscr{A}(X, R, w)$ by $\langle S\rangle$, $S\Gamma(S, w)$, $\mathscr{A}(S, w)$, respectively. For presentations, see Stephen [19].

For any non-empty set X, an *inverse word graph* Γ *over* X is a connected graph with edges labeled over the set $X \cup X^{-1}$, such that for any edge from v_1 to v_2 labeled by y, there is an *inverse edge* from v_2 to v_1 labeled by y^{-1}. The inverse word graph Γ is *deterministic* if no two distinct edges have the same initial vertex and label. We denote the vertex and edge by $V(\Gamma)$ and $E(\Gamma)$, respectively.

A *(birooted) inverse automaton over* X is a triple $\mathscr{A} = (\alpha, \Gamma, \beta)$, where Γ is an inverse word graph over X and α, β are vertices, called the *initial* and *terminal* roots of $\mathscr{A}$, respectively. The *language* $L[\mathscr{A}]$ of the automaton $\mathscr{A}$ is the set of all words labeling paths from α to β. An inverse automaton $\mathscr{A}$ over X is called an *approximate automaton* of $\mathscr{A}(X, R, w)$ if $L[\mathscr{A}] \subseteq L[\mathscr{A}(X, R, w)]$, and there is some word $w_1 \in L[\mathscr{A}]$ with $w_1 = w$ in $S = Inv\langle X \mid R\rangle$, written $\mathscr{A} \rightsquigarrow \mathscr{A}(X, R, w)$. The notation $\cong$ is used to indicate when two inverse word graphs (automata) are isomorphic.

If Γ and Γ_1 are disjoint inverse word graphs, $v_1, v_2 \in V(\Gamma)$ and $\alpha_1, \beta_1 \in V(\Gamma_1)$ then we *sew on* $(\alpha_1, \Gamma_1, \beta_1)$ *from* v_1 *to* v_2 by taking the quotient of $\Gamma \cup \Gamma_1$ by the V-equivalence generated by $\{(v_1, \alpha_1), (v_2, \beta_1)\}$. The *linear automaton* of $w = z_1 z_2 \cdots z_n \in (X \cup X^{-1})^+$, for $z_k \in X \cup X^{-1}$, is the inverse automaton with vertices $v_0 = \alpha_w, v_1, \ldots, v_{n-1}, v_n = \beta_w$ and edges $v_{k-1} \to^{z_k} v_k$, $v_k \to^{z_k^{-1}} v_{k-1}$, for $k = 1, 2, \ldots, n$. If (r, s) is a relation in R and there is a path $v_1 \to^r v_2$ in Γ, with no path $v_1 \to^s v_2$, then we perform an *elementary expansion*, relative to $\langle X \mid R\rangle$, by sewing on the linear automaton of s from v_1 to v_2. A deterministic inverse word graph (automaton) over X is *closed relative to* $\langle X \mid R\rangle$ if no elementary expansion can be performed.

If Γ is an inverse graph over X, then we say *there is a path from vertex* v_1 *to vertex* v_2 *labeled by* $s \in S$, written $v_1 \to^s v_2$, if there is a path $v_1 \to^w v_2$, for some $w \in (X \cup X^{-1})^+$ such that $w\tau = s$ in S. If Γ is closed, relative to $\langle X \mid R\rangle$, and we have a path $v_1 \to^w v_2$, for some $w \in (X \cup X^{-1})^+$ with $w\tau = s$, then we also have a path $v_1 \to^y v_2$, for any $y \in (X \cup X^{-1})^+$ with $y\tau \geq s$.

3. HNN extensions of inverse semigroups

The theory of lower bounded HNN extensions has been generalized by the authors in [5]. An inverse subsemigroup U is called *lower bounded in* S if, for any $u \in U$ and $e \in E(S)$ with $u \geq e$ in S, there exists $f \in E(U)$ with $u \geq f \geq e$ in S. The lower bounded inverse subsemigroup condition is illustrated in Fig. 1. We review some definitions and results from [5].

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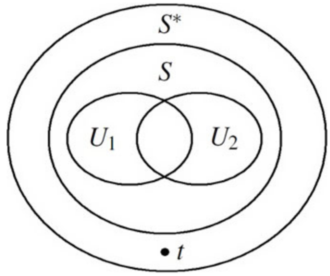

Figure 2. *The HNN extensions $S^* = [S; U_1, U_2; \phi]$.*

We consider an HNN extension $S^* = [S; U_1, U_2; \phi]$ of an inverse semigroup S where U_1 and U_2 are inverse monoids that are lower bounded in S, with respective identities e_1 and e_2, and $\phi : U_1 \to U_2$ is an isomorphism. If U_1 and U_2 are only inverse subsemigroups that are lower bounded in S, then we can study the HNN extension $[S'; U_1', U_2'; \phi']$, where $S' = S \cup \{1\}$, the element 1 is disjoint from S and is the identity of $S \cup \{1\}$, $U_1' = U_1 \cup \{1\}$ and $U_2' = U_2 \cup \{1\}$ are inverse monoids that are lower bounded in $S \cup \{1\}$ and ϕ' is the isomorphism $U_1' \to U_2'$ induced by $\phi : U_1 \to U_2$ and $1 \to 1$.

Let S have inverse semigroup presentation $\langle X \mid R \rangle$. We also denote this presentation for S by $\langle S \rangle$. Let t be disjoint from S. The free product $S * FIS(t)$ in the variety of inverse semigroups has presentation $\langle X \cup \{t\} \mid R \rangle$, where $FIS(t)$ is the free inverse semigroup on $\{t\}$. We also denote this presentation for $S * FIS(t)$ by $\langle S \cup \{t\} \rangle$. The HNN extension S^* has inverse semigroup presentation $\langle X \cup \{t\} \mid R \cup R^* \rangle$, where R^* consists of the relations $tt^{-1} = e_1$, $t^{-1}t = e_2$ and $t^{-1}ut = (u)\phi$, for $u \in U_1$. We also denote this presentation for S^* by $\langle S^* \rangle$. In [20], it was proved that S is embedded into S^*. The HNN extension is illustrated in Fig. 2. For $w \in (X \cup X^{-1})^*$, we let $S\Gamma(S, w)$ and $\mathscr{A}(S, w)$ denote the Schützenberger graph and automaton of w, respectively, relative to $\langle S \rangle$. For $w \in (X \cup X^{-1} \cup \{t, t^{-1}\})^*$, we let $S\Gamma(S^*, w)$ and $\mathscr{A}(S^*, w)$ denote the Schützenberger graph and automaton of w, respectively, relative to $\langle S^* \rangle$.

We briefly describe the algorithm given in [5] for constructing the Schützenberger automata of S^*. Let Γ be an inverse word graph over $X \cup \{t\}$. An $\langle S \rangle$-*lobe* of Γ is a maximal connected subgraph with edges labeled over $X \cup X^{-1}$. A $\langle t \rangle$-*lobe* of Γ is a maximal connected subgraph with edges labeled over $\{t, t^{-1}\}$. The $\langle S \rangle$-lobe containing $v \in V(\Gamma)$ is denoted by $\Delta(v)$. Any path $v_1 \to^t v_2$ is called a *t-edge*. If $v_1 \to^t v_2$ is a t-edge where v_1 and v_2 belong to distinct $\langle S \rangle$-lobes $\Delta(v_1)$ and $\Delta(v_2)$, respectively, then $\Delta(v_1)$ and $\Delta(v_2)$ are called *adjacent* and we say $\Delta(v_2)$ *is connected to* $\Delta(v_1)$ *by a t-edge*.

An $\langle S \rangle$-*lobe path* is a finite sequence of $\langle S \rangle$-lobes $\Delta_1, \Delta_2, \ldots, \Delta_n$, where Δ_k is adjacent to Δ_{k+1}, for $1 \leq k \leq n-1$. The $\langle S \rangle$-lobe path is *reduced* if it is not of the form $\Delta_1, \Delta_2, \Delta_1$ and the $\langle S \rangle$-lobes are distinct, except possibly the first and last. There is a unique reduced $\langle S \rangle$-lobe path between any two $\langle S \rangle$-lobes if and only if there are no non-trivial reduced $\langle S \rangle$-lobe loops. The $\langle S \rangle$-*lobe graph* of Γ is the graph with vertices consisting of the $\langle S \rangle$-lobes of Γ and edges consisting of all pairs (Δ_1, Δ_2) of adjacent $\langle S \rangle$-lobes, where there is a t-edge $v_1 \to^t v_2$ from a vertex v_1 of Δ_1 to a vertex v_2 of Δ_2. The $\langle S \rangle$-lobe graph of Γ is a tree if and only if there are no non-trivial reduced $\langle S \rangle$-lobe loops. An $\langle S \rangle$-lobe of Γ is a called *extremal* if it is adjacent to precisely one other $\langle S \rangle$-lobe.

We say Γ is *t-cactoid* if it has finitely many $\langle S \rangle$-lobes, every t-edge $v_1 \to^t v_2$ connects distinct $\langle S \rangle$-lobes, for any such t-edge there are loops $v_1 \to^{e_1} v_1$ and $v_2 \to^{e_2} v_2$ in Γ, where e_1 and e_2 are the identities of U_1 and U_2, respectively, adjacent $\langle S \rangle$-lobes are connected by precisely one t-edge and the $\langle S \rangle$-lobe graph of Γ is a finite tree. An inverse automaton over $X \cup \{t\}$ is t-cactoid if its underlying graph is.

Construction 3.1. [5, Construction 3.5] *Let $\mathscr{A}$ be a t-cactoid inverse automaton over $X \cup \{t\}$ that is closed, relative to $\langle X \cup \{t\} \mid R \rangle$. Suppose $v_1 \to^t v_2$ is a t-edge of $\mathscr{A}$ and we have a loop $v_1 \to^f v_1$ in $\Delta(v_1)$, for some $f \in E(U_1)$, and no loop $v_2 \to^{(f)\phi} v_2$ in $\Delta(v_2)$. Let $\mathscr{A}'$ be the closed form, relative to $\langle X \cup \{t\} \mid R \rangle$, of the automaton obtained from $\mathscr{A}$ by sewing on the linear automaton of any word that defines $(f)\phi$ in S at v_2. The construction is illustrated in Fig. 3, where the circles represent $\langle S \rangle$-lobes, the dots represent*

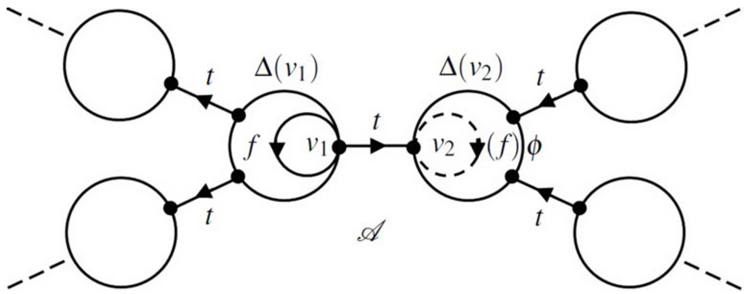

Figure 3. *Construction 3.1 illustrated.*

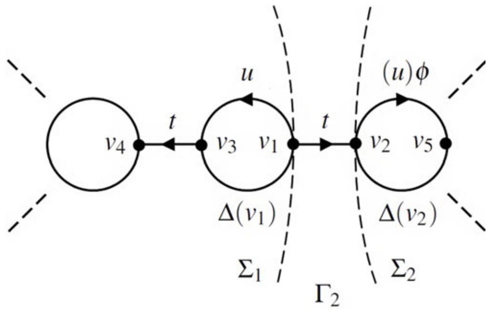

Figure 4. *Construction 3.2 illustrated.*

vertices of $\mathscr{A}$, the arrows represent paths, and the dashed arrow represents the linear automaton of $(f)\phi$. We have an analogous construction when we have a loop $v_2 \to^{(f)\phi} v_2$ in $\Delta(v_2)$, for some $f \in E(U_1)$, and no loop $v_1 \to^f v_1$ in $\Delta(v_1)$.

Construction 3.2. [5, Construction 3.12] *Let $\mathscr{B} = (\alpha_2, \Gamma_2, \beta_2)$ be a t-cactoid inverse automaton over $X \cup \{t\}$ that is closed, relative to $\langle X \cup \{t\} \mid R\rangle$. Suppose there are t-edges $v_1 \to^t v_2$, $v_3 \to^t v_4$ and paths $v_1 \to^u v_3$, $v_2 \to^{(u)\phi} v_5$ in Γ_2, for some $u \in U_1$. The situation is illustrated in Fig. 4.*

Since the $\langle S\rangle$-lobe graph of Γ_2 is a tree, the unique reduced $\langle S\rangle$-lobe path from an $\langle S\rangle$-lobe of Γ_2 to $\Delta(v_1)$ either contains $\Delta(v_2)$ or does not. Let Σ_1 be the subgraph of Γ_2 containing $\Delta(v_1)$ and any $\langle S\rangle$-lobe where the unique reduced $\langle S\rangle$-lobe path to $\Delta(v_1)$ does not contain $\Delta(v_2)$, including all t-edges connecting these $\langle S\rangle$-lobes. Similarly, let Σ_2 be the subgraph of Γ_2 containing $\Delta(v_2)$ and any $\langle S\rangle$-lobe where the unique reduced $\langle S\rangle$-lobe path to $\Delta(v_2)$ does not contain $\Delta(v_1)$, including all t-edges connecting these $\langle S\rangle$-lobes. Thus $\Sigma_1 \cup \Sigma_2$ is equal to Γ_2, minus the t-edge $v_1 \to^t v_2$, and $\Sigma_1 \cap \Sigma_2 = \emptyset$.

Let Σ_1^* and Σ_2^* denote disjoint copies of Σ_1 and Σ_2, respectively. Let α^* and β^* denote the unique respective images of α_2 and β_2 in $\Sigma_1^* \cup \Sigma_2^*$. Then, let η denote the V-equivalence on $\Sigma_1^* \cup \Sigma_2^*$ generated by $\{(v_4, v_5)\}$, letting v_4 and v_5 denote their unique images in $\Sigma_1^* \cup \Sigma_2^*$. Put $\mathscr{C} = (\alpha^*\eta, (\Sigma_1^* \cup \Sigma_2^*)/\eta, \beta^*\eta)$. Let $\mathscr{B}'$ denote the closed form of $\mathscr{C}$, relative to $\langle X \cup \{t\} \mid R\rangle$.

We have an analogous construction if there are t-edges $v_2 \to^t v_1$, $v_4 \to^t v_3$ and paths $v_1 \to^{(u)\phi} v_3$, $v_2 \to^u v_5$ in Γ_2, for some $u \in U_1$.

Let Γ be an inverse word graph over $X \cup \{t\}$. The graph Γ has the *idempotent property* if for every loop $v \to^s v$ in Γ, where $s \in S$, there is a loop $v \to^e v$, for some $e \in E(S)$ with $s \geq e$ in S. The graph Γ has the *equality property* if, for every t-edge $v_1 \to^t v_2$ in Γ, connecting two distinct $\langle S\rangle$-lobes, there is a loop $v_1 \to^u v_1$ in $\Delta(v_1)$ if and only if there is a loop $v_2 \to^{(u)\phi} v_2$ in $\Delta(v_2)$, for all $u \in U_1$.

For an t-edge $v_1 \to^t v_2$ of Γ, the set of *related pairs* of $v_1 \to^t v_2$ consists of (v_1, v_2) and all pairs (v_3, v_4) of vertices for which we have a path $v_1 \to^u v_3$ in $\Delta(v_1)$ and a path $v_2 \to^{(u)\phi} v_4$ in $\Delta(v_2)$, for some $u \in U_1$. If

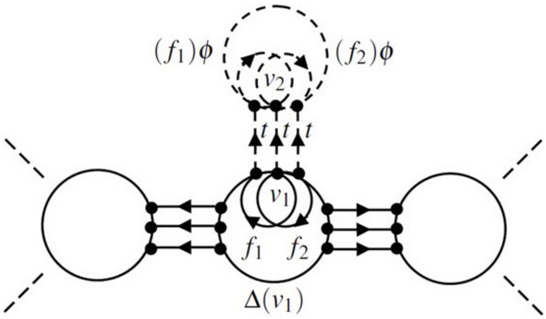

Figure 5. *Construction 3.3 illustrated.*

(v_3, v_4) is a related pair of $v_1 \to^t v_2$, then v_3 and v_4 are called its *first and second coordinates*, respectively. The graph Γ has the *separation property* if the related pairs of any two t-edges, connecting different pairs of $\langle S\rangle$-lobes, share no common first coordinates and no common second coordinates.

We say that a t-edge $v_1 \to^t v_2$ of Γ has *identified related pairs* if there is a t-edge $v_3 \to^t v_4$ for every related pair (v_3, v_4) of $v_1 \to^t v_2$. If, in addition, the pair (v_3, v_4) is a related pair of $v_1 \to^t v_2$, for every t-edge $v_3 \to^t v_4$ from $\Delta(v_1)$ to $\Delta(v_2)$, then we say $\Delta(v_1)$ and $\Delta(v_2)$ are *t-saturated* by $v_1 \to^t v_2$. The graph Γ has the *t-saturation property* if any two adjacent $\langle S\rangle$-lobes are t-saturated by some t-edge.

If Γ has the equality property, then the related pairs of any t-edge $v_1 \to^t v_2$ define a partial one-one map between $V(\Delta(v_1))$ and $V(\Delta(v_2))$. If Γ has the equality and separation properties and $v_1 \to^t v_2$ is the only t-edge from $\Delta(v_1)$ to $\Delta(v_2)$, then we can *t-saturate* $\Delta_1(v)$ and $\Delta_2(v)$ by sewing on a t-edge from v_3 to v_4, for every related pair (v_3, v_4) of $v_1 \to^t v_2$, other than (v_1, v_2). If Γ has the equality and separation properties and there is precisely one t-edge connecting adjacent $\langle S\rangle$-lobes, then the *t-saturated form* of Γ is obtained by t-saturating every pair of adjacent $\langle S\rangle$-lobes.

The graph Γ is *t-opuntoid* if every t-edge connects two distinct $\langle S\rangle$-lobes, the idempotent, equality and t-saturation properties hold and there are no non-trivial reduced $\langle S\rangle$-lobe loops. A *t-subopuntoid subgraph* of a t-opuntoid graph Γ is a connected subgraph that is also t-opuntoid and is formed by a collection of the $\langle S\rangle$-lobes of Γ. If Γ is t-opuntoid, then a $v \in V(\Gamma)$ is a *bud* if there is a loop $v \to^f v$ in $\Delta(v)$, for some $f \in E(U_1)$, and no t-edge $v \to^t v'$, or if there a loop $v \to^{(f)\phi} v$ in $\Delta(v)$, for some $f \in E(U_1)$, and no t-edge $v' \to^t v$. Any of the above graph properties holds for an inverse automaton over $X \cup \{t\}$ if it holds for its underlying graph.

Construction 3.3. [5, Construction 3.17] *Let $\mathscr{D}$ be a t-opuntoid automaton that is closed, relative to $\langle X \cup \{t\} \mid R\rangle$, and has a bud v_1. If $v_1 \in V(\mathscr{D})$ and there is a loop $v_1 \to^f v_1$ in $\Delta(v_1)$, for some $f \in E(U_1)$, and no t-edge $v_1 \to^t v_2$, then we form the automaton $\mathscr{E}$ from $\mathscr{D}$ by sewing on a t-edge $v_1 \to^t v_2$ and then sewing the linear automaton of any word that defines $(f)\phi$ in S^* at v_2, for every $f \in E(U_1)$ that labels a loop at v_1 in $\Delta(v_1)$. In Fig. 5, the dashed arrows represent the automata that are sewed and the dashed circle represents the new $\langle S\rangle$-lobe created. Let $\mathscr{E}'$ denote the closed form of $\mathscr{E}$, relative to $\langle S \cup \{t\}\rangle$. Let $v_1' \to^t v_2'$ denote the image of $v_1 \to^t v_2$ in $\mathscr{E}$. Then, $\mathscr{E}'$ is obtained from $\mathscr{E}$ by closing $\Delta(v_2')$, relative to $\langle S\rangle$. Let $v_1'' \to^t v_2''$ denote the image of $v_1' \to^t v_2'$ in $\mathscr{E}'$. Then, let $\mathscr{D}'$ be the automaton obtained from $\mathscr{E}'$ by sewing on a t-edge from v_3 to v_4, for every related pair (v_3, v_4) of $v_1'' \to^t v_2''$, other than (v_1'', v_2'').*

We have an analogous construction if we have a vertex $v_2 \in V(\mathscr{D})$ and there is a loop $v_2 \to^{(f)\phi} v_2$ in $\Delta(v_2)$, for some $f \in E(U_1)$, and no t-edge $v_1 \to^t v_2$.

A t-opuntoid graph Γ is *complete* if it has no buds. A complete t-opuntoid graph is illustrated in Fig. 6, where the circles represent $\langle S\rangle$-lobes and the arrows represent t-edges.

Lemma 3.4. [5, Lemmas 3.18, 3.19] *Let $\mathscr{D}$ be a t-opuntoid automaton, and let $\mathscr{D}'$ be obtained from $\mathscr{D}$ by Construction 3.3. Then $\mathscr{D}'$ is a t-opuntoid automaton and $\mathscr{D}$ is a t-subopuntoid subautomaton of $\mathscr{D}'$. Further, if $\mathscr{D} \rightsquigarrow \mathscr{A}(S^*, w)$ then $\mathscr{D}' \rightsquigarrow \mathscr{A}(S^*, w)$. We have a directed system of all automata obtained from*

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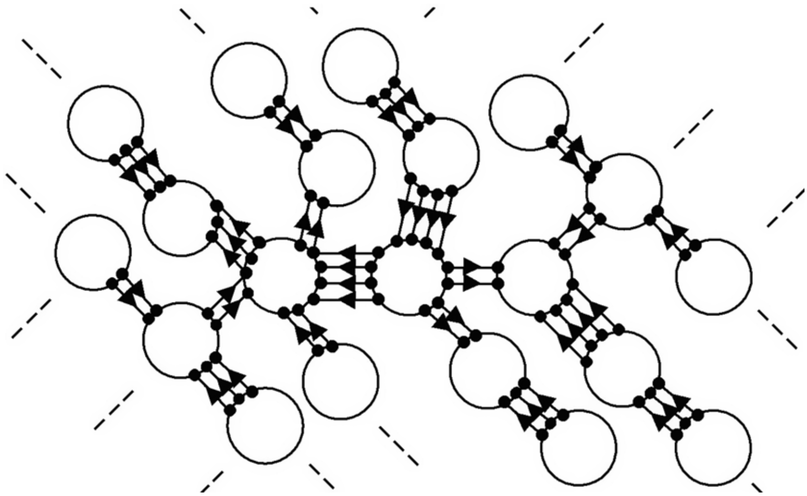

Figure 6. *A complete t-opuntoid graph.*

$\mathscr{D}$ by a finite number of applications of Construction 3.3. The direct limit $\mathscr{E}$ is a complete t-opuntoid automaton. Thus, if $\mathscr{D} \rightsquigarrow \mathscr{A}(S^*, w)$ and we have a loop $v_1 \to^{e_1} v_1$ for every t-edge $v_1 \to^t v_2$ then $\mathscr{E} \cong \mathscr{A}(S^*, w)$.

Algorithm 3.5. [5, Algorithm 3.20] *For $w \in (X \cup X^{-1} \cup \{t, t^{-1}\})^*$, the Schützenberger automaton of w, relative to $\langle S^* \rangle$, is constructed as follows:*

- *(i) Construct $\mathscr{A} = \mathscr{A}(S \cup \{t\}, w)$, using [15]. We can assume $\mathscr{A}$ is t-cactoid and has the idempotent property.*
- *(ii) Construct the direct limit $\mathscr{B}$ of the directed system of all automata obtained from $\mathscr{A}$ by a finite number of applications of Construction 3.1. Then, $\mathscr{B}$ is t-cactoid, has the idempotent and equality propertie,s and has at most as many $\langle S \rangle$-lobes as $\mathscr{A}$.*
- *(iii) If necessary, construct $\mathscr{B}'$ from $\mathscr{B}$ using Construction 3.2. Then ,$\mathscr{B}'$ is t-cactoid, has the idempotent and equality properties, and has fewer $\langle S \rangle$-lobes than $\mathscr{B}$.*
- *(iv) Steps (ii) and (iii) can be applied at most a finite number of times. The resulting automaton $\mathscr{C}$ is t-cactoid and has the idempotent, equality, and separation properties.*
- *(v) The t-saturated form $\mathscr{D}$ of $\mathscr{C}$ is t-opuntoid and has finite $\langle S \rangle$-lobes.*
- *(vi) Construct the direct limit $\mathscr{E}$ of the directed system of all automata obtained from $\mathscr{D}$ by a finite number of applications of Construction 3.3. Then, $\mathscr{E}$ is a complete t-opuntoid automaton and $\mathscr{E} \cong \mathscr{A}(S^*, w)$.*

Let Γ be a t-opuntoid graph. Let Δ_1 and Δ_2 be adjacent $\langle S \rangle$-lobes of Γ. Then, Δ_2 *feeds off* Δ_1 if there is a t-edge $v_1 \to^t v_2$ of Γ from Δ_1 to Δ_2 such that, for any loop $v_2 \to^y v_2$ in Δ_2, there is a loop $v_2 \to^g v_2$ in Δ_2, for some $g \in E(U_2)$ with $y \geq g$ in S. We also say that Δ_2 *feeds off* Δ_1 if there is a t-edge $v_2 \to^t v_1$ of Γ from Δ_2 to Δ_1 such that, for any loop $v_2 \to^y v_2$ in Δ_2, there is a loop $v_2 \to^f v_2$ in Δ_2, for some $f \in E(U_1)$ with $y \geq f$ in S. For non-adjacent $\langle S \rangle$-lobes Δ_1 and Δ_n of Γ, we say Δ_n *feeds off* Δ_1 if there is a sequence of $\langle S \rangle$-lobes $\Delta_1, \Delta_2, \ldots, \Delta_n$, where Δ_{k+1} is adjacent to Δ_k and Δ_{k+1} feeds off Δ_k, for $1 \leq k \leq n-1$,

Let Γ' be a t-subopuntoid subgraph of Γ. An $\langle S \rangle$-lobe of Γ that does not belong to Γ' is called *external* to Γ'. An extremal $\langle S \rangle$-lobe of Γ' is called a *parasite* if it feeds off the unique $\langle S \rangle$-lobe of Γ' to which it is adjacent. The subgraph Γ' is *parasite-free* if it has no parasites. The subgraph Γ' is a *host* of Γ if it has finitely many $\langle S \rangle$-lobes, is parasite-free, and every $\langle S \rangle$-lobe of Γ that is external to Γ' feeds off some $\langle S \rangle$-lobe of Γ'.

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Theorem 3.6. [5, Theorem 3.26] *Let $S^* = [S; U_1, U_2; \phi]$ be an HNN extension of an inverse semigroup S, where U_1 and U_2 are inverse monoids that are lower bounded in S. Then, the Schützenberger automata of S^* are complete t-opuntoid automata with a host.*

Lemma 3.7. [5, Lemma 3.23] *Let Γ be a t-opuntoid graph. Then, a host of Γ is a maximal parasite-free t-subopuntoid subgraph. If Γ has more than one host, then every host is an $\langle S\rangle$-lobe of Γ. The unique reduced $\langle S\rangle$-lobe path between any two hosts, in the $\langle S\rangle$-lobe tree of Γ, consists entirely of $\langle S\rangle$-lobes that are hosts.*

Thus, we can associate a number with a t-opuntoid graph Γ that has a host, by defining $n(\Gamma)$ to be the number of $\langle S\rangle$-lobes in any host. Either Γ has one host, in which case $n(\Gamma) \geq 1$, or every host of Γ is an $\langle S\rangle$-lobe, in which case $n(\Gamma) = 1$.

Lemma 3.8. [5, Lemma 3.24] *Let $\mathscr{D}$ be a t-opuntoid automaton with finitely many $\langle S\rangle$-lobes and a host Σ. If $\mathscr{D}'$ is obtained from $\mathscr{D}$ by Construction 3.3, then Σ is also a host of $\mathscr{D}'$.*

Lemma 3.9. [5, Corollary 3.29] *Let Γ and Γ' be complete t-opuntoid graphs that have hosts and let Σ be any host of Γ. Then, every isomorphism from Σ onto some host of Γ' extends (uniquely) to an isomorphism of Γ onto Γ'.*

Lemma 3.10. [5, Lemma 3.31] *If Γ is a t-opuntoid graph with finitely many $\langle S\rangle$-lobes, then the automorphism group of Γ is embedded into the automorphism group of some $\langle S\rangle$-lobe of Γ.*

Lemma 3.11. [5, Lemma 3.32] *Let Γ be a complete t-opuntoid graph that has a host. Let Γ' be the subgraph that consists of the $\langle S\rangle$-lobes of every host of Γ and the t-edges connecting them. Then, Γ' is a t-subopuntoid subgraph of Γ and the automorphism group of Γ is isomorphic to the automorphism group of Γ'.*

4. Lower bounded HNN extensions

In this section, let U_1 and U_2 denote inverse monoids of an inverse semigroup S, with respective identities e_1 and e_2, let $\phi : U_1 \to U_2$ be an isomorphism and let $S^* = [S; U_1, U_2; \phi]$ be a lower bounded HNN extension, as defined in [12]. That is, for each $e \in E(S)$ and $i \in \{1, 2\}$, the set $\{u \in U_i : u \geq e\}$ is either empty or has a minimal element, denoted by $f_i(e)$, and there does not exist an infinite sequence $\{u_k\}$, where $u_k \in E(U_i)$ and $u_k > f_i(eu_k) > u_{k+1}$, for all k. The monoids U_1 and U_2 are also lower bounded in S, as defined in Section 3, thus we can use the results of Section 3.

Theorem 4.1. *Let $S^* = [S; U_1, U_2; \phi]$ be a lower bounded HNN extension. Then, the Schützenberger automata of S^* are, up to isomorphism, the complete t-opuntoid automata that have a host, a loop $v_1 \to^{e_1} v_1$ for every t-edge $v_1 \to^{t} v_2$ and $\langle S\rangle$-lobes isomorphic to Schützenberger graphs of $\langle S\rangle$.*

Proof. The result is a restatement of Jajcayová [14, Theorem 4.1] using the definitions of Section 3. □

The Bass-Serre theory can be used to describe the maximal subgroups of S^*, as follows; see Cohen [7] and Dicks and Dunwoody [9, Chapters 1, 2, 3] for notation and definitions.

Notation 4.2. *Let Γ be a complete t-opuntoid graph that has a host, and let Γ' denote the t-subopuntoid subgraph of Γ that consists of the $\langle S\rangle$-lobes of every host and all t-edges connecting these hosts. Let $T(\Gamma')$ denote the $\langle S\rangle$-lobe tree of Γ', and let G denote the automorphism group of Γ'. For each $\alpha \in G$*

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and $\langle S\rangle$-lobe Δ of Γ', we define the action of α on Δ, written $\alpha\cdot\Delta$, to be $(\Delta)\alpha$, the image of Δ under α, which is also an $\langle S\rangle$-lobe of Γ'. We extend this action to an action of $T(\Gamma')$ by defining the action of α on the edge (Δ_1, Δ_2) to be equal to $((\Delta_1)\alpha, (\Delta_2)\alpha)$.

The quotient graph $G\backslash T(\Gamma')$ is the graph of orbits of the action of G on $T(\Gamma')$ and is connected. There exist subsets $T_0 \subseteq T \subseteq T(\Gamma')$ such that T_0 is a subtree of $T(\Gamma')$, T_0 and T have the same vertices, the initial vertex of every edge of T is also a vertex of T and T, is a G-transversal in $T(\Gamma')$; that is, T meets each G-orbit exactly once, and thus, the map $T \to G\backslash T(\Gamma') : y \to G\cdot y$, defined for all edges and vertices y, is bijective.

We can make T into a graph by specifying the initial vertex of each edge (Δ_1, Δ_2) to be Δ_1 and specifying the terminal vertex to be the unique vertex Δ_2' of T which lies in the same G-orbit as Δ_2. It then follows that the graph T is isomorphic to $G\backslash T(\Gamma')$ under the above map $y \to G\cdot y$, and T_0 is a maximal subtree of T, as well as a subtree of $T(\Gamma')$.

For any edge $y = (\Delta_1, \Delta_2)$ of T, the $\langle S\rangle$-lobes Δ_2 and Δ_2' lie in the same G-orbit, and thus, we can choose an element $\alpha_y \in G$ such that $\alpha_y\cdot\Delta_2' = \Delta_2$. If $y \in E(T_0)$, then $\Delta_2 \in V(T_0) = V(T)$ and so $\Delta_2' = \Delta_2$, in which case we take α_y as the identity of G. Next, let $G(y)$ denote the stabilizer group of y under the action of G; that is, the group $G(y)$ is the subgroup of G consisting of all automorphisms of Γ' which map Δ_1 onto itself and Δ_2 onto itself. Similarly, for each vertex, we let $G(\Delta)$ denote the stabilizer group of Δ under the action of G. For any edge $y = (\Delta_1, \Delta_2)$ of T, we have $G(y) \subseteq G(\Delta_1)$ and the map $t_y : G(y) \to G(\Delta_2') : \alpha \to \alpha_y \circ \alpha \circ \alpha_y^{-1}$ defines a group monomorphism.

We have a graph of groups $(G(-), T)$. Since $T(\Gamma')$ is a tree, the fundamental group $\Pi(G(-), T, T_0)$ of the graph of groups $(G(-), T)$ is then isomorphic to G. By Lemma 3.11, the automorphism group of Γ is isomorphic to G. Hence, the automorphism group of Γ is isomorphic to $\Pi(G(-), T, T_0)$. The group $\Pi(G(-), T, T_0)$ is generated by the disjoint union of $E(T)$ and the vertex groups of $(G(-), T)$, subject to the relation $y^{-1}\cdot\alpha\cdot y = (\alpha)t_y$, for all $y \in E(T)$ and all $\alpha \in G(y)$, and the relation $y = 1$, for all $y \in E(T_0)$.

Notation 4.3. *We define a graph of groups $(H(-), Y)$ for the HNN extension $S^* = [S; U_1, U_2; \phi]$, as follows. The graph Y has vertices $V(Y)$ the $\mathscr{D}$-classes of S. The graph Y has edges $E(Y)$ the set of all triples (D_1, D, D_2), where D is a $\mathscr{D}$-class of U_1, D_1 is the $\mathscr{D}$-class of S containing D, and D_2 is the $\mathscr{D}$-class of S containing $(D)\phi$.*

We specify an $\mathscr{H}$-class group within each $\mathscr{D}$-class of S and specify an $\mathscr{H}$-class group within each $\mathscr{D}$-class of U_1. Let $y = (D_1, D, D_2)$ be an edge of Y and let H_g, H_f and H_h be the specified $\mathscr{H}$-class groups of D_1, D, and D_2, containing idempotents g, f, and h, respectively. Fix $d_1 \in D_1$ such that $f\mathscr{R}d_1\mathscr{L}g$ in S and fix $d_2 \in D_2$ such that $(f)\phi\mathscr{R}d_2\mathscr{L}h$ in S. The maps $H_f \to H_g : s \to d_1^{-1}sd_1$ and $H_{(f)\phi} \to H_g : s \to d_2^{-1}sd_2$ are group monomorphisms. Then, $H(y) = d_1^{-1}H_fd_1$ is a subgroup of H_g and the map $t_y : H(y) \to H_h : d_1^{-1}sd_1 \to d_2^{-1}\cdot(s)\phi\cdot d_2$ is a group monomorphism.

The construction of the graph of groups $(H(-), Y)$ is completed by defining the vertex group $H(D)$, of each vertex D, to be the specified $\mathscr{H}$-class group of D and defining the edge group and monomorphism of each edge y to be the group $H(y)$ and the monomorphism t_y, respectively, as indicated above.

For $e \in E(S)$, let Y_e denote the connected component of Y containing, as a vertex, the $\mathscr{D}$-class of e in T. If $e \in E(U_1)$, then the $\mathscr{D}$-class of e in S and the $\mathscr{D}$-class of $(e)\phi$ in S are connected by an edge in Y and are thus in the same connected component. Let $(H_e(-), Y_e)$ denote the restriction of $(H(-), Y)$ to Y_e.

The following result generalizes Yamamura [21, Theorem 5.2], on locally full HNN extensions, and overlaps with Ayyash [1, Theorem 5.4.1], on HNN extensions of finite inverse semigroups.

Theorem 4.4. *Let $S^* = [S; U_1, U_2; \phi]$ be a lower bounded HNN extension and let e be an idempotent of S. Then, the maximal subgroup of S^* containing e is isomorphic to the fundamental group of the graph of groups $(H_e(-), Y_e)$.*

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Proof. Let $\Gamma = S\Gamma(S^*, e)$. By Theorem 4.1, the graph Γ is a complete t-opuntoid graph with a host such that the $\langle S\rangle$-lobes are isomorphic to Schützenberger graphs of $\langle S\rangle$. From Notation 4.2, the fundamental group $\Pi(G(-), T, T_0)$ of the graph of groups $(G(-), T)$ is isomorphic to the automorphism group of Γ and thus is also isomorphic to the maximal subgroup of S^* containing e. Hence, the theorem is completed by showing that the graphs of groups $(G(-), T)$ and $(H_e(-), Y_e)$ are conjugate isomorphic. We need to define a graph isomorphism between T and Y_e and define isomorphisms between the corresponding vertex and edge groups, such that the group isomorphisms commute with the corresponding edge monomorphisms.

From Algorithm 3.5, the Schützenberger graph $S\Gamma(S, e)$ is embedded onto a host of Γ. Thus $n(\Gamma) = 1$, as defined in Section 3, and every host is an $\langle S\rangle$-lobe, by Lemma 3.7. We define a graph map $\psi : T \to Y$ as follows.

Let $\Delta \in V(T)$. If $v_1, v_2 \in V(\Delta)$, then $(v_i, \Delta, v_i) \cong \mathscr{A}(S, e(v_i))$, for $i = 1, 2$, and $e(v_1)\mathscr{D}e(v_2)$ in S. Thus, we can define $(\Delta)\psi$ to be equal to the $\mathscr{D}$-class of S containing $e(v)$, for any vertex v of Δ.

Let (Δ_1, Δ_2) be an edge of T. Then, Δ_1 and Δ_2 are hosts of Γ and thus feed off each other, and there is a t-edge from a vertex of Δ_1 to a vertex of Δ_2. If $v_1 \to^t v_2$ and $v_1' \to^t v_2'$ are t-edges, where v_1, v_1' are vertices of Δ_1 and v_2, v_2' are vertices of Δ_2, then we have $e(v_1), e(v_1') \in E(U_1)$, $e(v_2) = (e(v_1))\phi$, $e(v_2') = (e(v_1'))\phi$, with $e(v_1)\mathscr{D}e(v_1')$ in U_1 and $e(v_2)\mathscr{D}e(v_2')$ in U_2. Thus, we can define $(\Delta_1, \Delta_2)\psi$ to be the edge (D_1, D, D_2), where D is the $\mathscr{D}$-class of U_1 containing $e(v_1)$, D_1 is the $\mathscr{D}$-class of S containing $e(v_1)$, and D_2 is the $\mathscr{D}$-class of S containing $(e(v_1))\phi$, for any t-edge $v_1 \to^t v_2$, from a vertex v_1 of Δ_1 to a vertex v_2 to Δ_2.

Let Γ' be the t-subopuntoid subgraph of Γ consisting of every host and let $G = AUT(\Gamma')$. As described in Notation 4.2, the initial vertex of the edge (Δ_1, Δ_2) is Δ_1 and the terminal vertex of the edge (Δ_1, Δ_2) is the unique vertex $\Delta_2' \in V(T)$ that lies in the same G-orbit as Δ_2. Let $v_1 \to^t v_2$ be a t-edge from a vertex v_1 of Δ_1 to a vertex v_2 of Δ_2. Let $e(v_1) \in D_1$ and $e(v_2) \in D_2$. Then, $(\Delta_1)\psi = D_1$ and $(\Delta_2')\psi = D_2$, since $\Delta_2' \cong \Delta_2$. Thus, the map $\psi : T \to Y$ defines a graph homomorphism.

We now show that ψ defines a monomorphism. Suppose Δ and Δ' are vertices of T with $(\Delta)\psi = (\Delta')\psi$. Let v denote a vertex of Δ and let v' denote a vertex of Δ'. Then $(\Delta)\psi = (\Delta')\psi$ implies that $e(v)\mathscr{D}e(v')$ in S, in which case we have $\Delta \cong \Delta'$. Since Δ and Δ' are hosts, the isomorphism between them extends to an automorphism of Γ, by Lemma 3.9. Thus Δ and Δ' are in the same G-orbit and so $\Delta = \Delta'$, since T is a transversal. We have shown that ψ is one-one on vertices. Since T is a tree, the map ψ must also be one-one on the edges. Since Δ has a host that is isomorphic to $S\Gamma(S, e)$, we have that $(T)\psi$ is a connected subgraph of Y_e.

We now show that $(T)\psi = Y_e$. Let (D_1, D, D_2) be an edge of Y_e, where $D_1 = (\Delta_1)\psi$, for some vertex Δ_1 of T. Let f be an idempotent of U_1 that is in D. There exists a vertex v_1 of Δ_1 such that $e(v_1) = f$. Then, there must be a t-edge $v_1 \to^t v_2$, where v_2 is a vertex of an $\langle S\rangle$-lobe Δ_2. Since Δ_1 is a host of Γ and $e(v_2) = (f)\phi$, it follows that Δ_2 is also a host of Γ. Since T meets each G-orbit of $T(\Gamma')$ exactly once, there exists an edge (Δ_1', Δ_2') of T that lies in the same G-orbit as (Δ_1, Δ_2). Since $\Delta_1 \in V(T)$, we must have $\Delta_1 = \Delta_1'$. Then, there exists a t-edge $v_1' \to^t v_2'$ from a vertex v_1' of Δ_1 to a vertex v_2' of Δ_2', such that $e(v_1') = f$ and $e(v_2') = (f)\phi$. We then have $(\Delta_1.\Delta_2')\psi = (D_1, D, D_2)$. A similar proof shows that if (D_1, D, D_2) is an edge of Y_e, where $D_2 = (\Delta_2)\psi$, for some vertex Δ_2 of T, then $(\Delta_1'.\Delta_2)\psi = (D_1, D, D_2)$, for some edge $(\Delta_1'.\Delta_2)$ of T. It now follows that $(T)\psi$ is a maximal connected subgraph of Y_e. We have shown that $\psi : T \to Y$ defines a graph monomorphism onto Y_e.

We now define the vertex group isomorphisms. Let Δ denote a vertex of T. Let $H((\Delta)\psi) = H_g$, the $\mathscr{H}$-class group of S with identity g. If v is a vertex of Δ then $e(v)\mathscr{D}g$ in S. Thus, we have an isomorphism $\pi : \Delta \to S\Gamma(S, g)$. The group $G(\Delta)$ is the stabilizer group of Δ, under the action of G. Since Δ is a host of Γ, any automorphism of Δ extends (uniquely) to an automorphism of Γ, by Lemma 3.9. Thus, we have an isomorphism $G(\Delta) \to AUT(\Delta)$, under the mapping $\alpha \to \alpha_\Delta$, where α_Δ denotes the restriction of α to Δ. We then have an isomorphism $AUT(\Delta) \to AUT(S\Gamma(S, g))$, defined by $\alpha \to \pi^{-1} \circ \alpha_\Delta \circ \pi$. We have an isomorphism $AUT(S\Gamma(S, g)) \to H_g$, under the mapping $\beta \to (g)\beta$; the set of vertices of $S\Gamma(S, g)$ is the $\mathscr{R}$-class of S containing g. Hence we have an isomorphism $\psi : G(\Delta) \to H((\Delta)\psi)$, defined by

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$\alpha \to (g)\pi^{-1} \circ \alpha_\Delta \circ \pi$. The map ψ may be expressed by saying that $(\alpha)\psi = s$, where $s \in S$ such that $\mathscr{A}(S, s) \cong (v, \Delta, (v)\alpha)$, for any vertex v of Δ with $e(v) = g$.

We now define the edge group isomorphisms. Let $y = (\Delta_1, \Delta_2)$ be an edge in T and let $(y)\psi = (D_1, D, D_2)$. Let H_g and H_f denote the specified $\mathscr{H}$-class groups of D_1 and D, containing the identities g and f, respectively. Thus, $H(D_1) = H_g$ and $H((y)\psi) = d_1^{-1}H_f d_1$, where d_1 is the fixed element of D_1 such that $f\mathscr{R}d_1\mathscr{L}g$ in S. Let $v_1 \to^t v_2$ be a t-edge from a vertex v_1 of Δ_1 to a vertex v_2 of Δ_2. Then, $e(v_1)\mathscr{R}a\mathscr{L}f$, for some $a \in U_1$. Thus, we have a path $v_1 \to^a v_3$, where $e(v_3) = f$, and a path $v_3 \to^{d_1} v_4$, where $e(v_4) = g$.

Let $\alpha \in G(y)$. Then, α stabilizes Δ_1 and Δ_2 and so $(v_1)\alpha \to^t (v_2)\alpha$ is a t-edge from Δ_1 to Δ_2. Since Δ_1 and Δ_2 are t-saturated, there is a path $v_1 \to^b (v_1)\alpha$, for some $b \in U_1$. Since, we have a path $v_1 \to^{ad_1} v_4$ in Δ_1, we have a path $(v_1)\alpha \to^{ad_1} (v_4)\alpha$ in Δ. Thus, $(v_4, \Delta_1, (v_4)\alpha) \cong \mathscr{A}(S, s)$, where $s = d_1^{-1}(fa^{-1}ba)d_1$ and $fa^{-1}ba \in H_f$. Hence, $\psi : G(\Delta_1) \to H((\Delta_1)\psi)$ maps $G(y)$ into $H((y)\psi)$.

Conversely, let $c \in H_f$. Since $\psi : G(\Delta_1) \to H((\Delta_1)\psi)$ is an isomorphism, there exists $\alpha \in G(\Delta_1)$ such that $(v_4, \Delta, (v_4)\alpha) \cong \mathscr{A}(S, d_1^{-1}cd_1)$. Then we have $(v_3, \Delta_1, (v_3)\alpha) \cong \mathscr{A}(S, c)$ and $(v_1, \Delta_1, (v_1)\alpha) \cong \mathscr{A}(S, afca^{-1})$. Thus the t-edge $(v_1)\alpha \to^t (v_2)\alpha$ must also be a t-edge from Δ_1 to Δ_2. This implies $\alpha \in G(y)$. Thus the isomorphism $\psi : G(\Delta_1) \to H((\Delta_1)\psi)$ maps $G(y)$ onto $H((y)\psi)$.

Finally, we show that the isomorphisms between the vertex and edge groups of $(G(-), T)$ and $(H(-), Y_e)$ commute with the edge monomorphisms. Let $y = (\Delta_1, \Delta_2)$ be an edge of T, and let $(y)\psi$ be equal to (D_1, D, D_2). Let H_g, H_f and H_h denote the specified $\mathscr{H}$-class groups of D_1, D and D_2, containing idempotents g, f and h, respectively. Let d_1 and d_2 be the fixed elements of D_1 and D_2, respectively, such that $f\mathscr{R}d_1\mathscr{L}g$ and $(f)\phi\mathscr{R}d_2\mathscr{L}h$ in S. The map $t_{(y)\psi} : H((y)\psi) \to H_h$ defined by $d_1^{-1}sd_1 \to d_2^{-1} \cdot (s)\phi \cdot d_2$, for $s \in H((y)\psi)$, is the edge monomorphism for $(y)\psi$. Let Δ_2' be the unique vertex of T that belongs in the same G-orbit as Δ_2, and let $\alpha_y \in G$ such that $\alpha_y \cdot \Delta_2' = \Delta_2$. The edge monomorphism $t_y : G(y) \to G(\Delta_2')$ for y is given by $\alpha \to \alpha_y \circ \alpha \circ \alpha_y^{-1}$.

The composition of the edge map t_y with $\psi : G(\Delta_2') \to H((\Delta_2')\psi)$ is the map $t_y \circ \psi : G(y) \to H((\Delta_2')\psi) : \alpha \to s$, with $\mathscr{A}(S, s) \cong (v, \Delta_2', (v)\alpha_y \circ \alpha \circ \alpha_y^{-1})$, for any vertex v of Δ_2' such that $e(v) = h$. Since α_y maps Δ_2' isomorphically onto Δ_2, we can redefine this map by saying $(\alpha)t_y \circ \psi = s$, where $s \in S$ such that $\mathscr{A}(S, s) \cong (v, \Delta_2, (v)\alpha)$, for some vertex v of Δ_2 with $e(v) = h$.

The composition of $\psi : G(y) \to H((y)\psi)$ with the edge map $t_{(y)\psi}$ is given by $\psi \circ t_{(y)\psi} : G(y) \to H((\Delta_2')\psi) : \alpha \to d_2^{-1} \cdot (r)\phi \cdot d_2$, where $r \in H_f$ such that $\mathscr{A}(S, r) \cong (v_1, \Delta_1, (v_1)\alpha)$, for some t-edge $v_1 \to^t v_2$ from a vertex v_1 of Δ_1 to a vertex v_2 of Δ_2, with $e(v_1) = f$.

Since $e(v_2) = (f)\phi$ and $(f)\phi\mathscr{R}d_2$ in S, there exists a vertex v_2' of Δ_2 such that $(v_2, \Delta_2, v_2') \cong \mathscr{A}(S, d_2)$. Since we have a path $v_1 \to^r (v_1)\alpha$ in Δ_1, we have a path $v_2 \to^{(r)\phi} (v_2)\alpha$ in Δ_2. Then, $(v_2, \Delta_2, (v_2)\alpha) \cong \mathscr{A}(S, (r)\phi)$. Now $((v_2)\alpha, \Delta_2, (v_2')\alpha) \cong \mathscr{A}(S, d_2)$ and so $(v_2', \Delta_2, (v_2')\alpha) \cong \mathscr{A}(S, d_2^{-1} \cdot (r)\phi \cdot d_2)$. We have $e(v_2') = h$ and so $(\alpha)t_y \circ \psi = d_2^{-1} \cdot (r)\phi \cdot d_2 = \psi \circ t_{(y)\psi}$, as required, and the proof of the theorem is complete. □

Notation 4.5. *We define an equivalence $\sim_i$ on S by $s_1 \sim_i s_2$ if and only if $s_1 = s_2$ or $s_1\mathscr{R}s_2$ with $s_1 = s_2u$, for some $u \in U_i$, for $s_1, s_2 \in S$, for $i = 1, 2$.*

Theorem 4.6. *Let $S^* = [S; U_1, U_2; \phi]$ be a lower bounded HNN extension, and let e be an idempotent of S^* that is not $\mathscr{D}$-related to any element of S. Then, the maximal subgroup of S^* containing e is isomorphic to a subgroup H of S, whose quotient $H/\sim_i$ is finite, for some $i \in \{1, 2\}$.*

Proof. Let $\Gamma = S\Gamma(S^*, e)$. Then, the maximal subgroup of S^* containing e is isomorphic to the automorphism group of Γ. If $n(\Gamma) = 1$, as defined in Section 3, then there exists an $\langle S\rangle$-lobe Δ that is a host of Γ. Since Δ is isomorphic to a Schützenberger graph of S, we then have $e\mathscr{D}g$ in S^*, for some $g \in E(S)$, a contradiction. Thus, $n(\Gamma) > 1$ and Γ has precisely one host Σ, consisting of at least two $\langle S\rangle$-lobes.

The automorphism group of Γ is isomorphic to the automorphism group of Σ, by Lemma 3.11. From Lemma 3.10, the automorphism group of Σ is embedded into the automorphism group of some $\langle S\rangle$-lobe Δ of Σ, under the embedding $\alpha \to \alpha_\Delta$, where α_Δ denotes the restriction of α to Δ. Let v be a vertex of

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Δ. We have $\Delta \cong S\Gamma(S, g)$, where $g = e(v)$. Then, the map $\psi : AUT(\Gamma) \to H_g$ defined by $\alpha \to s$, where $(v, \Delta, (v)\alpha) \cong \mathscr{A}(S, s)$ and H_g is the $\mathscr{H}$-class of S containing g, defines a group monomorphism.

Let H denote the image of $AUT(\Gamma)$ under ψ. Let $v_1 \to^t v_2$ be a t-edge of Σ, where one of the vertices v_1, v_2 belongs to Δ. Suppose $v_1 \in V(\Delta)$. We can assume $v = v_1$. Now let $\alpha_1, \alpha_2 \in AUT(\Gamma)$, $(\alpha_1)\psi = s_1$ and $(\alpha_2)\psi = s_2$. If $(v_1)\alpha_1 \to^t (v_2)\alpha_1$ and $(v_1)\alpha_2 \to (v_2)\alpha_2$ are t-edges from Δ to an $\langle S\rangle$-lobe Δ' of Σ, then we have $s_2 = s_1 u$, for some $u \in U_1$, and so $s_1 \sim_1 s_2$. Thus, the number of $\sim_1$-classes in H is at most the number of $\langle S\rangle$-lobes in Σ that are adjacent to Δ. Since Σ has finitely many $\langle S\rangle$-lobes, the group H has finitely many $\sim_1$-classes. If $v_2 \in V(\Delta)$, then a similar proof shows that the group H has finitely many $\sim_2$-classes. □

Theorems 4.4 and 4.6 tell us that every maximal subgroup of S^* is either isomorphic to the fundamental group of some graph of groups $(H_e(-), Y_e)$, where the vertex and edge groups are subgroups of S, or is isomorphic to a subgroup of S.

Corollary 4.7. *Let $S^* = [S; U_1, U_2; \phi]$ be a lower bounded HNN extension.*

- *(i) If S is combinatorial then every maximal subgroup of S^* is a free group.*
- *(ii) S^* is combinatorial if and only if S is combinatorial and Y is a forest.*

Proof. The results are immediate from Theorem 4.4. □

Corollary 4.8. *Let $S^* = [S; U_1, U_2; \phi]$ be a lower bounded HNN extension. Let $f \in E(U_1)$ and H_f, $H_{(f)\phi}$, G_f denote the maximal subgroups containing f, $(f)\phi$, f in S, S, U_1, respectively. Assuming $f\mathscr{D}g$ in S implies $f\mathscr{D}g$ in U_1, for $g \in E(U_1)$:*

- *(ii) If $(f)\phi\mathscr{R}d\mathscr{L}f$ in S, for $d \in S$, then the maximal subgroup of S^* containing f is isomorphic to the group HNN extension $[H_f; G_f, d^{-1} \cdot (G_f)\phi \cdot d]$.*
- *(iii) If $(f)\phi\mathscr{D}f$ in S, then the maximal subgroup of S^* containing f is isomorphic to the amalgamated free product of the group amalgam $[H_f, H_{(f)\phi}; G_f]$.*

Proof. Suppose $(f)\phi\mathscr{R}d\mathscr{L}f$ in S, for $d \in S$. Assuming $f\mathscr{D}g$ in S implies $f\mathscr{D}g$ in U_1, for $g \in E(U_1)$, the component Y_f consists of one vertex D_1 and one edge $y = (D_1, D, D_1)$. We may assume that the vertex group $H(y)$ is G_f and the vertex group $H(D_1)$ is H_f. The group monomorphism $t_y : H(y) \to H(D_1)$ is given by $s \to d^{-1} \cdot (s)\phi \cdot d$. By Theorem 4.4, the maximal subgroup of S^* containing e is isomorphic to the HNN extension of groups $[H(D_1); H(y), (H(y))t_y; t_y]$

Suppose $(f)\phi\mathscr{D}f$ in S. Assuming $f\mathscr{D}g$ in S implies $f\mathscr{D}g$ in U_1, for $g \in E(U_1)$, the component Y_f consists of two vertices D_1 and D_2 connected by a single edge $y = (D_1, D, D_2)$. We may assume that the vertex group $H(y)$ is G_f, the vertex group $H(D_1)$ is H_f, and the vertex group $H(D_2)$ is $H_{(f)\phi}$. The group monomorphism $t_y : H(y) \to H(D_2)$ is given by $s \to (s)\phi$. Then, by Theorem 4.4, the maximal subgroup of S^* containing e is isomorphic to the amalgamated free product of the group amalgam $[H(D_1), H(D_2); H(y) \cong H(y)t_y]$. □

Notation 4.9. *Similar to Ayyash and Cherubini [2], we define a binary relation $\prec_S$ on $E(U_1) \cup E(U_2)$. For $f, g \in E(U_1) \cup E(U_2)$, we write $f \prec_S g$ iff $f\mathscr{D}h \le g$ in S, for some $h \in E(S)$. We then let $\prec$ denote the transitive closure of $\prec_S$ and the set $\{(f, (f)\phi), ((f)\phi, f) : f \in E(U_1)\}$. As the next result shows, we are interested in when the intersection of $\prec$ and $\succ_S$ is contained in $\prec_S$.*

An inverse semigroup is *completely semisimple* if two distinct idempotents in any $\mathscr{D}$-class are not comparable, under the natural partial order. Equivalently, from [3, Lemma 10], an inverse semigroup is completely semisimple if and only if the endomorphism monoid and the automorphism group coincide

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for every Schützenberger graph. We have the following result for lower bounded HNN extensions, which has been generalized in [5, Theorem 3.30].

Theorem 4.10 *(2, Ayyash and Cherubini, Theorem 5.3). Let $S^* = [S; U_1, U_2; \phi]$ be a lower bounded HNN extension. Then, S^* is completely semisimple if and only if S is completely semisimple and $\prec \cap \succ_S \subseteq \prec_S$.*

Corollary 4.11. *Let $S^* = [S; U_1, U_2; \phi]$ be a lower bounded HNN extension. Suppose S is completely semisimple and $f\mathscr{D}(f)\phi$, for all $f \in E(U_1)$. Then, $\prec \cap \succ_S \subseteq \prec_S$ and so S^* is completely semisimple.*

Proof. Let $f_1, g_1, f_2, g_2, \ldots f_n, g_n \in E(U_1) \cup E(U_2)$, for $n \geq 1$, where at least one of $f_k = g_k$, $(f_k)\phi = g_k$ and $(f_k)\phi^{-1} = g_k$ holds, for $1 \leq k \leq n$, and $g_k \prec_S f_{k+1}$, for $1 \leq k \leq n-1$. Thus, we have $f_1 \prec g_n$. Assuming $f\mathscr{D}(f)\phi$, for all $f \in E(U_1)$, we have $f_k \prec_S g_k$, for $1 \leq k \leq n$. It then follows that $f_1 \prec_S g_n$, as $\prec_S$ is transitive. Thus, we have $\prec = \prec_S$. Hence S^* is completely semisimple, by Theorem 4.10. □

Corollary 4.12. *Let $S^* = [S; U_1, U_2; \phi]$ be a lower bounded HNN extension. Suppose S is completely semisimple and the following hold, for all $f, g \in E(U_1)$:*

(i) We do not have $f \prec_S (g)\phi$ in S and so $E(U_1) \cap E(U_2) = \emptyset$.
(ii) $f \prec_S g$ implies $f\mathscr{R}u\mathscr{L}u_1^{-1}u_1 \leq g$, for some $u \in U_1$.
(iii) $(f)\phi \prec_S (g)\phi$ implies $(f)\phi\mathscr{R}(u)\phi\mathscr{L}(u^{-1}u)\phi \leq (g)\phi$, for some $u \in U_1$.

Then $\prec \cap \succ_S \subseteq \prec_S$ and so S^* is completely semisimple.

Proof. Let $f_1, g_1, f_2, g_2, \ldots f_n, g_n \in E(U_1) \cup E(U_2)$, for $n \geq 1$, where at least one of $f_k = g_k$, $(f_k)\phi = g_k$ and $(f_k)\phi^{-1} = g_k$ holds, for $1 \leq k \leq n$, and $g_k \prec_S f_{k+1}$, for $1 \leq k \leq n-1$. If $f_k = g_k$, then $g_{k-1} \prec_S f_k = g_k \prec_S f_{k+1}$, and we can shorten the sequence. Thus we can assume $f_k \neq g_k$.

Suppose $f_1 \in E(U_1)$ and $(f_1)\phi = g_1 \in E(U_2)$. From condition (i) and $g_1 \prec_S f_2$, we have $f_2 \notin E(U_1)$ and so $(f_2)\phi^{-1} = g_2 \in E(U_1)$. From condition (iii) and $g_1 \prec_S f_2$, we have $g_1\mathscr{R}(u_1)\phi\mathscr{L}(u_1^{-1}u_1)\phi \leq f_2$, for some $u_1 \in U_1$. Then applying ϕ^{-1}, we have $f_1\mathscr{R}u_1\mathscr{L}u_1^{-1}u_1 \leq g_2$.

From condition (i) and $g_2 \prec_S f_3$, we have $f_3 \notin E(U_2)$ and so $(f_3)\phi = g_3 \in E(U_2)$. From condition (ii) and $g_2 \prec_S f_3$, we have $g_2\mathscr{R}u_2\mathscr{L}u_2^{-1}u_2 \leq f_3$, for some $u_2 \in U_1$. Thus, $f_1\mathscr{R}u_1u_2\mathscr{L}u_2^{-1}u_1^{-1}u_1u_2 \leq f_3$, where $u_1u_2 \in U_1$.

Continuing in this manner, we have $f_1\mathscr{R}u\mathscr{L}u^{-1}u \leq f_{2k+1}$, for some $u \in U_1$, for $k \geq 1$. Thus, if we also have $f_1 \succ_S f_{2k+1}$ then $f_1\mathscr{D}f_{2k+1}$ in U_1, since S is completely semisimple. Similarly, if $f_1 \in E(U_2)$ and $f_1 \succ_S f_{2k+1}$ then $f_1\mathscr{D}f_{2k+1}$ in U_2. Hence, $\prec \cap \succ_S \subseteq \prec_S$ and so S^* is completely semisimple, by Theorem 4.10. □

We now establish a result that provides sufficient conditions for the HNN extension S^* to have finite $\mathscr{R}$-classes. For S^* to have finite $\mathscr{R}$-classes it is necessary for S to have finite $\mathscr{R}$-classes. Since the bicyclic inverse semigroup has infinite $\mathscr{R}$-classes, an inverse semigroup with finite $\mathscr{R}$-classes cannot contain a copy of the bicyclic inverse semigroup and so must be completely semisimple.

Definition 4.13. *The relation $\prec$ is reflexive and transitive on $E(U_1) \cup E(U_2)$. It follows that $\prec \cap \succ$ defines an equivalence on $E(U_1) \cup E(U_2)$. The $\prec \cap \succ$ equivalence classes are partially ordered by $[f] \leq [g]$ if and only if $f \prec g$, where $[f]$ and $[g]$ denote the $\prec \cap \succ$-classes of $f, g \in E(U_1)$, respectively. We say that $E(U_1) \cup E(U_2)$ is finite $\prec \cap \succ$-above if every strictly ascending chain of $\prec \cap \succ$-classes is finite.*

Lemma 4.14. *Let $S^* = [S; U_1, U_2; \phi]$ be any HNN extension where S is completely semisimple and $\prec \cap \succ_S \subseteq \prec_S$. If $f \prec \cap \succ g$, where $f, g \in E(U_1) \cup E(U_2)$, then f and g are related by the equivalence on $E(U_1) \cup E(U_2)$ generated by the $\mathscr{D}$-relation on S and the mapping ϕ.*

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Proof. For $f \prec \cap \succ g$, where $f, g \in E(U)$, we have:

(i) As $f \prec g$, there exists $f_1, g_1, f_2, g_2, \ldots f_n, g_n \in E(U_1) \cup E(U_2)$, $n \geq 1$, where $f = f_1$, $g = g_n$, at least one of $f_k = g_k$, $(f_k)\phi = g_k$ and $(f_k)\phi^{-1} = g_k$ holds, for $1 \leq k \leq n$, and $g_k \prec_S f_{k+1}$, for $1 \leq k \leq n-1$.
(ii) Since $f \succ g$, there exists $h_1, j_1, h_2, j_2, \ldots h_n, j_m \in E(U_1) \cup E(U_2)$, $m \geq 1$, where $g = h_1$, $f = j_m$, at least one of $h_k = j_k$, $(h_k)\phi = j_k$ and $(h_k)\phi^{-1} = j_k$ holds, for $1 \leq k \leq m$, and $j_k \prec_S h_{k+1}$, for $1 \leq k \leq m-1$.
(iii) Since $\prec \cap \succ_S \subseteq \prec_S$ and $g_k \prec_S f_{k+1} \prec g \prec f = f_1 \prec g_k$, for $1 \leq k \leq n-1$, we then have $g_k \succ_S f_{k+1}$, for $1 \leq k \leq n-1$,
(iv) As $\prec \cap \succ_S \subseteq \prec_S$ and $j_k \prec_S h_{k+1} \prec f \prec g = h_1 \prec j_k$, for $1 \leq k \leq m-1$, we then have $j_k \succ_S h_{k+1}$, for $1 \leq k \leq m-1$.
(v) Since S is completely semisimple. the relation $\prec_S \cap \succ_S$ is the $\mathscr{D}$-relation on S. Thus $g_k \mathscr{D} f_{k+1}$, for $1 \leq k \leq n-1$, and $j_k \mathscr{D} h_{k+1}$, for $1 \leq k \leq m-1$.

Hence f and g are related by the equivalence on $E(U_1) \cup E(U_2)$ generated by the $\mathscr{D}$-relation on S and the mapping ϕ. □

The fundamental group of a graph of groups whose underlying graph is a finite tree is obtained inductively by a process of repeating amalgamated free products of groups or HNN extensions of groups, one for each edge. The fundamental group is then referred to as a finite tree product of the vertex groups.

Lemma 4.15. *Let $S^* = [S; U_1, U_2; \phi]$ be a lower bounded HNN extension. If S^* has finite $\mathscr{R}$-classes, then every component of Y is a finite tree and the resulting tree product is not proper.*

Proof. Suppose S^* has finite $\mathscr{R}$-classes. Let $e \in E(S)$. Then, from Theorem 4.4, the fundamental group $\Pi(H_e(-), Y_e)$ is isomorphic to the $\mathscr{H}$-class of S^* containing e and so $\Pi(H_e(-), Y_e)$ finite. If Y_e is not a tree then $\Pi(H_e(-), Y_e)$ is necessarily infinite, since it contains a free group. Thus Y_e is a tree.

Suppose Y_e is an infinite tree. From Theorem 4.4, the graph Y_e is isomorphic to the graph of orbits $AUT(\Gamma')\backslash T(\Gamma')$, where Γ' denotes the t-subopuntoid subgraph of $\Gamma = S\Gamma(S^*, e)$ that consists of all $\langle S\rangle$-lobes of Γ that are hosts and $T(\Gamma')$ is the $\langle S\rangle$-lobe tree of Γ'. Thus, $AUT(\Gamma')\backslash T(\Gamma')$, and hence $T(\Gamma')$, has infinitely many vertices. This implies that Γ has infinitely many $\langle S\rangle$-lobes and we reach a contradiction, since Γ has as many vertices as the $\mathscr{R}$-class of S^* containing e. Hence, Y_e is a finite tree. Any proper amalgamated free product of groups is necessarily infinite. Since $\Pi(H_e(-), Y_e)$ is finite, it cannot be a proper tree product. □

In contrast with the situation for an HNN extension of a finite group, which is always infinite, an HNN extension of a finite inverse semigroup can have finite $\mathscr{R}$-classes.

Theorem 4.16. *Let $S^* = [S; U_1, U_2; \phi]$ be a lower bounded HNN extension. Suppose S is completely semisimple, with finite $\mathscr{R}$-classes, $\prec \cap \succ_S \subseteq \prec_S$ holds and $E(U_1) \cup E(U_2)$ is finite $\prec \cap \succ$-above.*

(i) If every component of Y is a finite tree and has a tree product that is not proper, then S^* has finite $\mathscr{R}$-classes.
(ii) If $e_1 = (e_1)\phi$ belongs to a trivial $\mathscr{D}$-class of S and every component of Y, except for Y_{e_1}, is a finite tree and has a tree product that is not proper, then each Schützenberger graph of $\langle S^*\rangle$ has finitely many $\langle S\rangle$-lobes that have more than one vertex.

Proof. Suppose every component of Y is a finite tree and has a tree product that is not proper. Every such tree product is isomorphic to a maximal subgroup of S and thus is finite, since S has finite $\mathscr{H}$-classes. To prove that S^* has finite $\mathscr{R}$-classes, we show that every Schützenberger graph of $\langle S^*\rangle$ has finitely many $\langle S\rangle$-lobes. We first show that every such graph has finitely many hosts.

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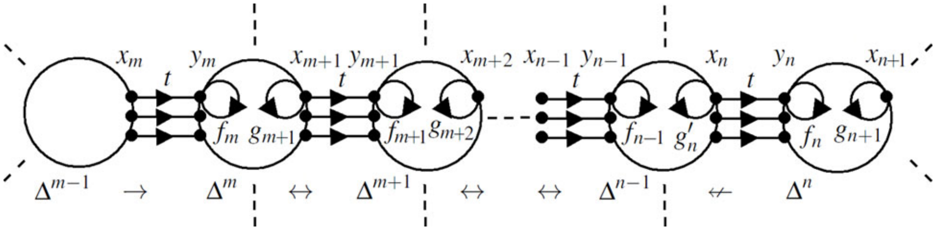

Figure 7. *The $\langle S\rangle$-lobes $\Delta^{m-1}, \Delta^m, \ldots, \Delta^n$ of Γ.*

Let Γ be a Schützenberger graph of $\langle S^*\rangle$. From Theorem 4.1, the graph Γ is a complete t-opuntoid graph that has a host, where the $\langle S\rangle$-lobes are isomorphic to Schützenberger graphs of $\langle S\rangle$. Let Γ' denote the t-subopuntoid subgraph of Γ that consists of the $\langle S\rangle$-lobes of every host of Γ and all t-edges connecting these hosts. If $n(\Gamma) > 1$ then, by Lemma 3.7, the graph Γ has precisely one host and so, since a host has finitely many $\langle S\rangle$-lobes, the subgraph Γ' has finitely many $\langle S\rangle$-lobes.

If $n(\Gamma) = 1$, then the graph of orbits $AUT(\Gamma)\backslash T(\Gamma')$ is isomorphic to some connected component of Y, from the proof of Theorem 4.4, and is thus finite, by assumption. The tree product of this connected component is not proper, by assumption, and so $AUT(\Gamma)$ is isomorphic to a maximal subgroup of S and is finite. Thus, the set of orbits of any vertex or edge of $T(\Gamma')$ is also finite. It now follows that $T(\Gamma')$ has finitely many vertices and so Γ' has finitely many $\langle S\rangle$-lobes. Hence Γ has finitely many hosts.

We now choose an $\langle S\rangle$-lobe Δ of Γ' and show that there is a bound on the length of any reduced $\langle S\rangle$-lobe path in Γ which starts in Δ. Let $\Delta = \Delta^1, \Delta^2, \ldots$ be a reduced $\langle S\rangle$-lobe path in Γ. Since Γ' has finitely many $\langle S\rangle$-lobes, there is a least positive integer $m > 1$ such that Δ^m is external to Γ'. Since either Γ' or Δ^{m-1} is a host of Γ, we have $\Delta^k \to \Delta^{k+1}$, for $k \geq m-1$. The situation is illustrated in Fig. 7.

Let $x_m \to^t y_m$ be a t-edge from a vertex x_m of Δ^{m-1} to a vertex y_m of Δ^m. The case when we have a t-edge $y_m \to^t x_m$ from a vertex y_m of Δ^m to a vertex x_m of Δ^{m-1} is similar. Since the $\langle S\rangle$-lobes are isomorphic to Schützenberger graphs of $\langle S\rangle$, by Theorem 4.1, and $\Delta^{m-1} \to \Delta^m$, we have $(y_m, \Delta^m, y_m) \cong \mathscr{A}(S, f_m)$, for some $f_m \in E(U_2)$.

Next, suppose we have $\Delta^k \leftrightarrow \Delta^{k+1}$, for all $k \geq m$. For each $k \geq m$, the reduced $\langle S\rangle$-lobe path $\Delta^m, \Delta^{m+1}, \ldots, \Delta^k$, including the t-edges connecting the $\langle S\rangle$-lobes, forms a t-opuntoid graph Σ_k, where each $\langle S\rangle$-lobe is a host of Σ_k. Since $\Delta^k \leftrightarrow \Delta^{k+1}$, for all $k \geq m$, the graph Σ_k can be obtained from Δ^m by repeated applications of Construction 3.3.

Since $(y_m, \Delta^m, y_m) \cong \mathscr{A}(S, f_m)$, we have $(y_m, \Sigma_k, y_m) \rightsquigarrow \mathscr{A}(S^*, f_m)$, by Lemma 3.4. Using Lemmas 3.4 and 3.8, it follows that (y_m, Σ_k, y_m) is embedded onto a t-subopuntoid subautomaton of $\mathscr{A}(S^*, f_m)$, where the image of each $\langle S\rangle$-lobe of Σ_k is also a host of $\mathscr{A}(S^*, f_m)$. Since $S\Gamma(S^*, f_m)$ has finitely many hosts, as proved above, the sequence of graphs Σ_k is bounded. Thus there exists a least positive integer $n > m$ such that $\Delta^{n-1} \not\leftarrow \Delta^n$.

Let $x_n \to^t y_n$ be a t-edge from a vertex x_n of Δ^{n-1} to a vertex y_n of Δ^n. The case when we have a t-edge $y_n \to^t x_n$ from a vertex y_n of Δ^n to a vertex y_n of Δ^{n-1} is similar. Since the $\langle S\rangle$-lobes are isomorphic to Schützenberger graphs of $\langle S\rangle$, by Theorem 4.1, we have $(y_n, \Delta^n, y_n) \cong \mathscr{A}(S, f_n)$, for some $f_n \in E(U_2)$. We show that $[f_m] < [f_n]$. where $[f_k]$ denotes the $\prec \cap \succ$-class of f_k, for $k = m, n$.

Without loss of generality, we assume that we also have a t-edge $x_k \to^t y_k$ from a vertex x_k of Δ^{k-1} to a vertex y_k of Δ^k and let $f_k \in E(U_2)$ such that $(y_k, \Delta^k, y_k) \cong \mathscr{A}(S, f_k)$, for $m+1 \leq k \leq n$. Put $g_k = (f_k)\phi^{-1}$, for $m \leq k \leq n$. Since $\Delta^k \leftrightarrow \Delta^{k+1}$, for $m \leq k \leq n-2$, we have $(x_{k+1}, \Delta^k, x_{k+1}) \cong \mathscr{A}(S, g_{k+1})$, for $m \leq k \leq n-2$. Since $\Delta^{n-1} \not\leftarrow \Delta^n$, we have $(x_n, \Delta^{n-1}, x_n) \cong \mathscr{A}(S, g'_n)$ such that $g'_n < g_n$ in S, for some $g'_n \in E(S)$. Now $f_k \mathscr{D} g_{k+1}$ in S, for $m \leq k \leq n-2$, and $f_{n-1} \mathscr{D} g'_n < g_n$ in S.

Thus, the idempotents $f_m, g_{m+1}, \ldots, f_{n-2}, g_{n-1}, f_{n-1}$ are all $\prec \cap \succ$-related and we also have $f_{n-1} \prec_S g_n$, and so $f_{n-1} \prec g_n$, by the definitions of $\prec$ and $\prec_S$. Suppose we have $f_{n-1} \succ g_n$. Then $f_{n-1} \succ_S g_n$, since it is assumed that $\prec \cap \succ_S \subseteq \prec_S$. As S is assumed completely semisimple, we then have $f_{n-1} \mathscr{D} g'_n = g_n$, a contradiction. Thus, we do not have $f_{n-1} \succ g_n$ and so $[f_m] = [f_{n-1}] < [g_n] = [f_n]$.

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Similarly, there is a least positive integer $q > n$ with $\Delta^{q-1} \not\leftarrow \Delta^q$. Continuing in this manner, we obtain a strictly ascending sequence $[f_m] < [f_n] < [f_q] \cdots$. Since S is finite $\prec \cap \succ$-above, the above sequence must be finite and terminates in $[e_1]$. Thus, there is a bound on the length of any reduced $\langle S \rangle$-lobe path in Γ starting in Δ. Since S has finite $\mathscr{R}$-classes, the number of $\langle S \rangle$-lobes in Γ that are adjacent to any given $\langle S \rangle$-lobe is also finite. It now follows that Γ has finitely many $\langle S \rangle$-lobes and part (i) is proved.

Assuming $e_1 = (e_1)\phi$ belongs to a trivial $\mathscr{D}$-class, the connected component Y_{e_1} of Y consists of one vertex and one loop, with all vertex and edge groups trivial. Every $\langle S \rangle$-lobe of the Schützenberger graph $S\Gamma(S^*, e_1)$ has precisely one vertex. Using a proof similar to that in part (i), any Schützenberger graph Γ of $\langle S^* \rangle$, other than $S\Gamma(S^*, e_1)$, has finitely many hosts. Then, since any $\langle S \rangle$-lobe that feeds off a trivial $\langle S \rangle$-lobe must also be trivial, the proof that Γ has finitely many non-trivial $\langle S \rangle$-lobes is also similar to that in (i). □

In Jajcayova [13], it was shown that an HNN extension of a free inverse semigroup S is lower bounded, and if U_1 and U_2 are finitely generated, then the HNN extension has decidable word problem.

Corollary 4.17. *Let $S^* = [S; U_1, U_2; \phi]$ be an HNN extension of a free inverse monoid and suppose the following hold, for all $f, g \in E(U_1)$:*

(i) We do not have $f \prec_S (g)\phi$ in S and so $E(U_1) \cap E(U_2) = \emptyset$.
(ii) $f \prec_S g$ implies $f \mathscr{R} u \mathscr{L} u_1^{-1} u_1 \leq g$, for some $u \in U_1$.
(iii) $(f)\phi \prec_S (g)\phi$ implies $(f)\phi \mathscr{R} (u)\phi \mathscr{L} (u^{-1}u)\phi \leq (g)\phi$, for some $u \in U_1$.

Then, S^* is completely semisimple, combinatorial, and with finite $\mathscr{R}$-classes. If $E(U_1) \cap E(U_2) = \{1\}$, the identity of S, and (i), (ii), and (iii) hold for $f, g \in E(U_1) \backslash \{1\}$, then S^* is completely semisimple, combinatorial, and there is a bound on the number of elements of S needed to express the elements as a product, within each $\mathscr{R}$-class of S^*.

Proof. We recall why the lower bounded properties hold. If $u \geq e$, where $u \in U_i$ and $e \in E(S)$, then we have $u \in E(U_i)$, for $i = 1, 2$, since S is free. For $e \in E(S)$, there are finitely many idempotents $f \in E(U_i)$ with $f \geq e$, for $i = 1, 2$. Then, for $e \in E(S)$, the set $\{u \in U_i : u \geq e\}$ is either empty or has a least element $f_i(e)$, for $i = 1, 2$. Thus, the first condition of a lower bounded HNN extension is satisfied.

Let $e \in E(S)$, $i \in \{1, 2\}$ and $\{u_k\}$ be a sequence of idempotents in $E(U_i)$ such that $u_k \geq f_i(eu_k) \geq u_{k+1}$, for all k. We have monomorphisms from $\mathscr{A}(S, e)$, $\mathscr{A}(S, u_k)$ and $\mathscr{A}(S, f(eu_k))$ into $\mathscr{A}(S, eu_k)$, for each k, which we regard as inclusions. Let $\Sigma_k = S\Gamma(S, e) \cap S\Gamma(S, f(eu_k))$. Suppose $\Sigma_k = \Sigma_{k+1}$, for some k. Now $u_k \geq f(ue_k) \geq u_k \geq f(ue_{k+1})$. Then if $w \in E(U_I)$ and $w \geq eu_{k+1}$ in S, then we must have $w \geq u_{k+1}$. Thus, we have $f(eu_{k+1}) = u_{k+1}$. Conversely, since $S\Gamma(S, e)$ is finite, we can have $\Sigma_k \subsetneq \Sigma_{k+1}$ at most a finite number of times. Thus, the second condition of a lower bounded HNN extension is satisfied. Hence, the HNN extension $S^* = [S; U_1, U_2; \phi]$ is lower bounded.

Since S is a free inverse semigroup, it is completely semisimple and has finite $\mathscr{R}$-classes. From Corollary 4.12, we have $\prec \cap \succ_S \subseteq \prec_S$ and S^* is completely semisimple. Further, the relation $\prec \cap \succ$ is the $\mathscr{D}$-relation in U_1 on $E(U_1)$ and the $\mathscr{D}$-relation in U_2 on $E(U_2)$. If $f, g \in E(U_1)$ and $f \mathscr{R} u \mathscr{L} u^{-1} u < g$, for some $u \in U_1$, then $[f] < [g]$, since S is completely semisimple. Since a free inverse monoid is finite $\mathscr{J}$-above, we then have that $E(U_1) \cup E(U_2)$ is finite $\prec \cap \succ$-above.

Conditions (i), (ii), and (iii) imply that every component Y_f of Y consists of two vertices, the $\mathscr{D}$-class of S containing f and the $\mathscr{D}$-class of S containing $(f)\phi$, and one edge, the $\mathscr{D}$-class of U_1 containing f, for $f \in E(U_1)$. Since a free inverse monoid is combinatorial, we now have that S^* has finite $\mathscr{R}$-classes, by Theorem 4.16 (i).

If $E(U_1) \cap E(U_2)$ consists of the identity of S, and (i), (ii), and (iii) hold for $f, g \in E(U_1) \backslash \{1\}$, then S^* is completely semisimple, combinatorial and has finite $\mathscr{R}$-classes, by the above. Since $e_1 = 1 = e_2$, each Schützenberger graph of $\langle S^* \rangle$ has finitely many $\langle S \rangle$-lobes that have more than one vertex, from

Theorem 4.16 (ii). Thus, if $r \in S^*$ then all the elements of the $\mathscr{R}$-class of S^* containing r can be expressed as a product involving fewer than N elements of S, for some $N \geq 1$. □

An inverse semigroup S is *residually finite* if for every finite non-empty subset $F \subseteq S$ there exists a homomorphism from S into some finite inverse semigroup T which separates the elements of F. Any inverse semigroup with finite $\mathscr{R}$-classes is residually finite, from [15, Lemma 5.3].

Corollary 4.18. *Let $S^* = [S; U_1, U_2; \phi]$ be an HNN extension, where S is finite, combinatorial, and conditions (i), (ii), (iii) of Corollary 4.17 hold. Then, S^* has finite $\mathscr{R}$-classes and so is residually finite.*

Proof. Let $i \in \{1, 2\}$. Since U_i is finite, if $e \in E(S)$ then there exists a least idempotent $f \in E(U_i)$ with $e \leq f$. If $u \in U_i$ with $u \geq e$. then $f \mathscr{R} fu \mathscr{L} u^{-1}fu$ in U_i and $u^{-1}fu \geq f$, since $u^{-1}fu \in E(U_i)$ and $u^{-1}fu \geq e$. As S is finite, and so completely semisimple, we must have $u^{-1}fu = f$. Then, fu belongs to the maximal subgroup of U_1 containing f, which is trivial. Hence $fu = f$ and it follows that the HNN extension $S^* = [S; U_1, U_2; \phi]$ is lower bounded.

From Corollary 4.12, we have $\prec \cap \succ_S \subseteq \prec_S$ and S^* is completely semisimple. As in the proof of Corollary 4.17, the relation $\prec \cap \succ$ is the $\mathscr{D}$-relation in U_1 on $E(U_1)$ and the $\mathscr{D}$-relation in U_2 on $E(U_2)$. If $f, g \in E(U_1)$ and $f \mathscr{R} u \mathscr{L} u^{-1}u < g$, for some $u \in U_1$, then $[f] < [g]$, since S is completely semisimple. Since S finite, we then have that $E(U_1) \cup E(U_2)$ is finite $\prec \cap \succ$-above.

Conditions (i), (ii), and (iii) imply that every component Y_f of Y consists of two vertices, the $\mathscr{D}$-class of S containing f and the $\mathscr{D}$-class of S containing $(f)\phi$, and one edge, the $\mathscr{D}$-class of U_1 containing f, for $f \in E(U_1)$. Since S is combinatorial, we now have that S^* has finite $\mathscr{R}$-classes, by Theorem 4.16 (i). □

An inverse semigroup S is E-unitary if $s \geq e$ implies $s \in E(S)$, for all $s \in S$ and $e \in E(S)$. From [19, Theorem 3.8], we have that S is E-unitary if and only if there exists a monomorphism from $\mathscr{A}(S, s_1)$ into $\mathscr{A}(S, s_2)$, whenever $s_1 \geq s_2$ in S. Equivalently, the inverse semigroup S is E-unitary if and only if homomorphisms between Schützenberger graphs are monomorphic. For S^* to be E-unitary, the homomorphisms between Schützenberger graphs of S^* must induce embeddings of the respective lobe trees.

Theorem 4.19. *Let $S^* = [S; U_1, U_2; \phi]$ be a lower bounded HNN extension where S is E-unitary, $su \in E(S)$ implies $ss^{-1} \sim_1 s$, and $s \cdot (u)\phi \in E(S)$ implies $ss^{-1} \sim_2 s$, for all $s \in S$ and $u \in U_1$. Then S^* is E-unitary.*

Proof. Let Γ and Γ' be Schützenberger graphs of S^* and let $\alpha : \Gamma \to \Gamma'$ be a homomorphism. Let β denote the homomorphism $T(\Gamma) \to T(\Gamma')$ between the lobe trees induced by α. We first show that β is an embedding.

Let $\Delta^1, \Delta^2, \ldots, \Delta^n$ be a reduced $\langle S \rangle$-lobe path in Γ where $(\Delta^1)\beta = (\Delta^n)\beta$. Then, $(\Delta^1)\beta, (\Delta^2)\beta, \ldots, (\Delta^n)\beta$ denote the vertices of a loop in $T(\Gamma')$. Since $T(\Gamma')$ is a tree, there exist some $k \leq n-1$ such that $(\Delta^{k-1})\beta = (\Delta^{k+1})\beta$. The t-edges between $(\Delta^{k-1})\beta$ and $(\Delta^k)\beta$ all start in one $\langle S \rangle$-lobe and end in the other. Without loss of generality, assume all these t-edges start in $(\Delta^{k-1})\beta$ and end in $(\Delta^k)\beta$. Then, there is a t-edge $x_1 \to^t y_1$ from a vertex x_1 of Δ^{k-1} to a vertex y_1 of Δ^k and a t-edge $x_2 \to^t y_2$ from a vertex x_2 of Δ^{k+1} to a vertex y_2 of Δ^k. The situation is illustrated in Fig. 8.

Let $s \in S$ such that $(y_1, \Delta^k, y_2) \cong \mathscr{A}(S, s)$. Then, we have t-edges $(x_1)\alpha \to^t (y_1)\alpha$ and $(x_2)\alpha \to^t (y_2)\alpha$ from $(\Delta^{k-1})\beta$ to $(\Delta^k)\beta$ in Γ'. By the t-saturation property of t-opuntoid graphs, there exists a path $(y_2)\alpha \to^{(u)\phi} (y_1)\alpha$ in $(\Delta^k)\beta$, for some $u \in U_1$. Let $r \in S$ such that $((y_1)\alpha, (\Delta^k)\beta, (y_2)\alpha) \cong \mathscr{A}(S, r)$. Then, we have $s \geq r$ and $(u^{-1})\phi \geq r$ in S. Thus $rr^{-1} \leq s \cdot (u)\phi$.

Since S is E-unitary we that $s \cdot (u)\phi$ is idempotent. By the conditions of the statement of the theorem, we then have $ss^{-1} \sim_2 s$. This implies that either $ss^{-1} = s$ or $ss^{-1} = sv$, for some $v \in U_2$. The first case implies $y_1 = y_2$ and so $\Delta^{k-1} = \Delta^{k+1}$, a contradiction since the original $\langle S \rangle$-lobe path was reduced. The second case implies that there is a path $y_1 \to^v y_2$ in Δ^k and so, by the t-saturation property, we must have $\Delta^{k-1} = \Delta^{k+1}$, again a contradiction.

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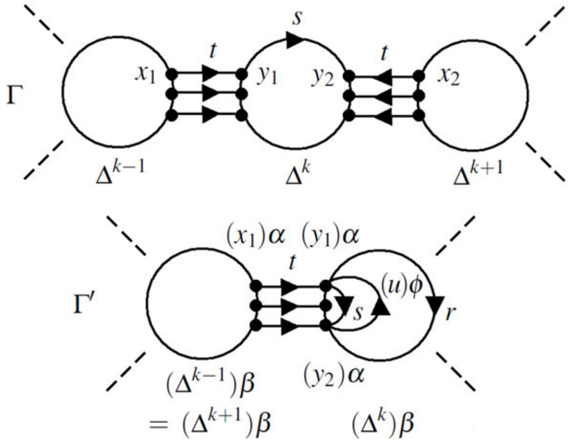

Figure 8. *The Schützenberger graphs* Γ *and* Γ'.

Hence, β must be one-one on the vertices of $T(\Gamma)$. Since $T(\Gamma)$ is a tree, this implies that β is an embedding. Since S is E-unitary, the homomorphisms between Schützenberger graphs of $\langle S\rangle$ are monomorphisms. Thus, each $\langle S\rangle$-lobe of Γ is embedded, under α, into some $\langle S\rangle$-lobe of Γ'. It now follows that α must be monomorphic. Hence, the HNN extension S^* is E-unitary. □

A subsemigroup U of an inverse semigroup S is a *unitary subsemigroup* if we have $us \in U$ implies $s \in U$, and $su \in U$ implies $s \in U$, for all $s \in S$ and $u \in U$. We note a few observations in the following corollary.

Corollary 4.20. *Let* $S^* = [S; U_1, U_2; \phi]$ *be an HNN extension where S is an E-unitary inverse semigroup. If S is a monoid and* U_1, U_2 *are subgroups of the groups of units of S then* S^* *is E-unitary. If* U_1 *and* U_2 *are semilattices satisfying the descending chain condition or are full unitary inverse subsemigroups of S then* S^* *is E-unitary.*

Proof. Suppose S is a monoid and U_1, U_2 are subgroups of the groups of units. If $u \geq e$, for some $u \in U_1$ and $e \in E(S)$, then $u = 1$, the identity of the monoid, since S is E-unitary. Similarly, if $(u)\phi \geq e$, for some $u \in U_1$ and $e \in E(S)$, then $(u)\phi = 1$. It follows that $S^* = [S; U_1, U_2; \phi]$ is a lower bounded HNN extension. Suppose $su \in E(S)$, for some $s \in S$ and $u \in U_1$. Then, $su = suu^{-1}s^{-1} = ss^{-1}$, as $uu^{-1} = 1$, and so $ss^{-1} \sim_1 s$. Similarly, if $s \cdot (u)\phi \in E(S)$, for some $s \in S$ and $u \in U_1$, then $ss^{-1} \sim_2 s$. Thus, S^* is E-unitary, from Theorem 4.19.

Suppose U_1 and U_2 are semilattices satisfying the descending chain condition. It is immediate that $S^* = [S; U_1, U_2; \phi]$ is a lower bounded HNN extension. If $su \in E(S)$, for some $s \in S$ and $u \in U_1$, then $u \in U_1 = E(U_1)$ implies $s \in E(S)$, since S is E-unitary. Then $ss^{-1} = s$ implies $ss^{-1} \sim_1 s$. Similarly, if $s \cdot (u)\phi \in E(S)$, for some $s \in S$ and $u \in U_1$, then $ss^{-1} \sim_2 s$. Thus S^* is E-unitary, from Theorem 4.19.

Suppose U_1 and U_2 are full unitary inverse subsemigroups of S. Since U_1 and U_2 are full in S, we have $E(U_1) = E(U_2) = E(S)$, and it is then immediate that $S^* = [S; U_1, U_2; \phi]$ is a lower bounded HNN extension. If $su \in E(S) = E(U_1)$, for some $s \in S$ and $u \in U_1$, then $s \in U_1$, since U_1 is a unitary subsemigroup. Then, $ss^{-1} = s \cdot (s^{-1})$, where $s^{-1} \in U_1$, and so $ss^{-1} \sim_1 s$. Similarly, if we have $s \cdot (u)\phi \in E(S)$, for some $s \in S$ and $u \in U_1$, then $ss^{-1} \sim_2 s$. Hence, S^* is E-unitary, from Theorem 4.19. □

An inverse semigroup S is 0-*E-unitary* if $s \geq e$ implies $s \in E(S)$, for all $s \in S\backslash\{0\}$ and $e \in E(S)\backslash\{0\}$. The inverse semigroup S is *strongly* 0-*E-unitary* if it admits an idempotent pure partial homomorphism to a group.

Corollary 4.21. *Let* $S^* = [S; U_1, U_2; \phi]$ *be a lower bounded HNN extension where S,* U_1, *and* U_2 *are* 0-*E-unitary, sharing a common* 0. *If* $su \in E(S)$ *implies* $ss^{-1} \sim_1 s$, *and* $s \cdot (u)\phi \in E(S)$ *implies* $ss^{-1} \sim_2 s$, *for all* $s \in S\backslash\{0\}$ *and* $u \in U_1\backslash\{0\}$, *then* S^* *is* 0-*E-unitary.*

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Proof. The proof is similar to that of Theorem 4.19. □

The polycyclic monoid P_n is the inverse monoid with zero that has the following presentation $\langle a_1, a_2, \dots, a_n, 0, 1 \mid a_i^{-1}a_i = 1, a_i^{-1}a_j = 0, i \neq j\rangle$, as an inverse monoid with zero. Non-zero elements can be written in the unique form xy^{-1}, where x, y are elements of A_n^*, the free monoid on $A_n = \{a_1, \dots, a_n\}$. Multiplication is then defined by:

$$xy^{-1} \cdot uv^{-1} = \begin{cases} xzv^{-1} & \text{if } u = yz, \text{ for some word } z \\ x(vz)^{-1} & \text{if } y = uz, \text{ for some word } z \\ 0 & \text{otherwise} \end{cases}$$

Idempotents are given by xx^{-1}, where $x \in A_n^*$, and $xy^{-1}\mathscr{R}xx^{-1}$. The monoid P_n is 0-E-unitary, 0-bisimple, and combinatorial. For $m \leq n$, we have a natural embedding of P_m into P_n, induced by the injection from A_m into A_n. Polycyclic inverse monoids are used to construct C^*-algebras [8]. Nearly all the inverse semigroup studied in C^*-algebra theory are strongly 0-E-unitary [17, Section 5].

Corollary 4.22. *Let $S^* = [S; U_1, U_2; \phi]$ be an HNN extension where $S = P_n$, $U_1 = P_m$, for $m \leq n$, are the polycyclic inverse monoids and ϕ is induced by any injection from A_m into A_b. Then, S^* is* 0-E-*unitary with group of units isomorphic to a free group on a singleton and all other maximal subgroups are trivial.*

Proof. Let $xx^{-1} \in E(S)$. Since S is 0-E-unitary, if $xx^{-1} \leq zy^{-1}$, where $zy^{-1} \in S$, then zy^{-1} is idempotent. If $xx^{-1} \leq yy^{-1}$ in S then $x = yz$, for some word z. Thus, there are finitely many idempotents yy^{-1} with $xx^{-1} \leq yy^{-1}$. Hence, the set $\{yy^{-1} \in U_i : yy^{-1} \geq xx^{-1}\}$ has a least element $f_1(xx^{-1})$, possibly 1, for $i = 1, 2$. We have:

$$xx^{-1} \cdot yy^{-1} = \begin{cases} xx^{-1} & \text{if } x = yz, \text{ for some word } z \\ yy^{-1} & \text{if } y = xz, \text{ for some word } z \\ 0 & \text{otherwise} \end{cases}$$

Let $yy^{-1} \in E(U_i)$ with $xx^{-1} \nleq yy^{-1}$. Then either $xx^{-1} \cdot yy^{-1} = 0$ or we have $xx^{-1} \cdot yy^{-1} = yy^{-1}$. Assume $xx^{-1} \geq yy^{-1}$ and so $f_i(xx^{-1} \cdot yy^{-1}) = yy^{-1}$. If $y_1y_1^{-1} \in E(U_i)$ with $yy^{-1} \geq y_1y_1^{-1}$ then $f_i(xx^{-1} \cdot y_1y_1^{-1}) = f_i(xx^{-1} \cdot yy^{-1} \cdot y_1y_1^{-1}) = yy^{-1} \cdot y_1y_1^{-1} = y_1y_1^{-1}$. It now follows that $S^* = [S; U_1, U_2; \phi]$ is a lower bounded HNN extension.

Let $xy^{-1} \in S$ and $uv^{-1} \in U_i$ such that $xy^{-1} \cdot uv^{-1}$ is idempotent, for $i \in \{1, 2\}$. Suppose $u = yz$, for some word z. Then, $y \in U_i$. Since $xy^{-1} \cdot uv^{-1} = (xz)v^{-1}$ is idempotent, we have $xz = v$ and so $x \in U_i$. Thus $xy^{-1} \in U_i$, $xx^{-1} \cdot xy^{-1} = xy^{-1}$ and so $xx^{-1} \sim_i xy^{-1}$. Suppose we have $y = uz$, for some word z. Then, since $xy^{-1} \cdot uv^{-1} = x(vz)^{-1}$ is idempotent, we have $x = vz$. Thus we have $xx^{-1} \cdot vu^{-1} = vz(vz)^{-1} \cdot vu^{-1} = vz(uz)^{-1} = xy^{-1}$ and so $xx^{-1} \sim_i xy^{-1}$. It follows that S^* is 0-E-unitary, by Corollary 4.21.

Since S is 0-bisimple, the component Y_1 of the graph of groups Y, as defined in Notation 4.3, consists of one vertex and one edge, where 1 is the identity of S. Since S is combinatorial, the fundamental group of the graph of groups $(H_1(-), Y_1)$ is isomorphic to the free group on a singleton. The maximal subgroup of S^* containing 1 is isomorphic to the fundamental group of the graph of groups $(H_1(-), Y_1)$, by Theorem 4.4. All other maximal subgroups of S^* are trivial, by Theorem 4.6. □

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