On *k*-folding map-germs and hidden symmetries of surfaces in the Euclidean 3-space

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We obtain a complete topological classification of *k*-folding map-germs on generic surfaces in $\mathbb{R}^3$, discover new robust features of surfaces and recover, in a unified way, many of the robust features that were obtained previously by considering the contact of a surface with lines, planes or spheres.

1. Introduction

The aim of this work is to study *k*-folding map-germs on complex surfaces in $\mathbb{C}^3$ and relate them to the extrinsic differential geometry of smooth surfaces in $\mathbb{R}^3$.

The standard Whitney fold of order *k* with respect to the plane $\pi_0 : y = 0$ in $\mathbb{C}^3$ is the map $\omega_k : \mathbb{C}^3 \to \mathbb{C}^3$, given by $\omega_k(x, y, z) = (x, y^k, z)$. The map ω_k 'folds' the space $\mathbb{C}^3$ along the plane π_0, gluing the points (x, y, z), $(x, \xi y, z), \ldots, (x, \xi^{k-1}y, z)$, where $\xi = e^{2\pi i/k}$ is a primitive *k*th-root of unity. The Whitney fold of order *k* with respect to any plane π, denoted by ω_k^π, is defined similarly in § 2.

Let *M* be a complex surface in $\mathbb{C}^3$. We call the restriction of ω_k^π to *M* the *k*-folding map on *M* with respect to π. As our study is local, given a point *p* on *M* and a plane π in $\mathbb{C}^3$ through *p*, we choose a coordinate system so that *M* is locally the graph of a function $z = f(x, y)$ and $\pi = \pi_0 : y = 0$ (see remark 2.3(4)). Then the germ at p_0 of the *k*-folding map is represented in standard form by the map-germ $F_k : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$, given by

$$F_k(x, y) = (x, y^k, f(x, y)). \tag{1.1}$$

For an analytic (resp. smooth) surface $M \subset \mathbb{R}^3$, the k-folding map at a point p on M is constructed by complexifying M (resp. a certain jet of a parametrization of M) at p. The singularities of a k-folding map-germ encode the local symmetries of M with respect to the (complex) reflection group of order k whose hyperplane arrangement consists of the single plane π.

The study of 2-folding map-germs on surfaces in $\mathbb{R}^3$ was carried out by Bruce and Wilkinson [**5**, **8**, **32**] (see [**6**, **12**–**15**, **32**] for more work on the subject), without resorting to complexification. The real map-germs in [**5**, **8**, **32**] are called folding map-germs and our 2-folding map-germs are their complexifications. Complexifying does not give extra information when $k = 2$. For $k \geqslant 3$, per contra, k-folding maps reveal a great deal of new geometric information. The local symmetries captured by these map-germs cannot be seen in the real case, which is why we call them *hidden symmetries* of $M \subset \mathbb{R}^3$. The loci of their singularities are visible on M and capture extrinsic geometric information of the surface.

Bruce and Wilkinson showed that folding maps capture the sub-parabolic and ridge curves, as well as umbilic points and other special points on these curves: these are robust features of the surface (i.e. they are special geometric features that can be traced on an evolving surface; see §6.2 for details). Passing to the complex setting, we show that the singularities of k-folding maps, $k \geqslant 2$, capture in a unified way, known robust features obtained by considering the contact of the surface with lines, planes or spheres (parabolic, sub-parabolic, ridge and flec-nodal curves, umbilic points, B_3, C_3 and S_3-points, A_2^*-points, cusps of Gauss (gulls-points) and butterfly-points). Our approach also reveals a new robust feature on surfaces: when k is divisible by 3, we obtain a new curve, which we call the H_3-curve. We also obtain new special points on previously known curves as well as on the H_3-curve. This motivates the following question: can the H_3-curve be obtained via the contact of the surface with some special geometric object? Further work is also required for understanding the link between local (hidden) symmetries of a surface and its contact with lines and planes.

The paper is organized as follows. In §2, we set notation and give some preliminaries. In §3, we obtain formulae for the invariants C, T, $\mu(\mathcal{D})$ and $r(\mathcal{D})$ of k-folding map-germs. These are respectively, the number of cross-caps, the number of triple points, the Milnor number and the number of branches of the double-point curve. These invariants determine the finite $\mathcal{A}$-determinacy and the topological class of a k-folding map-germ.

We show that there are no $\mathcal{A}$-simple k-folding map-germs for $k \geqslant 5$; also the modality of the map-germs is large for k large. This is why producing normal forms of finitely $\mathcal{A}$-determined germs, as is usually done in singularity theory, is not of great use here. Our classification is of strata in the jet space which are manifolds formed by unions of finitely $\mathcal{A}$-determined germs with the property that all germs in a given stratum are topologically equivalent.

We produce in §4 a stratification of the l-jet space of k-folding map-germs in standard form which is identified with the l-jet space of germs of functions $J^l(2,1)$. The stratification results are summarized as follows.

THEOREM 1.1. *For any integer $k \geqslant 2$, there is a stratification $\mathcal{S}_k$ of $J^{11}(2,1)$ such that, for any stratum S in $\mathcal{S}_k$ of codimension $\leqslant 4$, all k-folding map-germs in standard form with 11-jets in S are finitely $\mathcal{A}$-determined and are pairwise topologically equivalent.*

We relate in §6 the stratification of the jet space to the extrinsic differential geometry of surfaces in $\mathbb{R}^3$. After clarifying what it means for a surface to be generic, we deduce the following result about the topological classes of k-folding map-germs.

THEOREM 1.2. *Let $k \geqslant 3$ be an integer and let M be a generic smooth surface in $\mathbb{R}^3$ (or a complex surface in $\mathbb{C}^3$). Then, at any point p on M and for any plane π through p, the k-folding map-germ at p with respect to π is finitely $\mathcal{A}$-determined and is topologically equivalent to one of the following map-germs:*

$$
\begin{aligned}
\mathbf{M}_0^k \quad & (x,y) \mapsto (x, y^k, y), \\
\mathbf{M}_1^k \quad & (x,y) \mapsto (x, y^k, xy + y^2), \\
\mathbf{M}_l^k \quad & (x,y) \mapsto (x, y^k, y^2 + y^3 + x^l y), \ l = 2,3,4, \\
\mathbf{N}_l^k \quad & (x,y) \mapsto (x, y^k, y^2 + x^2 y + y^{2l-1}), \ l = 3,4, \\
\mathbf{O}_4^k \quad & (x,y) \mapsto (x, y^k, y^2 + x^3 y + xy^3), \\
\mathbf{P}_l^k \quad & (x,y) \mapsto (x, y^k, xy + y^3 + y^{3l-1}), \ l = 2,3,4, \\
\mathbf{Q}_3^k \quad & (x,y) \mapsto (x, y^k, xy + y^4 + y^5 + y^6), \\
\mathbf{Q}_4^k \quad & (x,y) \mapsto (x, y^k, xy + y^4 + y^6 + y^8), \\
\widetilde{\mathbf{Q}}_4^k \quad & (x,y) \mapsto (x, y^k, xy + y^4 + y^5 + y^7), \\
\mathbf{R}_4^k \quad & (x,y) \mapsto (x, y^k, xy + y^5 + y^6 + y^7), \\
\mathbf{U}_3^k \quad & (x,y) \mapsto (x, y^k, x^2 y + 2xy^2 + y^3 + y^4), \\
\mathbf{U}_4^k \quad & (x,y) \mapsto (x, y^k, x^2 y + 2xy^2 + y^3 + y^8), \\
\mathbf{V}_4^{k,j,j'} {\scriptstyle(\dagger)} \quad & (x,y) \mapsto (x, y^k, x^2 y + xy^2 + a_{j,j'} y^3 + b_{j,j'} x^3 y + y^4), \\
\mathbf{W}_4^{k,j} {\scriptstyle(\dagger)} \quad & (x,y) \mapsto (x, y^k, x^2 y + xy^2 + c_j y^3 + 4x^2 y^2 + y^4), \\
\mathbf{W}_4^{3q,q} {\scriptstyle(\dagger)} \quad & (x,y) \mapsto (x, y^k, x^2 y + y^3 + y^4 + y^5), \\
\mathbf{X}_4^k \quad & (x,y) \mapsto (x, y^k, xy^2 + y^3 + x^3 y + y^4), \\
\mathbf{Y}_4^k \quad & (x,y) \mapsto (x, y^k, -xy^2 + x^2 y + y^4 + y^5).
\end{aligned}
$$

(†) *The constants $a_{j,j'}, b_{j,j'}$ (resp. c_j) are as in proposition 4.23 (resp. 4.25) and the conditions on the indices are as in table 6.*

It is worth noting that 3-folding map-germs can have $\mathcal{A}$-simple singularities and their corresponding strata in $\mathcal{S}_3$ are $\mathcal{A}$-constant. For $k \geqslant 4$, none of the k-folding map-germs are $\mathcal{A}$-simple, except for immersions and for M_1^4, which is the C_3-singularity. The strata of $\mathcal{S}_k$ give rise to moduli of finitely determined map-germs with constant invariants $C, T, \mu(\mathcal{D})$ and $r(\mathcal{D})$.

The representatives $\mathbf{M}_0^k$ and $\mathbf{M}_1^k$ are quasi-homogeneous, and $\mathbf{M}_l^k$, with $2 \nmid k$, $\mathbf{Q}_3^k$, with $12 \nmid k$, $\mathbf{R}_4^k$, with $20 \nmid k$ and $\mathbf{U}_4^k$, with $3 \nmid k$, can be replaced by quasi-homogeneous representatives of the same topological class (see remarks 4.10). For these cases, the invariants can be calculated in terms of the weights and degrees [24] and the topological triviality of the strata in the jet space follows from equisingularity results in [9].
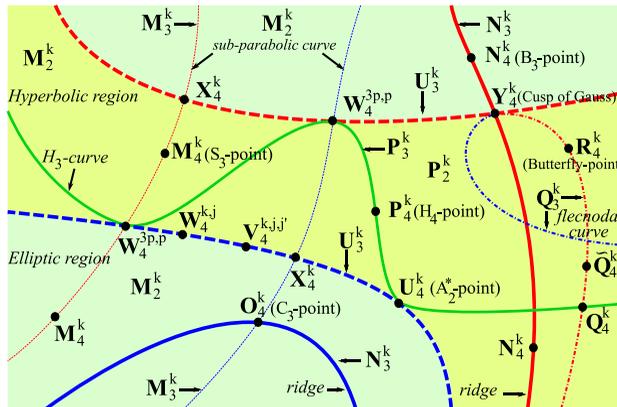
Figure 1. Robust features captured by k-folding map-germs away from umbilic points for $k \geqslant 4$, even and divisible by 3 (see theorem 6.5 and remarks 6.6 for the robust curves at umbilic points).

The robust features captured by k-folding map-germs on a generic surface are sketched in figure 1. An interesting finding is that, having studied symmetries of infinitely many orders (for any $k \geqslant 2$), we obtain a finite collection of robust features that occur along curves and a finite collection of special points on these curves if we discard the $\mathbf{V}^{k,j,j'}$ and $\mathbf{W}^{k,j}$-points.

2. Preliminaries

We introduce here k-folding map-germs and notation from singularity theory that are needed in the paper. We start with the singularity theory notation, see for example [25, 31] for more details.

2.1. Singularities of map-germs

We deal with germs $F\colon (M, p) \to (N, F(p))$ of holomorphic maps between complex manifolds. Taking coordinate charts, this is the same as dealing with map-germs $(\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$.

Let $\mathcal{O}_n$ be the local ring of germs of holomorphic functions $(\mathbb{C}^n, 0) \to \mathbb{C}$ and $\mathfrak{m}_n$ its maximal ideal (which is the subset of germs that vanish at the origin). Denote by $\mathcal{O}(n, p)$ the $\mathcal{O}_n$-module of holomorphic map-germs $(\mathbb{C}^n, 0) \to \mathbb{C}^p$, so $\mathcal{O}(n, p) = \bigoplus_p \mathcal{O}_n$.

Let $\mathcal{R}$ (resp. $\mathcal{L}$) be the group of bi-holomorphic germs $(\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ (resp. $(\mathbb{C}^p, 0) \to (\mathbb{C}^p, 0)$). The group $\mathcal{A} = \mathcal{R} \times \mathcal{L}$ of right-left equivalence acts on $\mathfrak{m}_n \cdot \mathcal{O}(n, p)$ by $(h_1, h_2) \cdot G = h_2 \circ G \circ h_1^{-1}$. Two germs H, G are said to be $\mathcal{A}$-equivalent, and write $H \sim_{\mathcal{A}} G$, if $H = (h_1, h_2) \cdot G$ for some $(h_1, h_2) \in \mathcal{A}$.

The l-jet space of map-germs in $\mathfrak{m}_n \cdot \mathcal{O}(n, p)$ is by definition

$$J^l(n, p) = \mathfrak{m}_n \cdot \mathcal{O}(n, p) / \mathfrak{m}_n^{l+1} \cdot \mathcal{O}(n, p).$$

Given a germ $G \in \mathfrak{m}_n \cdot \mathcal{O}(n,p)$, we identified its l-jet $j^l G$ with its Taylor polynomial of degree l at the origin. Let $\mathcal{A}_l$ be the subgroup of $\mathcal{A}$ whose elements have l-jets the germ of the identity. The group $\mathcal{A}_l$ is a normal subgroup of $\mathcal{A}$. Define $\mathcal{A}^{(l)} = \mathcal{A}/\mathcal{A}_l$. The elements of $\mathcal{A}^{(l)}$ are the l-jets of the elements of $\mathcal{A}$. The action of $\mathcal{A}$ on $\mathfrak{m}_n \cdot \mathcal{O}(n,p)$ induces an action of the jet group $\mathcal{A}^{(l)}$ on $J^l(n,p)$ as follows. For $j^l G \in J^l(n,p)$ and $j^l(h_1, h_2) \in \mathcal{A}^{(l)}$, $j^l(h_1, h_2) \cdot j^r G = j^l((h_1, h_2) \cdot G)$.

A germ G is said to be finitely $\mathcal{A}$-determined if there exist an integer l such that $G \sim_{\mathcal{A}} H$ for any H with $j^l H = j^l G$; $j^l G$ is then said to be a sufficient jet of G. The germ G is then said to be l-$\mathcal{A}$-determined. The least l satisfying this property is called the degree of determinacy of G.

There are classifications of finitely determined map-germs for various pairs (n,p). When $p = 1$, there is Arnold's extensive list of the $\mathcal{R}$-classification of germs of functions [1]. For $(n,p) = (2,2)$, classifications were carried out by several authors, the most extensive ones are given in [16, 29]. Here we need only the singularities of $\mathcal{A}_e$-codimension $\leqslant 2$, which we reproduce in table 9. For $(n,p) = (2,3)$, Mond [22] produced an extensive list of finitely $\mathcal{A}$-determined map-germs. We use in this paper the following singularities from [22]:

$$
\begin{array}{ll}
\text{Immersion} & (x, y, 0) \\
\text{Cross-cap} & (x, y^2, xy) \\
S_k & (x, y^2, y^3 + x^{k+1}y), k \geqslant 1 \\
B_k & (x, y^2, x^2 y + y^{2k+1}), k \geqslant 2 \\
C_3 & (x, y^2, xy^3 + x^3 y) \\
H_k & (x, xy + y^{3k-1}, y^3), k \geqslant 2 \\
X_4 & (x, y^3, x^2 y + xy^2 + y^4)
\end{array}
$$

The notion of a *simple germ* is defined in [1] as follows. Let X be a manifold and $\mathcal{G}$ a Lie group acting on X. The modality of a point $g \in X$ under the action of $\mathcal{G}$ on X is the least number m such that a sufficiently small neighbourhood of g may be covered by a finite number of m-parameter families of orbits. The point g is said to be *simple* if its modality is 0, that is, a sufficiently small neighbourhood intersects only a finite number of orbits. The modality of a finitely $\mathcal{A}$-determined map-germ is the modality of a sufficient jet in the jet-space under the action of the jet-group.

We also need the notion of topological equivalence. We say that two germs H, $G \in \mathfrak{m}_n \cdot \mathcal{O}(n,p)$ are topologically equivalent if $H = h_2 \circ G \circ h_1^{-1}$ for some germs of homeomorphisms h_1 and h_2 of, respectively, the source and target.

2.2. Reflections and k-folding maps

In all this paper, we fix the inner product $\langle a, b \rangle = \sum_i a_i \overline{b_i}$ in $\mathbb{C}^3$.

Let π be an element of the affine Grassmannian $\mathrm{Graff}(2,3)$ of planes in $\mathbb{C}^3$. A plane π has equation $\langle q, v \rangle = d$, where v is a fixed non-zero vector orthogonal to π and d is a fixed scalar. However, any non-zero scalar multiple of (d, v) gives an equation of π, so π is identified with the class $\overline{(d, v)} \in \mathbb{C}P^3$ of $(d, v) \in \mathbb{C}^4$.

Let $\pi : \langle q, v \rangle = d$ be a plane in $\mathbb{C}^3$. The orthogonal projection of a point $p \in \mathbb{C}^3$ to π along the vector v is the point $q = p + \lambda v \in \pi$ with $\lambda = (d - \langle p, v \rangle)/\langle v, v \rangle$.

Consider the map $\omega_k^\pi : \mathbb{C}^3 \to \mathbb{C}^3$ given by

$$\omega_k^\pi(p) = q + \lambda^k v = p + \lambda v + \lambda^k v.$$

If we take $(d', v') = (\alpha d, \alpha v)$, $\alpha \in \mathbb{C} \setminus 0$, as another representative of $\pi = \overline{(d, v)}$, then

$$p + \lambda' v' + \lambda'^k v' = p + \frac{\alpha(d - \langle p, v \rangle))}{\alpha^2 \langle v, v \rangle} \alpha v + \frac{\alpha^k (d - \langle p, v \rangle)^k)}{\alpha^{2k} \langle v, v \rangle} \alpha v$$

$$= p + \lambda v + \lambda^k \alpha^{1-k} v$$

$$= q + \lambda^k \alpha^{1-k} v.$$

Clearly, the map ω_k^π depends on the points on the line $(\alpha d, \alpha v) \in \mathbb{C}^4$ and not merely on the class of the line $\overline{(d, v)} \in \mathbb{C}P^3$. However, all these maps are $\mathcal{L}$-equivalent: the bi-holomorphic map $q - \lambda v \mapsto q - \lambda \alpha^{\frac{1-k}{k}} v$ composed (on the left) with the map ω_k^π with π represented by (d, v) gives the map ω_k^π with π represented by $(\alpha d, \alpha v)$. Therefore, the $\mathcal{L}$-class of ω_k^π depends only on π.

DEFINITION 2.1. *The Whitney fold of order k (k-fold for short) with respect to a plane $\pi \in \mathrm{Graff}(2, 3)$ is the $\mathcal{L}$-class of the map ω_π^k. We still denote by ω_k^π any representative of ω_k^π obtained by choosing a representative (p, v) of $\pi = \overline{(d, v)} \in \mathbb{C}P^3$.*

A k-fold may be viewed as generalization of the Whitney fold $(x, y^2, z) \mapsto (x, y^2, z)$. While the Whitney fold folds the space along the plane $\{y = 0\}$ and identifies the points (x, y, z) and $(x, -y, z)$, the Whitney fold of order k with respect to a plane π represented by (d, v) is a generically a k-to-1 branched cover, ramified along π, and identifies k-tuples of points $q - \lambda v, q - \xi \lambda v, \ldots, q - \xi^{k-1} \lambda v$, where $\xi = e^{2\pi i/k}$ is a primitive kth-root of unity and $q \in \pi$.

The map ω_k^π can also be viewed as the quotient map associated to the action of the cyclic group $\mathbb{Z}/k\mathbb{Z}$, regarded as a complex reflection group whose hyperplane arrangement consist of the single plane π. We regard $\mathbb{Z}/k\mathbb{Z}$ as the group generated by the order k complex reflection $q - \lambda v \mapsto q - \xi \lambda v$. Observe that, even though the plane π does not determine ω_k^π uniquely (it depends on the choice of a representative of $\pi = \overline{(d, v)} \in \mathbb{C}P^3$), the action of $\mathbb{Z}/k\mathbb{Z}$ on $\mathbb{C}^3$ is determined uniquely by π.

Given any subset $X \subseteq \mathbb{C}^3$, $\omega_k^\pi(X)$ encodes the order k reflectional symmetries of X with respect to π. See [17] for a recent work on singular maps related to reflection groups.

DEFINITION 2.2. *Let $M \subset \mathbb{C}^3$ be a complex surface, p a point on M and $k \geqslant 2$ an integer. Given $\pi \in \mathrm{Graff}(2, 3)$, the k-folding map-germ on M at p with respect to π is the $\mathcal{A}$-class of the restriction of ω_k^π to M at p. We denote any representative of the class by $F_k^\pi : (M, p) \to (\mathbb{C}^3, \omega_k^\pi(p))$.*

REMARKS 2.3.

1. All the map-germs F_k^π with π represented by $(\alpha d, \alpha v)$, $\alpha \in \mathbb{C}$, are $\mathcal{A}$-equivalent as the maps ω_k^π are $\mathcal{L}$-equivalent. Thus, the $\mathcal{A}$-class of F_k^π depends only on π and not on the choice of a representative (p, v) of $\pi = \overline{(d, v)} \in \mathbb{C}P^3$. In all the paper, we work with a representative of the $\mathcal{A}$-class of F_k^π.

2. If $p \notin \pi$, then F_k^π is the germ of an immersion. Thus, to obtain any meaningful local geometric information about the surface M we should take the plane π passing through the point $p \in M$.

3. The image of F_k^π is the image by ω_k^π of the germ (M, p), so for $p \in \pi \cap M$, F_k^π captures order k local symmetries of M with respect to π. The aim of this paper is to understand how these local symmetries are captured by the $\mathcal{A}$-singularities of F_k^π.

4. Let $p_0 \in \pi \cap M$ and (d, v) a representative of π. If $v \notin T_{p_0} M$, then F_k^π is a germ of an immersion and is $\mathcal{A}$-equivalent to $(x, y) \mapsto (x, y^k, y)$. Suppose that $v \in T_{p_0} M$. We choose a coordinates system in $\mathbb{C}^3$ so that p_0 is the origin, the z-axis is along a normal vector to M at p_0, the y-axis along v and the x-axis orthogonal to the previous two axes. Then we can take M locally at p_0 as the graph $z = f(x, y)$ of some holomorphic map f in a neighbourhood U of the origin. In this coordinate system, we have $\pi = \pi_0 : y = 0$. Consequently, the k-folding map-germ on M at p_0 is the germ $F_k = F_k^{\pi_0} : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$, given in standard form $F_k(x, y) = (x, y^k, f(x, y))$. In view of this, we shall always take a given k-folding map-germ in standard form (1.1).

5. Definition 2.2 is adapted as follows for the real case. When M is an analytic surface in $\mathbb{R}^3$, denote by $M_{\mathbb{C}, p}$ its local complexification at p and by $\pi_\mathbb{C}$ the complexification of π. The k-folding map-germ on M at p with respect to π is then defined as the $\mathcal{A}$-class of the restriction of $\omega_k^{\pi_\mathbb{C}}$ to $M_{\mathbb{C}, p}$ at p. When M is a smooth surface, we consider the k-folding map-germ of a given jet of (a parametrization of) M at p.

3. Topological invariants

We recall the definitions of some key $\mathcal{A}$-invariants of map-germs $(\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$. These are the Milnor number of the double-point curve $\mu(\mathcal{D})$, the number of cross-caps C and the number of triple points T. We give formulae for computing these invariants for k-folding map-germs, and use the invariants to study the finite $\mathcal{A}$-determinacy and topological equivalence of these germs.

3.1. The double-point curve

We start by recalling the definition of the double- and triple-point spaces of a corank one map-germ $F : (\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0)$ from [19]. Any such germ can be written in a suitable coordinate system in the form $F(x, y) = (x, f_n(x, y), f_{n+1}(x, y))$, with $x = (x_1, \ldots, x_{n-1}) \in (\mathbb{C}^{n-1}, 0)$ and $y \in (\mathbb{C}, 0)$.

Given $h \in \mathcal{O}_n$, the iterated divided differences of h are defined as

$$h[x, y, y'] = \frac{h(x, y') - h(x, y)}{y' - y} \in \mathcal{O}_{n+1},$$

$$h[x, y, y', y''] = \frac{h[x, y, y''] - h[x, y, y']}{y'' - y'} \in \mathcal{O}_{n+2}.$$

The multiple point ideals of a map-germ F as above are defined as

$$I^2(F) = \langle f_n[x, y, y'], f_{n+1}[x, y, y'] \rangle \subseteq \mathcal{O}_{n+1},$$

$$I^3(F) = \langle f_n[x, y, y'], f_n[x, y, y', y''], f_{n+1}[x, y, y'], f_{n+1}[x, y, y', y''] \rangle \subseteq \mathcal{O}_{n+2}.$$

The double- and triple-point spaces of F are, respectively,

$$\mathcal{D}^2(F) = V(I^2(F)),$$
$$\mathcal{D}^3(F) = V(I^3(F)).$$

By counting variables and generators, it follows that $\mathcal{D}^2(F)$ (resp. $\mathcal{D}^3(F)$) is a complete intersection whenever it has dimension $n-1$ (resp. $n-2$).

The double-point space $\mathcal{D}^2(F)$, as a subset of $(\mathbb{C}^{n-1} \times \mathbb{C} \times \mathbb{C}, 0)$, consists of points (x, y, y') such that either $y \neq y'$ and $F(x, y) = F(x, y')$ or $y' = y$ and F is singular at (x, y).

To define the source double-point space, we assume that F is finite. Then, the projection $\pi \colon \mathcal{D}^2(F) \to \mathbb{C}^{n-1} \times \mathbb{C}$ given by $(x, y, y') \mapsto (x, y)$ is also finite. As a consequence, the image of π can be given a complex structure as the 0-th Fitting ideal $\mathcal{F}_0(\pi_* \mathcal{O}_{\mathcal{D}^2(F)})$ of the push forward module $\pi_* \mathcal{O}_{\mathcal{D}^2(F)}$ (see [26] for details). The source double-point space $\mathcal{D}(F)$ is defined as the projection $\pi(\mathcal{D}^2(F))$ endowed with this complex space structure, that is,

$$\mathcal{D}(F) = V(\mathcal{F}_0(\pi_* \mathcal{O}_{\mathcal{D}^2(F)})) \subseteq \mathbb{C}^n.$$

To compute the source double-point space of a k-folding map-germ, we need the following result.

LEMMA 3.1. *Let Z be a germ of an n-dimensional Cohen–Macaulay space and let $h_1, \ldots, h_r$ in $\mathcal{O}_Z$ be regular elements. Write $X_j = V(h_j)$ and $X = V(h_1 \ldots h_r)$. Let $\phi \colon Z \to (\mathbb{C}^n, 0)$ be a germ of a morphism of complex spaces such that the restrictions $\phi|_{X_j} \colon X_j \to (\mathbb{C}^n, 0)$ are finite. Then $\mathcal{F}_0((\phi|_X)_* \mathcal{O}_X) = \prod_{j=1}^r \mathcal{F}_0((\phi|_{X_j})_* \mathcal{O}_{X_j})$.*

Proof. It is enough to prove the statement for the case $r = 2$. We can assume that X_1 and X_2 have no common irreducible component as topological spaces. Indeed, consider the two subspaces $\mathcal{X}_1 = V(h_1 - t)$ and $\mathcal{X}_2 = V(h_2)$ of $Z \times (\mathbb{C}, 0)$ and the map $\phi \times \mathrm{Id} \colon Z \times (\mathbb{C}, 0) \to (\mathbb{C}^{n+1}, 0)$. The spaces $\mathcal{X}_1$ and $\mathcal{X}_2$ have no common irreducible component. Moreover, if the statement holds for $\phi \times \mathrm{Id}$, then it holds for ϕ. This is a consequence of the fact that Fitting ideals commute with base change (see lemma 1.2 in [26]). Now consider the disjoint union $X_1 \sqcup X_2 \subseteq Z \sqcup Z$ and the commutative diagram

$$
\begin{array}{ccc}
X_1 \sqcup X_2 & \xrightarrow{\ \ \alpha\ \ } & X \\
& \searrow{\psi} \quad \swarrow{\phi|_X} & \\
& (\mathbb{C}^n, 0) &
\end{array}
$$

The map α is generically a local isomorphism (i.e. a local isomorphism on a Zariski open and dense subset) because X_1 and X_2 are assumed to have no common component. Moreover, both X and $X_1 \sqcup X_2$ are Cohen–Macaulay spaces, which implies that the ideals $\mathcal{F}_0(\psi_* \mathcal{O}_{X_1 \sqcup X_2})$ and $\mathcal{F}_0((\phi|_X)_* \mathcal{O}_X)$ are principal. Since α is generically a local isomorphism, they are necessarily equal. The statement then follows from the equalities

$$
\begin{aligned}
\mathcal{F}_0(\psi_* \mathcal{O}_{X_1 \sqcup X_2}) &= \mathcal{F}_0((\phi|_{X_1})_* \mathcal{O}_{X_1} \oplus (\phi|_{X_2})_* \mathcal{O}_{X_2}) \\
&= \mathcal{F}_0((\phi|_{X_1})_* \mathcal{O}_{X_1}) \cdot \mathcal{F}_0((\phi|_{X_2})_* \mathcal{O}_{X_2}). \qquad \square
\end{aligned}
$$

THEOREM 3.2. *For a k-folding map-germ $F_k(x, y) = (x, y^k, f(x, y))$, the double-point space $\mathcal{D}$ is the zero locus $V(\lambda)$, where $\lambda = \prod_{j=1}^{k-1} \lambda_j$ and*

$$\lambda_j = \frac{f(x, y) - f(x, \xi^j y)}{(1 - \xi^j)y},$$

for $1 \leqslant j \leqslant k - 1$. We have thus a decomposition $\mathcal{D} = \bigcup_{j=1}^{k-1} \mathcal{D}_j$, with $\mathcal{D}_j = V(\lambda_j)$.

Proof. The double-point space $\mathcal{D}^2(F_k)$ is the intersection of the zero loci of the divided differences

$$\frac{((y')^k - y^k)}{y' - y} \quad \text{and} \quad \frac{f(x, y) - f(x, y')}{y - y'}.$$

Since $((y')^k - y^k)/(y' - y) = \prod_{j=1}^{k-1}(y' - \xi^j y)$, we conclude that, as a set, the space $\mathcal{D}^2(F_k)$ is the union of the spaces

$$\mathcal{D}_j^2 = \left\{ (x, y, \xi^j y) \in (\mathbb{C}^{n-1} \times \mathbb{C} \times \mathbb{C}, 0) \mid \frac{f(x, y) - f(x, \xi^j y)}{(1 - \xi^j)y} = 0 \right\},$$

for $j = 1, \ldots, k - 1$. Each of the sets $\mathcal{D}_j^2$ projects to $V(\lambda_j)$, which shows that $\mathcal{D}(F_k) = V(\lambda)$ as sets.

To show the equality as complex spaces, observe that the possible dimension of $\mathcal{D}^2(F_k)$ is one or two. If $\mathcal{D}^2(F_k)$ has dimension two, then some branch $\mathcal{D}_j^2$ has dimension two. Therefore, the corresponding function λ_j is identically zero, which in turn implies $\lambda = 0$. Since F_k is finite by construction, the projection $\mathcal{D}^2(F_k)$ is finite, hence the image of $\mathcal{D}_j^2$ is a germ of a 2-dimensional analytic closed subset of $(\mathbb{C}^2, 0)$, so is equal to $(\mathbb{C}^2, 0)$. This implies that $\mathcal{D}(F_k) = (\mathbb{C}^2, 0) = V(0)$.

Suppose now that $\mathcal{D}^2(F_k)$ has dimension one. This implies that the functions $\prod_{j=1}^{k-1}(y' - \xi^j y)$ and $f[x, y, y']$ form a regular sequence. Applying lemma 3.1 with $Z = V(f[x, y, y'])$ gives $\mathcal{D}(F_k) = V(\prod_{j=1}^{k-1} \mathcal{F}_0((\pi|_{\mathcal{D}_j^2})_* \mathcal{O}_{\mathcal{D}_j^2}))$, where the $\mathcal{D}_j^2$, $j = 1, \ldots, k - 1$, are given the natural complex space structure. Each of the morphisms $\pi|_{\mathcal{D}_j^2} \colon \mathcal{D}_j^2 \to \mathbb{C}^2$ consists of forgetting the third coordinate of the tuple $(x, y, \xi^j y)$, and this implies $\mathcal{F}_0((\pi|_{\mathcal{D}_j^2})_* \mathcal{O}_{\mathcal{D}_j^2}) = \langle \lambda_j \rangle$. □

Now we introduce some results we use to check finite $\mathcal{A}$-determinacy of a k-folding map-germ F_k and topological triviality in families of such germs. The first of these results was proven in [**19**] for corank one map-germs, then extended to arbitrary corank in [**20**].

THEOREM 3.3. *A finite map-germ $F \colon (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ is finitely $\mathcal{A}$-determined if and only if its double-point curve $\mathcal{D}$ is reduced.*

The decomposition $\mathcal{D} = \bigcup_{j=1}^{k-1} \mathcal{D}_j$ in theorem 3.2 can be used to compute $\mu(\mathcal{D})$, making it easier to apply theorem 3.3 (and theorem 3.10 below). We denote by

$\mathcal{D}_j \cdot \mathcal{D}_{j'}$ the intersection multiplicity of two distinct branches of the double-point curve. Clearly, $\mathcal{D}_j \cdot \mathcal{D}_{j'} = \mathcal{D}_{j'} \cdot \mathcal{D}_j$, and

$$\mathcal{D}_j \cdot \mathcal{D}_{j'} = \dim_{\mathbb{C}} \frac{\mathcal{O}_2}{\langle \lambda_j, \lambda_{j'} \rangle}.$$

PROPOSITION 3.4. *A k-folding map-germ F_k is finitely $\mathcal{A}$-determined if and only if the Milnor numbers $\mu(\mathcal{D}_j)$, $j = 1, \ldots, k-1$, and the intersection multiplicities $\mathcal{D}_j \cdot \mathcal{D}_{j'}$ of all pairs $\mathcal{D}_j$ and $\mathcal{D}'_j$, with $j' \neq j$, are finite. In that case,*

$$\mu(\mathcal{D}(F_k)) = \sum_{j=1}^{k-1} \mu(\mathcal{D}_j) + 2 \sum_{\substack{j,j'=1 \\ j<j'}}^{k-1} \mathcal{D}_j \cdot \mathcal{D}_{j'} - k + 2.$$

Proof. By theorem 3.3, F_k is finitely $\mathcal{A}$-determined if and only if $\mu(\mathcal{D}(F_k))$ is finite, equivalently, $\mathcal{D}(F_k)$ has an isolated singularity. This occurs if and only if every branch $\mathcal{D}_j$ has an isolated singularity and no pair of branches $\mathcal{D}_j$ and $\mathcal{D}_{j'}$, with $j \neq j'$, have a common component. Using the formula $\mu = 2\delta - r + 1$ for plane curves (see [21]) and the property $\delta(X \cup Y) = \delta(X) + \delta(Y) + X \cdot Y$, we get

$$
\begin{aligned}
\mu(\mathcal{D}(F_k)) &= 2\delta(\mathcal{D}(F_k)) - r(\mathcal{D}(F_k)) + 1 \\
&= \sum_{j=1}^{k-1} (2\delta(\mathcal{D}_j) - r(\mathcal{D}_j) + 1) - k + 2 + 2 \sum_{j,j'=1, j<j'}^{k-1} \mathcal{D}_j \cdot \mathcal{D}_{j'} \\
&= \sum_{j=1}^{k-1} \mu(\mathcal{D}_j) + 2 \sum_{j,j'=1, j<j'}^{k-1} \mathcal{D}_j \cdot \mathcal{D}_{j'} - k + 2. \qquad \square
\end{aligned}
$$

REMARKS 3.5. 1. Suppose that $\mathcal{D}_j$ is a germ of a regular curve parametrized by a regular map-germ $\alpha \colon (\mathbb{C}, 0) \to (\mathbb{C}^2, 0)$. Then, $\mathcal{D}_j \cdot \mathcal{D}_{j'} = \mathrm{ord}(h_{j'} \circ \alpha)$, which is the degree of the first non-zero term in the Taylor expansion of $\lambda_{j'}(\alpha(t))$.

2. If both $\mathcal{D}_j$ and $\mathcal{D}_{j'}$ are regular curves, we refer to $\mathcal{D}_j \cdot \mathcal{D}_{j'}$ as the *order of contact between $\mathcal{D}_j$ and $\mathcal{D}_{j'}$*. We have $\mathcal{D}_j \cdot \mathcal{D}_{j'} = 1$ if and only if the two curves intersect transversally. Suppose they are tangential and parametrized, respectively, by $t \mapsto (t, \gamma_j(t))$ and $t \mapsto (t, \gamma_{j'}(t))$. Then $\mathcal{D}_j \cdot \mathcal{D}_{j'} = \mathrm{ord}(\gamma_j - \gamma_{j'})$.

3. Let F_k be a finitely $\mathcal{A}$-determined k-folding map-germ. Then any pair of branches $\mathcal{D}_j$ and $\mathcal{D}'_j$, with $j \neq j'$, cannot have any common irreducible component, otherwise $\mathcal{D}(F_k)$ would fail to be reduced. Hence, we have $r(\mathcal{D}) = \sum_{j=1}^{k-1} r(\mathcal{D}_j)$.

The defining functions λ_j of the branches $\mathcal{D}_j$ of the double-point curve play a major role in our study of finite $\mathcal{A}$-determinacy and topological equivalence of k-folding map-germs. We take $F_k(x, y) = (x, y^k, f(x, y))$ and write, for any given integer $p \geqslant 1$,

$$j^p f(x, y) = \sum_{q=1}^{p} \sum_{s=0}^{q} a_{qs} x^{q-s} y^s. \qquad (3.1)$$

Then,

$$j^{p-1}\lambda_j = \sum_{q=1}^{p}\sum_{s=1}^{q}\vartheta_{sj}a_{qs}x^{q-s}y^{s-1}, \tag{3.2}$$

with

$$\vartheta_{sj} = \frac{1-\xi^{sj}}{1-\xi^j} = 1 + \xi^j + \cdots + \xi^{(s-1)j}.$$

The constants ϑ_{sj} play a significant role in determining the singularity type of the germs λ_j and in computing $\mathcal{D}_j \cdot \mathcal{D}_{j'}$. The following properties are needed in § 4.

LEMMA 3.6. *The numbers ϑ_{sj} satisfy the following properties:*

(1) $\vartheta_{sj} = 0$ *if and only if* $k \mid sj$.

(2) $\vartheta_{sj} = 1$ *if and only if* $k \mid (s-1)j$.

(3) *If* $\vartheta_{sj} = \vartheta_{sj'}$, *then* ϑ_{sj} *is either* 0 *or* 1.

Proof. We observe that $\vartheta_{0j} = 0$, $\vartheta_{1j} = 1$ and, for any integers m, n, we have $\vartheta_{mj} = \vartheta_{(m+n)j}$ if and only if $k \mid nj$. For (1) we take $m = 0$ and $n = s$, and for (2) we take $m = 1$ and $n = s - 1$.

For (3), we observe that the constants ϑ_{sj} lie in the images of the curves $\gamma_n : S^1 \subset \mathbb{C} \to \mathbb{C}$ given by $\gamma_n(z) = (1 - z^{n+1})/(1 - z) = 1 + z + \cdots + z^n$. We show that the self intersection points of the curves γ_n are 0 and 1 (for $n \geqslant 3$).

Write $z = e^{i\theta}$, with $\theta \in [0, 2\pi)$. Then $\gamma_n(\theta) = x + iy$, with $(x, y) \in \mathbb{R}^2$, gives $1 - e^{i(n+1)\theta} = (x + iy)(1 - e^{i\theta})$. Therefore,

$$\cos((n+1)\theta) = 1 - x + x\cos(\theta) - y\sin(\theta),$$
$$\sin((n+1)\theta) = -y + y\cos(\theta) + x\sin(\theta).$$

Now the identity $\cos((n+1)\theta)^2 + \sin((n+1)\theta)^2 = 1$ gives

$$(1 - \cos(\theta))(x^2 + y^2 - x) - y\sin(\theta) = 0. \tag{3.3}$$

Suppose that $y = 0$. Then equation (3.3) becomes $(1 - \cos(\theta))x(x - 1) = 0$, so $x = 0$ or 1 or $\theta = 0$.

When $x = 0$, we have $1 - e^{i\theta} \neq 0$, so $1 - e^{i(n+1)\theta} = 0$. That gives $\theta = 2\pi j/(n + 1), j = 1, \ldots, n$. Therefore, γ_n passes n-times through the origin.

When $x = 1$, we get $e^{in\theta} = 1$, so $\theta = 2\pi j/n, j = 1, \ldots, n - 1$. Therefore, γ_n passes $(n - 1)$-times through the point 1.

When $\theta = 0$, we have $\gamma_n(0) = n + 1$ and the curve has no self-intersections at that point. See figure 2 for the cases $n = 5, 6, 7$.

Suppose now that $y \neq 0$. Then equation (3.3) can be written as $\cot(\theta/2) = (x^2 + y^2 - x)/y$. This shows that, for any $(x, y) \in \mathbb{R}^2$ with $y \neq 0$, there is at most one $\theta \in [0, 2\pi)$ satisfying $\gamma_n(\theta) = x + iy$. Therefore, the only self-intersection points of γ_n are 0 and 1. □
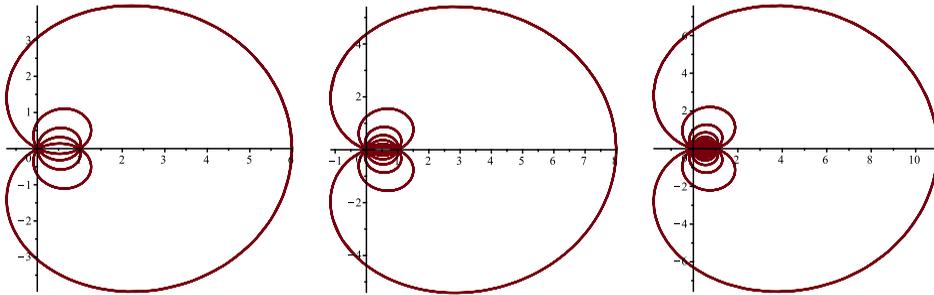
Figure 2. Maple plots of the curves γ_n for $n = 5, 6, 7$ (from left to right).

REMARK 3.7. As k cannot divide j, the condition $k \mid sj$ in lemma 3.6(1) can also be written as $d = \gcd(k, s) \neq 1$ and $j \in \{k/d, \ldots, ((d-1)k)/d\}$. Of course, the same applies to the condition $k \mid (s-1)j$ in lemma 3.6(2).

3.2. Cross-caps and triple points

The number of cross-caps and triple points are invariants of finitely $\mathcal{A}$-determined map-germs $F \colon (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ that can be described using stable deformations. A stable mapping $U \to (\mathbb{C}^3, 0)$ with U an open neighbourhood of the origin in $\mathbb{C}^2$ exhibits only regular points, transverse double points along curves, cross-caps and isolated transverse triple points. Every stable deformation F_t of F exhibits the same number C of cross-caps and T of triple points [**23**]. For a corank one map-germ, these are given by the formulae

$$C = \dim_{\mathbb{C}} \frac{\mathcal{O}_2}{JF}, \quad T = \frac{1}{6} \dim_{\mathbb{C}} \frac{\mathcal{O}_4}{I^3(F)},$$

where JF is the ideal generated by the 2×2 minors of the differential matrix of F [**23**] (the formula for C holds without the corank one assumption).

For $j, j' \in \{1, \ldots, k-1\}$, with $j \neq j'$, we set

$$\lambda_{j,j'} = \frac{\lambda_j - \lambda_{j'}}{y}$$

and define

$$T_{j,j'} = \dim_{\mathbb{C}} \frac{\mathcal{O}_2}{\langle \lambda_j, \lambda_{j,j'} \rangle}.$$

Observe that $T_{j,j'} = T_{j',j}$ for all $j \neq j'$.

PROPOSITION 3.8. *The number of cross-caps and of triple points of a finitely $\mathcal{A}$-determined k-folding map-germ F_k are given by*

$$C = \dim_{\mathbb{C}} \frac{\mathcal{O}_2}{\langle y^{k-1}, (\partial f/\partial y)(x, y) \rangle} \quad and \quad T = \frac{1}{3} \sum_{1 \leqslant j < j' \leqslant k-1} T_{j,j'}.$$

Proof. The formula for C is trivial. For T, we claim that

$$J = \left\langle \frac{y'^k - y^k}{y' - y}, \frac{\frac{y''^k - y^k}{y'' - y} - \frac{y'^k - y^k}{y' - y}}{y'' - y'} \right\rangle \subseteq I^3(F_k)$$

is a radical ideal of $\mathcal{O}_4$ whose zero locus is

$$Z = \{(x, y, y', y'') \mid y' = \xi^j y, y'' = \xi^{j'} y \text{ for some } j, j' \in \{1, \ldots, k-1\} \text{ with } j \neq j'\}.$$

Clearly, Z is contained in the zero locus $V(J)$. The converse inclusion follows by considering the degree of the generators of J and by the fact that $V(J)$ is generically reduced. Since $V(J)$ is also a complete intersection, it is reduced. Thus J is a radical ideal.

The irreducible decomposition of Z consists of the branches

$$Z_{j,j'} = \{(x, y, \xi^j y, \xi^{j'} y) \mid x, y \in \mathbb{C}\}.$$

Thus, we may decompose $\mathcal{D}^3(F_k)$, as a set, as the union of the (not necessarily irreducible) branches

$$\mathcal{D}^3_{j,j'}(F_k) = V(y' - \xi^j y, y'' - \xi^{j'} y, f[x, y, y'], f[x, y, y', y'']).$$

Eliminating y' and y'', gives $\mathcal{D}^3_{j,j'}(F_k) = \langle \lambda_j, \lambda_{j,j'} \rangle$. Therefore, we only need to show that $\dim_{\mathbb{C}}(\mathcal{O}_4/I^3(F_k)) = \sum_{j \neq j'} \dim_{\mathbb{C}}(\mathcal{O}_2/(\langle \lambda_j, \lambda_{j,j'} \rangle))$. It is clear that the same decomposition of $\mathcal{D}^3(F_k)$ applies to an unfolding $\tilde{F}_k(x, y, t) = (x, y^k, f_t(x, y), t)$ of F_k, and fixing a representative and a nonzero parameter δ, the same holds for $(\tilde{F}_k)_\delta$. This gives a decomposition of $\mathcal{D}^3((\tilde{F}_k)_\delta)$ into the branches $\mathcal{D}^3_{j,j'}((\tilde{F}_k)_\delta)$.

It is possible to choose an unfolding where the branches $\mathcal{D}^3_{j,j'}((\tilde{F}_k)_\delta)$ are pairwise disjoint. Indeed, it follows from their defining equations that two spaces $\mathcal{D}^3_{j,j'}((\tilde{F}_k)_\delta)$ and $\mathcal{D}^3_{s,s'}((\tilde{F}_k)_\delta)$ can only intersect on $\{y = 0\}$, so it is enough to find an unfolding where $\mathcal{D}^3_{j,j'}((\tilde{F}_k)_\delta) \cap \{y = 0\}$ is empty. Again, it follows from the defining equations that a point $(x, 0, 0, 0) \in \mathcal{D}^3_{j,j'}((\tilde{F}_k)_\delta)$ if $(\partial f_\delta / \partial x)(x, 0) = (\partial f_\delta^2 / \partial x^2)(x, 0) = 0$, a condition that can be avoided by choosing a suitable deformation f_t of f.

We take now an unfolding as above. Since Z is reduced, $\mathcal{D}^3((\tilde{F}_k)_\delta)$ is isomorphic to the union of the $\mathcal{D}^3_{j,j'}((\tilde{F}_k)_\delta)$ as complex spaces. The equality we need to show follows now from the constancy of the numbers involved under continuous deformations. Defining λ_j^δ and $\lambda_{j,j'}^\delta$ in the obvious way, we obtain

$$\dim_{\mathbb{C}} \frac{\mathcal{O}_4}{I^3(F_k)} = \sum_{(x,y,y',y'')} \dim_{\mathbb{C}} \frac{\mathcal{O}_4}{I^3((\tilde{F}_k)_\delta)_{(x,y,y',y'')}}$$

$$= \sum_{(x,y)} \sum_{j \neq j'} \dim_{\mathbb{C}} \frac{\mathcal{O}_2}{\langle \lambda_j^\delta, \lambda_{j,j'}^\delta \rangle} = \sum_{j \neq j'} \dim_{\mathbb{C}} \frac{\mathcal{O}_2}{\langle \lambda_j, \lambda_{j,j'} \rangle}.$$

Since $T_{j,j'} = T_{j',j}$, we add only the numbers $T_{j,j'}$, with $j < j'$, and replace $1/6$ by $1/3$. $\qquad\square$

3.3. Topological triviality

We say that two subspaces S and S' of a topological space X have the *same topological type* if there is a homeomorphism $X \to X$ restricting to an homeomorphism $S \to S'$. Milnor showed that two isolated hypersurface singularities with same topological type have the same Milnor number. In the case of our invariants, a similar result can be obtained using the results in [**28**, **30**] (see [**4**] for details).

PROPOSITION 3.9. *Let $F, G : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ be finitely $\mathcal{A}$-determined map-germs. If F and G are topologically equivalent, then $\mu(\mathcal{D}(F)) = \mu(\mathcal{D}(G))$, $r(\mathcal{D}(F)) = r(\mathcal{D}(G))$, $C(F) = C(G)$ and $T(F) = T(G)$.*

The Milnor number of the double-point curve is enough to determine the topological triviality of families of finitely $\mathcal{A}$-determined map-germs.

PROPOSITION 3.10 (Corollary 40 [**3**.)] *A family of finitely $\mathcal{A}$-determined map-germs $G_t \colon (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ is topologically trivial if and only if $\mu(\mathcal{D}(G_t))$ is constant along the parameter t.*

Using the upper semi-continuity of the numbers involved, proposition 3.10 can be combined with proposition 3.4 to yield the following result.

COROLLARY 3.11. *Let $F_k^t = (x, y^k, f_t(x, y))$ be a family of finitely $\mathcal{A}$-determined k-folding map-germs. The following statements are equivalent:*

(1) *The family F_k^t is topologically trivial.*

(2) *The numbers $\mu(\mathcal{D}_j)$ and $\mathcal{D}_j \cdot \mathcal{D}_{j'}$ are constant along the family F_k^t.*

(3) *The numbers $C, \mu(\mathcal{D}_j), \mathcal{D}_j \cdot \mathcal{D}_{j'}$ and $T_{j,j'}$ are constant along the family F_k^t.*

4. The jet space stratification

In this section, we study the singularities of k-folding map-germs which we take in standard form $F_k(x, y) = (x, y^k, f(x, y))$ (see remark 2.3(4)). We identify the set of such germs with the set $\mathcal{O}_2$ of germs, at the origin, of holomorphic functions f. For each k, we obtain a stratification $\mathcal{S}_k$ of $J^{11}(2, 1)$ (and hence of $J^l(2, 1)$ for $l \geqslant 11$). The stratification consists of the strata of codimension $\leqslant 4$ stated in theorem 1.1 together with the complement of their union (i.e. the union of strata of codimension $\geqslant 5$). Every stratum of codimension $\leqslant 4$ of $\mathcal{S}_k$ consists of finitely $\mathcal{A}$-determined and pairwise topologically equivalent k-folding map-germs. The jet level $l = 11$ is determined by the conditions defining the strata of $\mathcal{S}_k$ which involve the coefficients of f in (3.1) up to degree 11 (see tables 1, 2, 4, 6).

As pointed out in the Introduction, the case $k = 2$ was studied in [**5**, **8**, **32**]. The stratification $\mathcal{S}_2$ can be recovered from the results in this paper. The different $\mathcal{A}$-classes obtained in [**5**] correspond to different topological classes. This follows by analysing the invariants $C, T, \mu(\mathcal{D})$ and $r(\mathcal{D})$. We shall suppose here that $k \geqslant 3$ and write the jets of f as in (3.1).

Table 1. *Strata of $\mathcal{S}_3$ of codimension $\leqslant 4$*

Normal form	Defining equations and open conditions	Codim
Immersion	$a_{11} \neq 0$	0
S_1	$a_{11} = 0, a_{22}a_{21} \neq 0$	1
S_3	$a_{11} = a_{21} = 0, a_{22}a_{31} \neq 0$	2
S_5	$a_{11} = a_{21} = a_{31} = 0, a_{22}a_{41} \neq 0$	3
S_7	$a_{11} = a_{21} = a_{31} = a_{41} = 0, a_{22}a_{51} \neq 0$	4
H_2	$a_{11} = a_{22} = 0, a_{21} \neq 0, CndH_2 \neq 0$	2
H_3	$a_{11} = a_{22} = CndH_2 = 0, a_{21} \neq 0, CndH_3 \neq 0$	3
H_4	$a_{11} = a_{22} = CndH_2 = CndH_3 = 0, a_{21} \neq 0, CndH_4 \neq 0$	4
X_4	$a_{11} = a_{21} = a_{22} = 0, a_{31}a_{32}a_{44} \neq 0$	4
$\mathbf{U}_4^3$	$a_{11} = a_{21} = a_{22} = a_{44} = 0, a_{31}a_{32} \neq 0, CndUm_8 \neq 0$	4
$\mathbf{X}_4^3$	$a_{11} = a_{21} = a_{22} = a_{31} = 0, a_{32}a_{41}a_{44} \neq 0$	4
$\mathbf{W}_4^{3,1}$	$a_{11} = a_{21} = a_{22} = a_{32} = 0, a_{31}a_{44} \neq 0, a_{31}a_{55} - a_{42}a_{44} \neq 0$	4

$$CndH_2 = a_{32}a_{44} - a_{55}a_{21}$$

$$CndH_3 = a_{88}a_{21}^3 - (a_{77}a_{32} + a_{44}a_{65})a_{21}^2 + a_{44}(a_{42}a_{44} + a_{32}a_{54})a_{21} - a_{31}a_{44}^2a_{32}.$$

$$CndH_4 = a_{11,11}a_{21}^5 - (a_{44}a_{98} + a_{10,10}a_{32} + a_{77}a_{65})a_{21}^4$$
$$+ (a_{44}a_{87}a_{32} + a_{32}a_{54}a_{77} + a_{44}a_{54}a_{65} + 2a_{44}a_{77}a_{42} + a_{44}^2a_{75})a_{21}^3$$
$$- (a_{52}a_{44}^2 + a_{44}a_{31}a_{65} + 2a_{42}a_{44}a_{54} + a_{44}a_{64}a_{32}$$
$$+ 2a_{77}a_{32}a_{31} + a_{54}^2a_{32})a_{44}a_{21}^2$$
$$+ (2a_{44}a_{31}a_{42} + a_{44}a_{41}a_{32} + 3a_{32}a_{54}a_{31})a_{44}^2a_{21} - 2a_{31}^2a_{32}a_{44}^3$$

$$CndUm_8 = a_{55}(a_{31}a_{55} - a_{54}a_{32}) + a_{77}a_{32}^2 \text{ (see table 6)}$$

Table 2. *Strata of codimension $\leqslant 4$ in branch 2*

Name	Defining equations and open condition together with $a_{11} = a_{21} = 0$, $a_{22} \neq 0$	Codim
$\mathbf{M}_2^k, 2 \nmid k$	$a_{31} \neq 0$	2
$\mathbf{M}_2^k, 2 \mid k$	$a_{33}a_{31} \neq 0$	2
$\mathbf{M}_3^k, 2 \nmid k$	$a_{31} = 0, a_{41} \neq 0$	3
$\mathbf{M}_3^k, 2 \mid k$	$a_{31} = 0, a_{33}a_{41} \neq 0$	3
$\mathbf{M}_4^k, 2 \nmid k$	$a_{31} = a_{41} = 0, a_{51} \neq 0$	4
$\mathbf{M}_4^k, 2 \mid k$	$a_{31} = a_{41} = 0, a_{33}a_{51} \neq 0$	4
$\mathbf{N}_3^k, 2 \mid k$	$a_{33} = 0, a_{31} \neq 0, CndNA_3 \neq 0$	3
$\mathbf{N}_4^k, 2 \mid k$	$a_{33} = CndNA_3 = 0, a_{31} \neq 0, CndNA_5 \neq 0$	4
$\mathbf{O}_4^k, 2 \mid k$	$a_{31} = a_{33} = 0, a_{41}a_{43} \neq 0$	4

$CndNA_3 = a_{43}^2 - 4a_{31}a_{55}$
$CndNA_5 = 8a_{31}^3a_{77} - 4a_{65}a_{43}a_{31}^2 + 2a_{53}a_{43}^2a_{31} - a_{41}^2a_{43}^3$

Table 3. *Topological invariants of germs in the strata in table 2*

Name	C	T	$\mu(\mathcal{D})$	$r(\mathcal{D})$
$\mathbf{M}_2^k$, $2 \nmid k$	$2k-2$	0	$2(k-1)(k-2)-k+2$	$k-1$
$\mathbf{M}_2^k$, $2 \mid k$	$2k-2$	0	$(k-1)(k-2)+3-k$	k
$\mathbf{M}_3^k$, $2 \nmid k$	$3k-3$	0	$3(k-1)(k-2)-k+2$	$k-1$
$\mathbf{M}_3^k$, $2 \mid k$	$3k-3$	0	$(k-1)(k-2)+4-k$	$k-1$
$\mathbf{M}_4^k$, $2 \nmid k$	$4k-4$	0	$4(k-1)(k-2)-k+2$	$k-1$
$\mathbf{M}_4^k$, $2 \mid k$	$4k-4$	0	$(k-1)(k-2)+5-k$	k
$\mathbf{N}_3^k$, $2 \mid k$	$2k-2$	0	$(2k-3)(k-2)+3$	k
$\mathbf{N}_4^k$, $2 \mid k$	$2k-2$	0	$(2k-3)(k-2)+5$	k
$\mathbf{O}_4^k$, $2 \mid k$	$3k-3$	0	$(3k-4)(k-2)+4$	$k+1$

Table 4. *Strata of codimension $\leqslant 4$ in branch 3*

Name	Defining equations and open conditions together with $a_{11}=a_{22}=0$, $a_{21} \neq 0$	Codim
$\mathbf{P}_2^k$, $3 \nmid k$	$a_{33} \neq 0$	2
$\mathbf{P}_2^k$, $3 \mid k$	$a_{33} \neq 0$, $CndH_2 \neq 0$	2
$\mathbf{P}_3^k$, $3 \mid k$	$a_{33} \neq 0$, $CndH_2 = 0$, $CndH_3 \neq 0$	3
$\mathbf{P}_4^k$, $3 \mid k$	$a_{33} \neq 0$, $CndH_2 = CndH_3 = 0$, $CndH_4 \neq 0$	4
$\mathbf{Q}_3^k$, $3 \nmid k$, $4 \nmid k$	$a_{33} = 0$, $a_{44} \neq 0$	3
$\mathbf{Q}_3^k$, $3 \mid k$, $4 \nmid k$	$a_{33} = 0$, $a_{44} \neq 0$, $CndH_2 \neq 0$	3
$\mathbf{Q}_3^k$, $3 \nmid k$, $4 \mid k$	$a_{33} = 0$, $a_{44} \neq 0$, $CndQm_5 \neq 0$	3
$\mathbf{Q}_3^k$, $12 \mid k$	$a_{33} = 0$, $a_{44} \neq 0$, $CndH_2 \neq 0$, $CndQm_5 \neq 0$	3
$\mathbf{Q}_4^k$, $3 \mid k$, $4 \nmid k$	$a_{33} = CndH_2 = 0$, $a_{44} \neq 0$, $CndH_3 \neq 0$	4
$\mathbf{Q}_4^k$, $12 \mid k$	$a_{33} = CndH_2 = 0$, $CndH_3 \neq 0$, $CndQm_5 \neq 0$	4
$\widetilde{\mathbf{Q}}_4^k$, $3 \nmid k$, $4 \mid k$	$a_{33} = CndQm_5 = 0$, $a_{44} \neq 0$, $CndQm_6 \neq 0$	4
$\widetilde{\mathbf{Q}}_4^k$, $12 \mid k$	$a_{33} = CndQm_5 = 0$, $CndH_2 \neq 0$, $CndQm_6 \neq 0$	4
$\mathbf{R}_4^k$, $4 \nmid k$, $5 \nmid k$	$a_{33} = a_{44} = 0$, $a_{55} \neq 0$	4
$\mathbf{R}_4^k$, $4 \mid k$, $5 \nmid k$	$a_{33} = a_{44} = 0$, $a_{55} \neq 0$, $CndQm_5 \neq 0$	4
$\mathbf{R}_4^k$, $4 \nmid k$, $5 \mid k$	$a_{33} = a_{44} = 0$, $a_{55} \neq 0$, $CndRm_5 \neq 0$	4
$\mathbf{R}_4^k$, $20 \mid k$	$a_{33} = a_{44} = 0$, $a_{55} \neq 0$, $CndQm_5 \neq 0$, $CndRm_5 \neq 0$	4

$$
\begin{aligned}
CndQm_5 &= a_{32}a_{55} - a_{21}a_{66} \\
CndQm_6 &= a_{43}a_{55} - a_{21}a_{77} \\
CndRm_5 &= a_{32}a_{66} - a_{21}a_{77}
\end{aligned}
$$

It is clear that F_k is an immersion if and only if $a_{11} \neq 0$. All immersions are $\mathcal{A}$-finitely determined and pairwise topologically equivalent. Moreover, only immersions have $\mathcal{D} = \emptyset$, hence $r(\mathcal{D}) = 0$. We define $a_{11} \neq 0$ as the open stratum of $\mathcal{S}_k$ (corresponding to the jets of all immersions) and choose

$$
\mathbf{M}_0^k \colon (x, y) \mapsto (x, y^k, y)
$$

Table 5. *Topological invariants of germs in the strata in table 4*

Name	C	T	$\mu(\mathcal{D})$	$r(\mathcal{D})$
$\mathbf{P}_2^k,\ 3 \nmid k$	$k-1$	$\frac{(k-1)(k-2)}{6}$	$(2k-3)(k-2)$	$k-1$
$\mathbf{P}_2^k,\ 3 \mid k$	$k-1$	$\frac{(k-1)(k-2)+4}{6}$	$(2k-3)(k-2)+4$	$k-1$
$\mathbf{P}_3^k,\ 3 \mid k$	$k-1$	$\frac{(k-1)(k-2)+10}{6}$	$(2k-3)(k-2)+10$	$k-1$
$\mathbf{P}_4^k,\ 3 \mid k$	$k-1$	$\frac{(k-1)(k-2)+16}{6}$	$(2k-3)(k-2)+16$	$k-1$
$\mathbf{Q}_3^k,\ 3 \nmid k,\ 4 \nmid k$	$k-1$	$\frac{(k-1)(k-2)}{3}$	$(3k-4)(k-2)$	$k-1$
$\mathbf{Q}_3^k,\ 3 \mid k,\ 4 \nmid k$	$k-1$	$\frac{(k-1)(k-2)+1}{3}$	$(3k-4)(k-2)+2$	$k-1$
$\mathbf{Q}_3^k,\ 3 \nmid k,\ 4 \mid k$	$k-1$	$\frac{(k-1)(k-2)+6}{3}$	$(3k-4)(k-2)+12$	$k-1$
$\mathbf{Q}_3^k,\ 12 \mid k$	$k-1$	$\frac{(k-1)(k-2)+7}{3}$	$(3k-4)(k-2)+14$	$k-1$
$\mathbf{Q}_4^k,\ 3 \mid k,\ 4 \nmid k$	$k-1$	$\frac{(k-1)(k-2)+4}{3}$	$(3k-4)(k-2)+8$	$k-1$
$\mathbf{Q}_4^k,\ 12 \mid k$	$k-1$	$\frac{(k-1)(k-2)+10}{3}$	$(3k-4)(k-2)+20$	$k-1$
$\widetilde{\mathbf{Q}}_4^k,\ 3 \nmid k,\ 4 \mid k$	$k-1$	$\frac{(k-1)(k-2)+9}{3}$	$(3k-4)(k-2)+18$	$k-1$
$\widetilde{\mathbf{Q}}_4^k,\ 12 \mid k$	$k-1$	$\frac{(k-1)(k-2)+10}{3}$	$(3k-4)(k-2)+20$	$k-1$
$\mathbf{R}_4^k,\ 4 \nmid k,\ 5 \nmid k$	$k-1$	$\frac{(k-1)(k-2)}{2}$	$(4k-5)(k-2)$	$k-1$
$\mathbf{R}_4^k,\ 4 \mid k,\ 5 \nmid k$	$k-1$	$\frac{(k-1)(k-2)+2}{2}$	$(4k-5)(k-2)+6$	$k-1$
$\mathbf{R}_4^k,\ 4 \nmid k,\ 5 \mid k$	$k-1$	$\frac{(k-1)(k-2)+8}{2}$	$(4k-5)(k-2)+24$	$k-1$
$\mathbf{R}_4^k,\ 20 \mid k$	$k-1$	$\frac{(k-1)(k-2)+10}{2}$	$(4k-5)(k-2)+30$	$k-1$

as a normal form for topological equivalence of germs in this stratum. The strata corresponding to singular germs are organized into four branches according to the following result.

LEMMA 4.1. *For $k \geqslant 3$, every singular k-folding map-germ is $\mathcal{A}$-equivalent to a k-folding map-germ whose 2-jet is equal to $(x, 0, xy + y^2), (x, 0, y^2), (x, 0, xy)$ or $(x, 0, 0)$.*

Proof. As we are assuming F_k to be singular at the origin, we have $a_{11} = 0$. Then F_k is $\mathcal{A}$-equivalent to a germ whose 2-jet is $(x, 0, a_{21}xy + a_{22}y^2)$. Depending on the coefficients a_{21} and a_{22}, the 2-jet can be taken to one of the following forms:

$$
\begin{array}{lll}
(x, 0, xy + y^2) & \Longleftrightarrow & a_{11} = 0, a_{21}a_{22} \neq 0 \quad \text{(Branch 1)} \\
(x, 0, y^2) & \Longleftrightarrow & a_{11} = a_{21} = 0, a_{22} \neq 0 \quad \text{(Branch 2)} \\
(x, 0, xy) & \Longleftrightarrow & a_{11} = a_{22} = 0, a_{21} \neq 0 \quad \text{(Branch 3)} \\
(x, 0, 0) & \Longleftrightarrow & a_{11} = a_{21} = a_{22} = 0 \quad \text{(Branch 4)}
\end{array}
$$

$\square$

We have the following about $\mathcal{A}$-simplicity of germs of k-folding maps.

PROPOSITION 4.2. *There are no $\mathcal{A}$-simple k-folding map-germs for $k \geqslant 5$.*

Proof. It is enough to show that the orbit of a map-germ F_k with a 2-jet $(x, 0, xy + y^2)$ is not simple as the orbits of germs in the remaining branches in lemma

Table 6. *Strata of codimension $\leqslant 4$ in branch 4*

Name	Defining equations and open conditions together with $a_{11} = a_{21} = a_{22} = 0$	Codim
$\mathbf{U}_3^k,\ 3 \nmid k$	$a_{31}a_{33} \neq 0,\ \Delta_j^k \neq 0,\ \Omega_{j,k-j} \neq 0$	3
$\mathbf{U}_3^k,\ 3 \mid k$	$a_{31}a_{33} \neq 0,\ a_{44} \neq 0,\ \Delta_j^k \neq 0,\ \Omega_{j,j'}^k \neq 0$	3
$\mathbf{U}_4^k,\ 3 \mid k$	$a_{31}a_{33} \neq 0,\ a_{44} = 0,\ \Delta_j^k \neq 0,$ $\Omega_{j,j'}^k \neq 0,\ CndUm_8 \neq 0$	4
$\mathbf{V}_4^{k,j,j'},\ (*)$	$a_{31}a_{33} \neq 0,\ \Omega_{j,j'}^k = 0,\ a_{32} \neq 0,$ $a_{44} \neq 0,\ CndVm_{5j,j'} \neq 0$	4
$\mathbf{W}_4^{k,j},\ 3 \nmid k,\ (**)$	$a_{31}a_{33} \neq 0,\ \Delta_j^k = 0,\ CndWA_2 \neq 0$	4
$\mathbf{W}_4^{k,j},\ 3 \mid k,\ (**),\ (***)$	$a_{31}a_{33} \neq 0,\ \Delta_j^k = 0,\ CndWA_2 \neq 0,$ $a_{44} \neq 0$	4
$\mathbf{W}_4^{3p,p},\ k = 3p$	$a_{31}a_{33} \neq 0,\ a_{32} = 0,\ a_{44} \neq 0,$ $Cnd\widetilde{W}m_8 \neq 0$	4
$\mathbf{X}_4^k,\ 2 \nmid k,\ 3 \nmid k$	$a_{32}a_{33} \neq 0,\ a_{31} = 0,\ a_{41} \neq 0$	4
$\mathbf{X}_4^k,\ 2 \mid k,\ 3 \nmid k$	$a_{32}a_{33} \neq 0,\ a_{31} = 0,\ a_{41} \neq 0$	4
$\mathbf{X}_4^k,\ 2 \nmid k,\ 3 \mid k$	$a_{32}a_{33} \neq 0,\ a_{31} = 0,\ a_{41} \neq 0,\ a_{44} \neq 0$	4
$\mathbf{X}_4^k,\ 6 \mid k$	$a_{32}a_{33} \neq 0,\ a_{31} = 0,\ a_{41} \neq 0,\ a_{44} \neq 0$	4
$\mathbf{Y}_4^k,\ 2 \nmid k$	$a_{32}a_{31} \neq 0,\ a_{33} = 0,\ a_{44} \neq 0$	4
$\mathbf{Y}_4^k,\ 2 \mid k$	$a_{32}a_{31} \neq 0,\ a_{33} = 0,\ CndYA_3 \neq 0,$ $CndYm_{6j} \neq 0,\ a_{44} \neq 0$	4

$$
\begin{aligned}
\Delta_j^k &= (a_{32}^2 - 4a_{31}a_{33})\xi^{2j} + 2(a_{32}^2 - 2a_{31}a_{33})\xi^j + a_{32}^2 - 4a_{31}a_{33} \\
\Omega_{j,j'}^k &= a_{31}a_{33}(1 + \xi^j + \xi^{j'})^2 - a_{32}^2(\xi^j + \xi^{j'} + \xi^{j+j'}) \\
CndUm_8 &= a_{55}(a_{31}a_{55} - a_{54}a_{32}) + a_{77}a_{32}^2 \\
CndVm_{5j,j'} &= a_{32}^2 a_{44}\beta_{j,j'} + a_{32}a_{33}(2a_{33}a_{42} - a_{32}a_{43})\alpha_{j,j'} + a_{41}a_{33}^3,\ \text{with} \\
&\quad \alpha_{j,j'} = \frac{\xi^j + \xi^{j'} + \xi^{j+j'}}{(1 + \xi^j + \xi^{j'})^2}, \\
&\quad \beta_{j,j'} = \frac{\xi^{j+3j'} + \xi^{3j+j'} + \xi^{3j'} + 2\xi^{2j+j'} + 2\xi^{j+2j'} + \xi^{3j} + 2\xi^{j+j'} + \xi^j + \xi^{j'}}{(1 + \xi^j + \xi^{j'})^4}. \\
CndWA_2 &= (a_{41}a_{33} + a_{31}a_{43})a_{32}^3 - 2a_{31}(a_{42}a_{33} + 2a_{31}a_{44})a_{32}^2 + 8a_{31}^3 a_{33}a_{44} \\
Cnd\widetilde{W}m_8 &= a_{31}a_{55} - a_{42}a_{44} \\
CndYA_3 &= a_{43}^2 - 4a_{31}a_{55} \\
CndYm_{6j} &= (a_{31}a_{44}^2 + a_{55}a_{32}^2 - a_{43}a_{44}a_{32})\xi^{4j} \\
&\quad + a_{44}(2a_{31}a_{44} - a_{32}a_{43})\xi^{2j} + a_{31}a_{44}^2 \\
(*) &\quad j < j';\ (j,j') \neq (p,2p)\ when\ k = 3p \\
(**) &\quad If\ 2 \mid k\ then\ j \neq k/2 \\
(***) &\quad If\ k = 3p,\ then\ j \neq p, 2p
\end{aligned}
$$

4.1 are adjacent to it. For such a germ, we have $j^k F_k \sim_{\mathcal{A}^{(k)}} (x, (y - (1/2)x)^k, y^2)$. The result follows by theorem 1:1 in [**22**] as there are no $\mathcal{A}$-simple germs of the form $(x, y^2, f(x, y))$ with $j^4 f \equiv 0$.

When $k = 4$ and for F_4 in branch 1, we have $j^4 F_4 \sim_{\mathcal{A}^{(4)}} (x, xy^3 + x^3y, y^2)$. This is a C_3-singularity and is $\mathcal{A}$-simple [**22**]. For F_4 in branch 2, we have $j^4 F_4 \sim_{\mathcal{A}^{(4)}} (x, 0, y^2)$ so it leads to non $\mathcal{A}$-simple germs. By adjacency, the germs in branches

3 and 4 also lead to non-$\mathcal{A}$-simple germs. Therefore, the C_3-singularity is the only $\mathcal{A}$-simple singularity of 4-folding map-germs.

The case $k = 3$ is treated in §4.1 where there are several $\mathcal{A}$-simple singularities of 3-folding map-germs.　□

REMARKS 4.3. 1. The degree of $\mathcal{A}$-determinacy of a singular germ F_k is greater or equal to k (the germ $(x, y) \mapsto (x, 0, f(x, y))$ is not finitely $\mathcal{A}$-determined for any f).

2. For germs in branch 1 or branch 2, we have $F_k \sim_{\mathcal{A}} (x, yp(x, y^2), y^2)$ for some germ p. One can study the germ p, as was done in [**22**], instead of F_k, but this blurs the order of the original k-folding map-germ. Also, the approach in [**22**] of reducing the action of $\mathcal{A}$ on the set of 2-folding map-germs to the action of a subgroup of $\mathcal{K}$ on $\mathfrak{m}_2$ does not extend to germs of k-folding maps for $k \geqslant 3$.

4.1. The case $k = 3$

When $k = 3$ we get several $\mathcal{A}$-simple map-germs. For this reason, we treat this case separately.

THEOREM 4.4. *The only $\mathcal{A}$-simple singularities a 3-folding map-germ F_3 can have are those of type S_{2l-1}, $l \geqslant 2$, or H_s, $s \geqslant 3$. In the real case, the S_{2l-1}-singularities are of type S_{2l-1}^-. The strata of $\mathcal{S}_3$ of codimension $\leqslant 4$ are given in table 1.*

Proof. We can write $f(x, y) = f_0(x, y^3) + yf_1(x, y^3) + y^2 f_2(x, y^3)$, for some germs of holomorphic functions f_i, $i = 0, 1, 2$. Then $F_3 \sim_{\mathcal{A}} (x, y^3, yf_1(x, y^3) + y^2 f_2(x, y^3))$.

Suppose that $a_{10} = 0$ (F_3 is singular) and $a_{22} \neq 0$. Then

$$F_3 \sim_{\mathcal{A}} (x, y^3, y(g(x) + y^3 h(x, y^3)) + y^2(a_{22} + k(x, y^3)))$$

for some germs of holomorphic functions $g \in \mathfrak{m}_1$, $h \in \mathcal{O}_2$ and $k \in \mathfrak{m}_2$. We can make successive changes of coordinates in the target so that $j^p F_3 \sim_{\mathcal{A}^{(p)}} (x, y^3, yL(x) + a_{22}y^2)$ for any $p \geqslant 3$. It is not difficult to show that F_3 is finitely $\mathcal{A}$-determined if and only if $ord(L) = ord(g) = ord(f_y(x, 0))$ is finite. Suppose that this is the case and denote by l that order. Then $j^l L(x) = a_{l1}x^l$, $a_{l1} \neq 0$, and the change of coordinates $y \mapsto y - (a_{l1}/2a_{22})x^l$ in the source yields $j^{2l+1} F_3 \sim_{\mathcal{A}^{(2l+1)}} (x, y^3 - 3(a_{l1}/2a_{22})^2 yx^{2l}, a_{22}y^2)$. This is an S_{2l-1}-singularity, and since it is $(2l+1)$-$\mathcal{A}$-determined, we have $F_3 \sim_{\mathcal{A}} (x, y^3 - yx^{2l}, y^2)$. In the real case, this is an S_{2l-1}^--singularity.

The above calculations show, in particular, that we do not get the simple singularities $B_l^{\pm}$, $C_l^{\pm}$ and F_4 whose 2-jets are $\mathcal{A}^{(2)}$-equivalent to $(x, 0, y^2)$.

Similar calculations show that when $a_{11} = a_{22} = 0$ and $a_{21} \neq 0$, we get an H_l-singularity when the singularity of F_3 is finitely $\mathcal{A}$-determined.

The remaining cases are studied in the same way as in the case $k \geqslant 4$ (table 6). We get three strata with topological normal forms $\mathbf{U}_4^3$, $\mathbf{X}_4^3$ and $\mathbf{W}_4^{3,1}$ (the germs in these strata are not $\mathcal{A}$-simple) together with the stratum represented by the topological normal form $\mathbf{U}_3^3$, which is topologically equivalent to Mond's singularity X_4 (see §2 and [**22**]).　□

Table 7. *Topological invariants of germs in strata in table 6*

Name	C	T	$\mu(\mathcal{D})$	$r(\mathcal{D})$
$\mathbf{U}_3^k,\, 3 \nmid k$	$2k-2$	$\frac{(k-1)(k-2)}{3}$	$4(k-1)(k-2)+1$	$2k-2$
$\mathbf{U}_3^k,\, 3 \mid k$	$2k-2$	$\frac{(k-1)(k-2)+1}{3}$	$4(k-1)(k-2)+3$	$2k-2$
$\mathbf{U}_4^k,\, 3 \mid k$	$2k-2$	$\frac{(k-1)(k-2)+4}{3}$	$4(k-1)(k-2)+9$	$2k-2$
$\mathbf{V}_4^{k,j,j'},\, 3 \nmid k$	$2k-2$	$\frac{(k-1)(k-2)+3}{3}$	$4(k-1)(k-2)+7$	$2k-2$
$\mathbf{V}_4^{k,j,j'},\, 3 \mid k$	$2k-2$	$\frac{(k-1)(k-2)+4}{3}$	$4(k-1)(k-2)+9$	$2k-2$
$\mathbf{W}_4^{k,j},\, 3 \nmid k$	$2k-2$	$\frac{(k-1)(k-2)}{3}$	$4(k-1)(k-2)+3$	$2k-4$
$\mathbf{W}_4^{k,j},\, 3 \mid k$	$2k-2$	$\frac{(k-1)(k-2)+1}{3}$	$4(k-1)(k-2)+5$	$2k-4$
$\mathbf{W}_4^{3p,p},\, k=3p$	$2k-2$	$\frac{(k-1)(k-2)+4}{3}$	$4(k-1)(k-2)+11$	$2k-4$
$\mathbf{X}_4^k,\, 2 \nmid k, 3 \nmid k$	$3k-3$	$\frac{(k-1)(k-2)}{3}$	$5(k-1)(k-2)+1$	$2k-2$
$\mathbf{X}_4^k,\, 2 \mid k, 3 \nmid k$	$3k-3$	$\frac{(k-1)(k-2)}{3}$	$5(k-1)(k-2)+2$	$2k-3$
$\mathbf{X}_4^k,\, 2 \nmid k, 3 \mid k$	$3k-3$	$\frac{(k-1)(k-2)+1}{3}$	$5(k-1)(k-2)+3$	$2k-2$
$\mathbf{X}_4^k,\, 6 \mid k$	$3k-3$	$\frac{(k-1)(k-2)+1}{3}$	$5(k-1)(k-2)+4$	$2k-3$
$\mathbf{Y}_4^k,\, 2 \nmid k$	$2k-2$	$\frac{(k-1)(k-2)}{2}$	$5(k-1)(k-2)+1$	$2k-2$
$\mathbf{Y}_4^k,\, 2 \mid k$	$2k-2$	$\frac{k(k-2)}{2}$	$5(k-1)(k-2)+3(k-1)$	$2k-2$

REMARK 4.5. The invariants associated to the simple singularities can be found in [**23**]; those associated to $\mathbf{U}_4^3, \mathbf{X}_4^3, \mathbf{W}_4^{3,1}$ are given in table 7.

4.2. The case $k \geqslant 4$

We consider here the case when $k \geqslant 4$, which we divide into the four branches according to the $\mathcal{A}^{(2)}$-orbits in lemma 4.1. In all that follows, $j \in \{1, \ldots, k-1\}$, and subindices of singularities indicate the codimension of the stratum.

4.2.1. Branch 1: $a_{11} = 0$, $a_{21}a_{22} \neq 0$

THEOREM 4.6. *Any germ F_k of a k-folding map satisfying $a_{11} = 0$ and $a_{21}a_{22} \neq 0$ is finitely $\mathcal{A}$-determined and is topologically equivalent to*

$$\mathbf{M}_1^k \colon (x,y) \mapsto (x, y^k, xy + y^2).$$

The invariants take the values $\mu(\mathcal{D}) = (k-2)^2$, $C = k-1$, $T = 0$ and $r(\mathcal{D}) = k-1$ and the double-point curve of F_k is the union of $k-1$ regular curves intersecting transversally.

Proof. The functions defining the branches $\mathcal{D}_j$ of the double-point curve (see theorem 3.2) are given by $\lambda_j = a_{21}x + (1 + \xi^j)a_{22}y + O(2)$, where $O(l)$ denotes a remainder of order l. Clearly, all of the branches $\mathcal{D}_j$ are regular curves. As the scalars $1 + \xi^j$ are pairwise distinct, the space $\mathcal{D} = \mathcal{D}(F_k) = \bigcup_j \mathcal{D}_j$ consists of $k-1$ regular curves intersecting transversally at the origin.

We have $\mu(\mathcal{D}_j) = 0$ and $\mathcal{D}_j \cdot \mathcal{D}_{j'} = 1$, and from proposition 3.4 we obtain $\mu(\mathcal{D}) = (k-2)^2$. By theorem 3.3, any germ F_k satisfying the conditions in the statement of

the theorem is finitely $\mathcal{A}$-determined. These germs form a stratum of codimension 1 defined by $\{a_{11} = 0, a_{21}a_{22} \neq 0\}$. Since this stratum is path connected, we conclude by theorem 3.10 that it consists of topologically equivalent germs. We choose for a topological model the germ $\mathbf{M}_1^k$ given in the statement of the theorem.

By proposition 3.11, it is enough to compute C and $T_{j,j'}$ for $\mathbf{M}_1^k$ as these invariants are constant along the stratum. We have $C = \dim_{\mathbb{C}} \mathcal{O}_2/\langle y^{k-1}, x + 2y \rangle = k - 1$ and $T_{j,j'} = \dim_{\mathbb{C}} \mathcal{O}_2/\langle x + (1 + \xi^j)y, 1 \rangle = 0$. $\qquad\square$

REMARK 4.7. The singularity $\mathbf{M}_1^4$ is the $\mathcal{A}$-simple singularity C_3 (see the proof of proposition 4.2).

The proofs for the cases in the remaining branches follow by similar arguments used in the proof of theorem 4.6 (except for the calculations of T). To avoid repetition, we highlight only key differences in each case. The notation for the conditions that define the strata are those indicated in the tables.

4.2.2. Branch 2: $a_{11} = a_{21} = 0$, $a_{22} \neq 0$

THEOREM 4.8. *The strata of codimension $\leqslant 4$ of finitely $\mathcal{A}$-determined k-folding map-germs in the branch $a_{11} = a_{21} = 0$, $a_{22} \neq 0$ are those given in table 2. The invariants associated to the germs in each stratum are given in table 3.*

Proof. The result follows from propositions 4.9, 4.11 and 4.12. $\qquad\square$

For germs in this branch, the map-germ $(x, y) \mapsto (x, f(x, y))$ is finite and generically two-to-one. Therefore, $T = 0$ for any finitely $\mathcal{A}$-determined map-germ in this branch.

The germs of the functions defining the double-point branch $\mathcal{D}_j$ is given by

$$\lambda_j = \vartheta_{2j}a_{22}y + a_{31}x^2 + \vartheta_{2j}a_{32}xy + \vartheta_{3j}a_{33}y^2 + O(3).$$

The branch $\mathcal{D}_j$ is thus regular if and only if $\vartheta_{2j} \neq 0$. By lemma 3.6, $\vartheta_{2j} = 0$ when k is even and $j = k/2$.

PROPOSITION 4.9. *Any k-folding map-germ F_k satisfying $a_{11} = a_{21} = \cdots = a_{l1} = 0$, $a_{22}a_{(l+1)1} \neq 0$, for some $l \geqslant 2$, and $a_{33} \neq 0$ when $k = 2p$, is finitely $\mathcal{A}$-determined and is topologically equivalent to*

$$\mathbf{M}_l^k \colon (x, y) \mapsto (x, y^k, y^2 + y^3 + x^l y).$$

The invariants C, T, $\mu(\mathcal{D})$ and $r(\mathcal{D})$ are as in table 2. All the double-point branches are regular curves except for the branch $\mathcal{D}_p$, when $k = 2p$, which has an A_{l-1}-singularity. We have $\mathcal{D}_j \cdot \mathcal{D}_{j'} = l$, for all $j \neq j'$.

Proof. Fix an index j and assume that $2 \nmid k$ or that $k = 2p$ but $j \neq p$. Then $\mathcal{D}_j$ is a regular curve and can be parametrized by $t \mapsto (t, \gamma_j(t))$, with $\gamma_j(t) = -(a_{l+1,1}/((1 + \xi^j)a_{22}))t^l + O(l + 1)$. Clearly, any two distinct branches have order of contact equal to l.

Suppose now that $k = 2p$. As $\vartheta_p^2 = 0$, the coefficients of $x^s y$ in λ_p vanish for all $s \geqslant 1$. Moreover, since $\vartheta_p^3 = 1$, the function λ_p is of the form $\lambda_p = a_{l+1,1}x^l +$

$a_{33}y^2 + y^3h(y)$. This implies that λ_p is $\mathcal{R}$-equivalent to $y^2 + x^l$ if and only if $a_{33} \neq 0$, in which case it has an A_{l-1}-singularity. As for the contact between branches, $ord(\lambda_p(t, (-a_{l+1,1}/((1 + \xi^j)a_{22}))t^l + O(l+1)) = l$, hence $\mathcal{D}_j \cdot \mathcal{D}_p = l$. This determines $\mu(\mathcal{D})$, and hence the topological triviality and the constancy of the invariants along the stratum. □

REMARKS 4.10. 1. Branch 1 can be considered as a particular case of the strata $\mathbf{M}_l^k$ in proposition 4.9 (the condition $a_{33} \neq 0$ is not needed when $l = 1$).

2. When $2 \nmid k$, the term y^3 in $\mathbf{M}_l^k$ is irrelevant for topological equivalence. We include it to represent both k even and k odd by the same map-germ. We do this for all subsequent topological normal forms.

PROPOSITION 4.11. *Suppose that $k = 2p$. Any k-folding map-germ F_k satisfying $a_{11} = a_{21} = a_{33} = 0$, $a_{22}a_{31} \neq 0$ and the additional conditions in (a) or (b) below is finitely $\mathcal{A}$-determined and is topologically equivalent to one of the germs*

$$\mathbf{N}_l^k\colon (x,y) \mapsto (x, y^k, y^2 + x^2y + y^{2l-1}), \quad l = 3, 4.$$

The invariants $\mu(\mathcal{D}(F_k)), C, T, r(\mathcal{D}(F_k))$ are as in table 3. The branches $\mathcal{D}_j$ are regular curves for all $j \neq p$ and $\mathcal{D}_j \cdot \mathcal{D}_{j'} = 2$ for all $j \neq j'$.

(a) If $CndNA_3 \neq 0$, then the branch $\mathcal{D}_p$ has an A_3-singularity and the map-germ is topologically equivalent to $\mathbf{N}_3^k$.

(b) If $CndNA_3 = 0$ and $CndNA_5 \neq 0$, then $\mathcal{D}_p$ has an A_5-singularity and the map-germ is topologically equivalent to $\mathbf{N}_4^k$.

Proof. We have $\lambda_p = a_{31}x^2 + a_{41}x^3 + a_{43}xy^2 + a_{51}x^4 + a_{53}x^2y^2 + a_{55}y^4 + O(5)$. It has an A_3-singularity if and only if $CndNA_3 \neq 0$. When $CndNA_3 = 0$, we need to consider the 7-jet of λ_p. A calculation shows that λ_p has an A_5-singularity if and only if $CndNA_5 \neq 0$. In both cases, we have $ord(\lambda_p(t, -a_{31}/((1 + \xi^j)a_{22})t^2 + O(3))) = 2$ for all $j \neq p$. □

PROPOSITION 4.12. *Suppose that $k = 2p$. Any k-folding map-germ F_k satisfying $a_{11} = a_{21} = a_{31} = a_{33} = 0$ and $a_{22}a_{41}a_{43} \neq 0$ is finitely $\mathcal{A}$-determined and is topologically equivalent to*

$$\mathbf{O}_4^k\colon (x,y) \mapsto (x, y^k, y^2 + x^3y + xy^3).$$

The codimension of the stratum is 4 and the invariants $\mu(\mathcal{D}(F_k)), C, T, r(\mathcal{D}(F_k))$ are as in table 3. The branches $\mathcal{D}_j$, $j \neq p$, are regular curves and $\mathcal{D}_p$ has a D_4-singularity. We have $\mathcal{D}_j \cdot \mathcal{D}_{j'} = 2$ for the distinct regular branches and $\mathcal{D}_j \cdot \mathcal{D}_p = 3$, for $j \neq p$.

Proof. The result follows from the fact that $\lambda_p = a_{41}x^3 + a_{43}xy^2 + O(4)$. □

4.2.3. Branch 3: $a_{11} = a_{22} = 0$, $a_{21} \neq 0$

THEOREM 4.13. *The strata of codimension $\leqslant 4$ of finitely $\mathcal{A}$-determined k-folding map-germs in the branch $a_{11} = a_{22} = 0$, $a_{21} \neq 0$ are those given in table 4. The invariants of the germs in each stratum are given in table 5.*

Proof. The result follows from propositions 4.14, 4.16 and 4.18.   □

For any finitely $\mathcal{A}$-determined k-folding map-germ in this branch, we have $C = \dim_{\mathbb{C}} \mathcal{O}_2/\langle y^{k-1}, a_{21}x + O(2) \rangle = k - 1$.

The double-point branches are regular curves that can be parametrized by $t \mapsto (\gamma_j(t), t)$, with

$$\gamma_j(t) = -\frac{1}{a_{21}}\vartheta_{3j}a_{33}t^2 - \frac{1}{a_{21}^2}(\vartheta_{2j}\vartheta_{3j}a_{32}a_{33} - \vartheta_{4j}a_{21}a_{44})t^3$$

$$-\frac{1}{a_{21}^3}(\vartheta_{2j}^2\vartheta_{3j}a_{32}^2a_{33} - \vartheta_{2j}\vartheta_{4j}a_{32}a_{44}a_{21} + \vartheta_{3j}^2a_{33}(a_{31}a_{33} - a_{43}a_{21})$$

$$+\vartheta_{5j}a_{55}a_{21}^2)t^4 + O(5).$$

The strata are determined by the contact between the branches of the double-point curve which depend on ϑ_{sj} as well as on the coefficients a_{pq}. We start with the case $a_{33} \neq 0$, where the strata depend on the divisibility of k by 3.

PROPOSITION 4.14. *Suppose that $a_{11} = a_{22} = 0$ and $a_{21}a_{33} \neq 0$. Any k-folding map-germ F_k in case* (a) *or satisfying the additional conditions in* (b) *is finitely $\mathcal{A}$-determined and is topologically equivalent to one of the map-germs*

$$\mathbf{P}_l^k \colon (x, y) \mapsto (x, y^k, xy + y^3 + y^{3l-1}), \quad \text{for } l = 2, 3, 4.$$

The invariants $\mu(\mathcal{D}), C, T, r(\mathcal{D})$ are as in table 5. We have contact $\mathcal{D}_j \cdot \mathcal{D}_{j'} = 2$, except for $\mathcal{D}_p \cdot \mathcal{D}_{2p}$ when $k = 3p$ which is given in (b).

(a) *If $3 \nmid k$, then F_k is topologically equivalent to $\mathbf{P}_2^k$.*

(b) *If $k = 3p$, then the strata are as follows:*

   If $CndH_2 \neq 0$, then $\mathcal{D}_p \cdot \mathcal{D}_{2p} = 4$ and F_k is topologically equivalent to $\mathbf{P}_2^k$.

   If $CndH_2 = 0$ and $CndH_3 \neq 0$, then $\mathcal{D}_p \cdot \mathcal{D}_{2p} = 7$ and F_k is topologically equivalent to $\mathbf{P}_3^k$.

   If $CndH_2 = CndH_3 = 0$ and $CndH_4 \neq 0$, then $\mathcal{D}_p \cdot \mathcal{D}_{2p} = 10$ and F_k is topologically equivalent to $\mathbf{P}_4^k$.

Proof. If $3 \nmid k$, then by lemma 3.6 we have $\vartheta_{3j} \neq 0$ and $\vartheta_{3j} \neq \vartheta_{3j'}$ for all $j \neq j'$. This implies $\mathcal{D}_j \cdot \mathcal{D}_{j'} = 2$ for all $j \neq j'$.

If $k = 3p$, then by lemma 3.6 the equality $\vartheta_{3j} = \vartheta_{3j'}$ holds only when $\{j, j'\} = \{p, 2p\}$. Again, we obtain $\mathcal{D}_j \cdot \mathcal{D}_{j'} = 2$ for all $j \neq j'$ with $\{j, j'\} \neq \{p, 2p\}$.

We have $\vartheta_{3p} = \vartheta_{3(2p)} = 0$, $\vartheta_{4p} = \vartheta_{4(2p)} = 1$ and $\vartheta_{5p} = \vartheta_{2(2p)}$ (lemma 3.6). Using the parametrizations of $\mathcal{D}_p$ and $\mathcal{D}_{2p}$, we get $\mathcal{D}_p \cdot \mathcal{D}_{2p} = 4$ if and only if $a_{32}a_{44} - a_{21}a_{55} \neq 0$, equivalently, $CndH_2 \neq 0$.

When $CndH_2 = 0$, the exceptional branches $\mathcal{D}_p$ and $\mathcal{D}_{2p}$ are parametrized by $x = -(a_{44}/a_{21})y^3 + \beta_s y^7 + O(9)$, $s = 1, 2$, with $\beta_1 - \beta_2 \neq 0$ if and only if $CndH_3 \neq 0$. Then, $\mathcal{D}_p \cdot \mathcal{D}_{2p} = 7$.

When $CndH_2 = CndH_3 = 0$, the exceptional branches are parametrized by $x = -(a_{44}/a_{21})y^3 + \beta_s y^{10} + O(11)$, $s = 1, 2$, with $\beta_1 - \beta_2 \neq 0$ if and only if $CndH_4 \neq 0$. Then, $\mathcal{D}_p \cdot \mathcal{D}_{2p} = 10$.

The values of T can be computed using the models $\mathbf{P}_l^k$. We have $\lambda_j = x + \vartheta_{3j}y^2 + \vartheta_{(3l-1)j}y^{3l-2}$ and $\lambda_{j,j'} = (\vartheta_{3j} - \vartheta_{3j'})y + (\vartheta_{3l-1,j} - \vartheta_{3l-1,j'})y^{3l-3}$. By lemma 3.6, $T_{j,j'} = \dim_{\mathbb{C}}(\mathcal{O}_2/\langle x,y \rangle) = 1$ when $3 \nmid k$ or when $k = 3p$ but $\{j,j'\} \neq \{p,2p\}$. For $k = 3p$, we show that $\vartheta_{3l-1,p} \neq \vartheta_{3l-1,2p}$ so $T_{p,2p} = 3l - 3$. $\qquad\square$

REMARK 4.15. The singularity $\mathbf{P}_2^4 \colon (x,y) \mapsto (x, y^4, xy + y^3 + y^5)$ is topologically equivalent to the singularity $T_4 \colon (x,y) \mapsto (x, y^4, xy + y^3)$ in [**22**]. Observe that according to proposition 4.14, if k is not divisible by 3, then the y^{3l-1} term can be removed from the expression of $\mathbf{P}_l^k$, without changing the topological class of the germ.

PROPOSITION 4.16. *Suppose that* $a_{11} = a_{22} = a_{33} = 0$ *and* $a_{21}a_{44} \neq 0$. *Any k-folding map-germ* F_k *in case* (a) *or satisfying the additional conditions in* (b), (c) *or* (d) *is finitely $\mathcal{A}$-determined and is topologically equivalent to one of the following map-germs:*

$$\mathbf{Q}_3^k \colon (x,y) \mapsto (x, y^k, xy + y^4 + y^5 + y^6)$$

$$\mathbf{Q}_4^k \colon (x,y) \mapsto (x, y^k, xy + y^4 + y^6 + y^8)$$

$$\widetilde{\mathbf{Q}}_4^k \colon (x,y) \mapsto (x, y^k, xy + y^4 + y^5 + y^7)$$

The invariants are as in table 5. We have $\mathcal{D}_j \cdot \mathcal{D}_{j'} = 3$ *except when j and j' are in the sets J or J' below.*

(a) *If* $3 \nmid k$ *and* $4 \nmid k$, *then there are no exceptional branches and the germs in this stratum are topologically equivalent to* $\mathbf{Q}_3^k$.

(b) *If* $k = 3p$ *and* $4 \nmid k$, *then* $J = \{p, 2p\}$.
  (b1) *If* $CndH_2 \neq 0$, *then* $\mathcal{D}_p \cdot \mathcal{D}_{2p} = 4$ *and* F_k *is topologically equivalent to* $\mathbf{Q}_3^k$.

  (b2) *If* $CndH_2 = 0$ *and* $CndH_3 \neq 0$, *then* $\mathcal{D}_p \cdot \mathcal{D}_{2p} = 7$ *and* F_k *is topologically equivalent to* $\mathbf{Q}_4^k$.

(c) *If* $k = 4p$, $3 \nmid k$, *then* $J = \{p, 2p, 3p\}$.
  (c1) *If* $CndQm_5 \neq 0$, *then* $\mathcal{D}_j \cdot \mathcal{D}_{j'} = 5$, *for all* $j, j' \in J, j \neq j'$, *and* F_k *is topologically equivalent to* $\mathbf{Q}_3^k$.

  (c2) *If* $CndQm_5 = 0$ *and* $CndQm_6 \neq 0$, *then* $\mathcal{D}_j \cdot \mathcal{D}_{j'} = 6$, *for all* $j, j' \in J, j \neq j'$, *and* F_k *is topologically equivalent to* $\widetilde{\mathbf{Q}}_4^k$.

(d) *If* $k = 12p$, *the exceptional contact between double-point branches occurs when the indices are in* $J = \{4p, 8p\}$ *or* $J' = \{3p, 6p, 9p\}$. *There are three strata:*
  (d1) *If* $CndH_2 \neq 0$ *and* $CndQm_5 \neq 0$, *then* $\mathcal{D}_j \cdot \mathcal{D}_{j'} = 4$ (*resp.* $\mathcal{D}_j \cdot \mathcal{D}_{j'} = 5$) *for all distinct pairs with* j, j' *in* J (*resp.* J'), *and* F_k *is topologically equivalent to* $\mathbf{Q}_3^k$.

  (d2) *If* $CndH_2 = 0, CndH_3 \neq 0$ *and* $CndQm_5 \neq 0$, *then* $\mathcal{D}_j \cdot \mathcal{D}_{j'} = 7$ (*resp.* $\mathcal{D}_j \cdot \mathcal{D}_{j'} = 5$) *for all distinct pairs with* j, j' *in* J (*resp.* J'), *and* F_k *is topologically equivalent to* $\mathbf{Q}_4^k$.

(d3) *If $CndQm_5 = 0, CndQm_6 \neq 0$, and $CndH_2 \neq 0$, then $\mathcal{D}_j \cdot \mathcal{D}_{j'} = 4$ (resp. $\mathcal{D}_j \cdot \mathcal{D}_{j'} = 6$) for all distinct pairs with j, j' in J (resp. J'), and F_k is topologically equivalent to $\widetilde{\mathbf{Q}}_4^k$.*

Proof. The branches $\mathcal{D}_j$ can be parametrized by $t \mapsto (\gamma_j(t), t)$ with

$$\gamma_j(t) = -\frac{1}{a_{21}}\vartheta_{4j}a_{44}t^3 - \frac{1}{a_{21}^2}(\vartheta_{2j}\vartheta_{4j}a_{32}a_{44} - \vartheta_{5j}a_{21}a_{55})t^4$$
$$- \frac{1}{a_{21}^3}(\vartheta_{2j}^2\vartheta_{4j}a_{32}^2a_{44} - \vartheta_{3j}\vartheta_{4j}a_{43}a_{44}a_{21}$$
$$- \vartheta_{2j}\vartheta_{5j}a_{21}a_{32}a_{55} + \vartheta_{6j}a_{66}a_{21}^2)t^5 + O(6).$$

(a) If $3 \nmid k$, $4 \nmid k$, then $\vartheta_{4j} \neq \vartheta_{4j'}$ for all j, j' with $j \neq j'$ (lemma 3.6). Therefore, $\mathcal{D}_j \cdot \mathcal{D}_{j'} = 3$ for all distinct pairs.

(b) If $k = 3p$ and $4 \nmid p$, then $\vartheta_{4j} \neq \vartheta_{4j'}$ for all j, j' with j or j' not in $J = \{p, 2p\}$. For such pairs, $\mathcal{D}_j \cdot \mathcal{D}_{j'} = 3$.

Using the fact that $\vartheta_{6p} = \vartheta_{6(2p)} = \vartheta_{3p} = \vartheta_{3(2p)} = 0$, $\vartheta_{4p} = \vartheta_{4(2p)} = 1$ and $\vartheta_{5p} = \vartheta_{2p} \neq \vartheta_{5(2p)} = \vartheta_{2(2p)}$, the parametrization of the exceptional branch $\mathcal{D}_p$ becomes

$$\gamma_p(t) = -\frac{a_{44}}{a_{21}}t^3 - \frac{\vartheta_{2p}(a_{32}a_{44} - a_{21}a_{55})}{a_{21}^2}t^4 - \frac{\vartheta_{2p}^2 a_{32}(a_{32}a_{44} - a_{21}a_{55})}{a_{21}^3}t^5 + O(6).$$

A parametrization of $\mathcal{D}_{2p}$ is obtained by replacing ϑ_{2p} by $\vartheta_{2(2p)}$ in the expression of γ_p. Therefore, $\mathcal{D}_p \cdot \mathcal{D}_{2p} = 4$ if and only if $a_{32}a_{44} - a_{21}a_{55} \neq 0$, i.e. $CndH_2 \neq 0$.

When $CndH_2 = 0$, we have

$$\gamma_p^1 = -\frac{a_{44}}{a_{21}}t^3 - \frac{a_{31}a_{44}^2 - a_{21}a_{44}a_{54} + a_{21}^2 a_{77}}{a_{21}^3}t^6 + \frac{\vartheta_p^2 CndH_3}{a_{21}^4}t^7 + O(8).$$

For γ_{2p}, we replace ϑ_p^2 by ϑ_{2p}^2 in γ_p. It follows that $\mathcal{D}_p \cdot \mathcal{D}_{2p} = 7$ if and only if $CndH_3 \neq 0$.

(c) If $k = 4p$ and $3 \nmid p$, we have $\mathcal{D}_j \cdot \mathcal{D}_{j'} = 3$ for j or j' not in $\{p, 2p, 3p\}$.

The parametrization of $\mathcal{D}_p$ becomes $\gamma_p(t) = a_{55}/a_{21}t^4 + \vartheta_p^2(a_{32}a_{55} - a_{21}a_{66})/a_{21}^2 t^5 + O(6)$, and similarly for γ_{2p} and γ_{3p} replacing p by $2p$ and $3p$ respectively. Consequently, the branches $\mathcal{D}_p, \mathcal{D}_{2p}, \mathcal{D}_{3p}$ have pairwise order of contact 5 if and only if $a_{32}a_{55} - a_{21}a_{66} \neq 0$, i.e. $CndQm_5 \neq 0$.

When $CndQm_5 = 0$, $\gamma_p = -a_{55}/a_{21}t^4 - \vartheta_{2p}\vartheta_{3p}(a_{21}a_{77} - a_{43}a_{55})/a_{21}^2 t^6 + O(7)$, with similar adjustments as above for γ_{2p} and γ_{3p}. Therefore, the three exceptional branches have pairwise order of contact 6 if and only if $a_{21}a_{77} - a_{43}a_{55} \neq 0$, i.e. $CndQm_6 \neq 0$.

(d) This follows by lemma 3.6 and (b) and (c) above.

The contact between the branches determines $\mu(\mathcal{D})$, the topological types and their associated strata. It remains to compute T for each normal form.

For $\mathbf{Q}_3^k$ we have $\lambda_j = x + \vartheta_{4j}y^3 + \vartheta_{5j}y^5 + \vartheta_{6j}y^5$, and $\lambda_{j,j'} = (\vartheta_{4j} - \vartheta_{4j'})y^2 + (\vartheta_{5j} - \vartheta_{5j'})y^3 + (\vartheta_{6j} - \vartheta_{6j'})y^4$. Using the properties of ϑ_{sj} in lemma 3.6, we have $T_{j,j'} = 2$ unless $k = 3p$ and $j, j' \in \{p, 2p\}$, or $k = 4p$ and $j, j' \in \{p, 2p, 3p\}$. In the first case we get $T_{j,j'} = 3$ and in the second $T_{j,j'} = 4$. The invariant for the germ

$\mathbf{Q}_4^k$ differs from $\mathbf{Q}_3^k$ only when k is divisible by 3. For $k = 3p$, we have $T_{p,2p} = 6$ and $T_{j,j'} = 4$ if $k = 4q$ and $j, j' \in \{q, 2q, 3q\}$. All other indices j and j' give $T_{j,j'} = 2$. Similarly, for the germ $\widetilde{\mathbf{Q}}_4^k$ and for $k = 4p$, we get $T_{j,j'} = 5$ if $j, j' \in \{p, 2p, 3p\}$. If $k = 3q$, then $T_{q,2q} = 3$. All other indices j and j' give $T_{j,j'} = 2$. □

REMARK 4.17. When k is divisible by 12, the germs $\mathbf{Q}_4^k$ and $\widetilde{\mathbf{Q}}_4^k$ have the same invariants $C, T, \mu(D)$ and $r(D)$ but they are not topologically equivalent as their associated sets of contacts between double-points branches are distinct (see [**33**]).

PROPOSITION 4.18. *Suppose that $a_{11} = a_{22} = a_{33} = a_{44} = 0$ and $a_{21}a_{55} \neq 0$. Any k-folding map-germ F_k in case* (a) *or satisfying the additional conditions in* (b) *or* (c) *is finitely $\mathcal{A}$-determined and is topologically equivalent to*

$$\mathbf{R}_4^k \colon (x,y) \mapsto (x, y^k, xy + y^5 + y^6 + y^7).$$

The invariants associated to the germs in the stratum are as in table 5. We have $\mathcal{D}_j \cdot \mathcal{D}_{j'} = 4$ except for the exceptional pairs of branches below.

(a) *If $4 \nmid k$ and $5 \nmid k$, there are no additional conditions and no exceptional branches.*

(b) *If $k = 4p$, $5 \nmid k$ and $CndQm_5 \neq 0$, then $\mathcal{D}_j \cdot \mathcal{D}_{j'} = 5$ for all distinct pairs with j, j' in $J = \{p, 2p, 3p\}$.*

(c) *If $k = 5p$, $4 \nmid k$ and $CndRm_5 \neq 0$, $\mathcal{D}_j \cdot \mathcal{D}_{j'} = 5$ for all distinct pairs with j, j' in $J = \{p, 2p, 3p, 4p\}$.*

(d) *If $k = 20p$, $CndQm_5 \neq 0$ and $CndRm_5 \neq 0$, then $\mathcal{D}_j \cdot \mathcal{D}_{j'} = 5$ for all distinct pairs with j, j' in $J = \{5p, 10p, 15p\}$ or in $J' = \{4p, 8p, 12p, 16p\}$.*

Proof. The branches $\mathcal{D}_j$ can be parametrized by $t \mapsto (\gamma_j(t), t)$ with

$$\gamma_j(t) = -\frac{1}{a_{21}}\vartheta_{5j}a_{55}t^4 + \frac{1}{a_{21}^2}(\vartheta_{2j}\vartheta_{5j}a_{32}a_{55} - \vartheta_{6j}a_{21}a_{66})t^5$$

$$+ \frac{1}{a_{21}^3}(a_{21}(\vartheta_{2j}\vartheta_{6j}a_{32}a_{66} - \vartheta_{7j}a_{21}a_{77})$$

$$- ((\vartheta_{2j})^2 a_{32}^2 - \vartheta_{3j}a_{21}a_{43})\vartheta_{5j}a_{55})t^6 + O(7).$$

(a) If $4 \nmid k$ and $5 \nmid k$, then $\vartheta_{5j} \neq \vartheta_{5j'}$ for all j, j' with $j \neq j'$ (by lemma 3.6). Therefore, $\mathcal{D}_j \cdot \mathcal{D}_{j'} = 4$ for all distinct pairs.

(b) If $k = 4p$ and $5 \nmid k$, then $\vartheta_{5p} = 1$, $\vartheta_{6p} = \vartheta_{2p}$, and we can write $\gamma_p(t) = -a_{55}/a_{21}t^4 + \vartheta_{2p}CndQm_5/a_{21}^2 t^5 + O(6)$. We get similarly $\gamma_{2p}(t)$ and $\gamma_{3p}(t)$ by substituting ϑ_{2p} by, respectively, $\vartheta_{2(2p)}$ and $\vartheta_{2(3p)}$ in γ_p. The result follows as $\vartheta_{2s} \neq \vartheta_{2s'}$ for $s, s' \in \{p, 2p, 3p\}$, $s \neq s'$.

(c) If $k = 5p$ and $4 \nmid k$, then $\vartheta_{5p} = 0$, $\vartheta_{6p} = 1$, $\vartheta_{7p} = \vartheta_{2p}$, so $\gamma_p(t) = -a_{66}/a_{21}t^5 + \vartheta_{2p}CndRm_5/a_{21}^2 t^6 + O(7)$. The expressions γ_{sp}, $s = 2, 3, 4$, are obtaining by substituting ϑ_{2p} by $\vartheta_{2(sp)}$ in γ_p.

(d) The case $k = 20p$ follows in a similarly way to that of proposition 4.16(d).

The calculations for T are similar to those in the proof of proposition 4.16. We obtain $T_{j,j'} = 4$ if $k = 4p$ and $j, j' \in \{p, 2p, 3p\}$, and $T_{j,j'} = 5$ if $k = 5q$ and $j, j' \in \{q, 2q, 3q, 4q\}$. All remaining indices j and j' give $T_{j,j'} = 3$. □

REMARK 4.19. The singularities $\mathbf{Q}_3^4$ and $\mathbf{R}_4^4$ have the same invariants, but the contact of their double-point branches shows that they are not topologically equivalent.

4.2.4. Branch 4: $a_{11} = a_{21} = a_{22} = 0$

THEOREM 4.20. *The strata of codimension $\leqslant 4$ of finitely $\mathcal{A}$-determined k-folding map-germs in the branch $a_{11} = a_{22} = a_{21} = 0$ are those given in table 6. The invariants of the germs in each stratum are given in table 7.*

Proof. The result follows from propositions 4.22, 4.23, 4.25, 4.26 and 4.27. □

The functions defining the double-point branches $\mathcal{D}_j$ have the following initial terms:

$$\lambda_j(x, y) = a_{31} x^2 + a_{32} \vartheta_{2j} xy + a_{33} \vartheta_{3j} y^2 + O(3).$$

Consequently, the branches $\mathcal{D}_j$ are singular. A branch $\mathcal{D}_j$ has an A_1-singularity unless the discriminant Δ_j^k of the quadratic part of λ_j vanishes. We have

$$\Delta_j^k = (a_{32}^2 - 4a_{31}a_{33})\xi^{2j} + 2(a_{32}^2 - 2a_{31}a_{33})\xi^j + a_{32}^2 - 4a_{31}a_{33}. \qquad (4.1)$$

An A_1-singularity is a transverse intersection of two regular curves. Two branches $\mathcal{D}_j$ and $\mathcal{D}_{j'}$ with an A_1-singularity may have one or both of their components being tangential (i.e. the tangent cones of the two branches have a non-trivial intersection). Taking the resultant of $j^2\lambda_j$ and $j^2\lambda_{j'}$ with respect to one of the variables, we find that this happens if and only if $a_{31}a_{33} = 0$ or

$$\Omega_{j,j'}^k = a_{31}a_{33}(1 + \xi^j + \xi^{j'})^2 - a_{32}^2(\xi^j + \xi^{j'} + \xi^{j+j'}) = 0. \qquad (4.2)$$

We have $1 + \xi^j + \xi^{j'} = 0$ or $\xi^j + \xi^{j'} + \xi^{j+j'} = 0$ if and only if $k = 3p$ and $j, j' \in \{p, 2p\}$. Therefore, if $3 \nmid k$ or if $k = 3p$ and $j, j' \notin \{p, 2p\}$, $V(\Omega_{j,j'}^k)$ is a codimension 1 algebraic variety in $J^l(2, 1)$, for $l \geqslant 3$. For such pairs, we set

$$\alpha_{j,j'} = \frac{\xi^j + \xi^{j'} + \xi^{j+j'}}{(1 + \xi^j + \xi^{j'})^2} \quad \text{and} \quad \alpha = \frac{a_{32}^2}{a_{31}a_{33}}. \qquad (4.3)$$

We have the following properties of Δ_j^k and $\Omega_{j,j'}^k$ when $a_{31}a_{33} \neq 0$; the case $a_{31}a_{33} = 0$ is dealt with in propositions 4.26 and 4.27.

PROPOSITION 4.21. *Suppose that $a_{31}a_{33} \neq 0$ and that $3 \nmid k$ or $k = 3p$ and $j, j' \notin \{p, 2p\}$. Then:*

(1) $\Delta_j^k = \xi^{2j}\Delta_{k-j}^k$, *and if $\Delta_j^k = 0$ then $\Delta_s^k \neq 0$ for $s \notin \{j, k - j\}$, so the solutions of $\Delta_j^k = 0$ in the kth-roots of unity come in pairs.*

(2) For Δ_j^k to vanish requires α in (4.3) to belong to the real semi-line $(-\infty, 3)$.

(3) $\alpha_{j,j'} = \alpha_{j',j}$ *for all pairs* (j, j'). *We have* $\alpha_{j,j'} = \alpha_{k-j,j'-j} = \alpha_{k-j',j-j'}$ *and the pairs* $(j, j'), (k - j, j' - j), (k - j', j - j')$ *are pairwise distinct. Furthermore,* $\alpha_{l,q}^k = \alpha_{j,j'}^k$ *if and only if* (l, q) *or* (q, l) *is one of those 3 pairs.*

(4) $\alpha_{j,j'}$ *is real if and only if* $j' = k - j$ *or* $j' = 2j$. *In that case, by* (3), $\alpha_{k-j,2j}$ *is also real. Then,* $\alpha_{j,k-j} = \alpha_{j,2j} = \alpha_{k-j,2j} = (1 + \xi^j + \xi^{k-j})^{-1}$.

(5) *If* α *is real then* $\Omega_{j,j'}^k = 0$ *if and only if* $j' = k - j$ *or* $j' = 2j$. *Then by* (3), *we also have* $\Omega_{k-j,2j}^k = 0$.

(6) *If* α *is real, then* $\Omega_{j_0,k-j_0}^k = 0$ *implies* $\Delta_j^k \neq 0$ *for all* j. *Conversely, if* $\Delta_{j_0}^k = 0$, *then* $\Omega_{j,k-j}^k \neq 0$ *for all* j.

(7) *If* $\Delta_j^k \Delta_{j'}^k \neq 0$ *and* $\Omega_{j,j'}^k \neq 0$ *for all* j, j' *with* $j' \neq j$, *then* $\mathcal{D}_j \cdot \mathcal{D}_{j'} = 4$.

Proof. (1) As $\xi^{-j} = \xi^{k-j}$, factoring out ξ^{2j} in (4.1) gives $\Delta_j^k = \xi^{2j} \Delta_{k-j}^k$.

If $a_{32}^2 - 4a_{31}a_{33} = 0$, then for Δ_j^k to vanish requires $a_{32}^2 - 2a_{31}a_{33} = 0$. This would imply $a_{31}a_{33} = 0$. Therefore, under the hypothesis of the proposition, we can assume that $a_{32}^2 - 4a_{31}a_{33} \neq 0$. Then

$$\Delta_j^k = 0 \iff \xi^{2j} + \frac{2(\alpha - 2)}{\alpha - 4}\xi^j + 1 = 0$$

with α as in (4.3). If ξ^j is a solution of the above quadratic equation, then so is $\xi^{k-j} = \xi^{-j}$. Therefore, for α fixed, if $\Delta_j^k = 0$, then $\Delta_s^k \neq 0$ for $s \notin \{j, k - j\}$.

(2) When $a_{31}a_{33} \neq 0$, we can write $\Delta_j^k = a_{31}a_{33}((1 + \xi^j)^2\alpha - 4(1 + \xi^j + \xi^{2j}))$. Clearly, $\Delta_j^k \neq 0$ when $\xi^j = -1$. Thus, $\Delta_j^k = 0$ if and only if $\alpha = 4(1 + \xi^j + \xi^{2j})/(1 + \xi^j)^2$, which shows that if $\Delta_j^k = 0$ then α must be real. The discriminant of the quadratic equation $(\xi^j + 1)^2\alpha - 4(\xi^{2j} + \xi^j + 1) = 0$ in ξ^j is $4(\alpha - 3)$, so $\alpha < 3$ as the solutions ξ^j are not real.

(3) Clearly as $\alpha_{j,j'} = (\xi^j + \xi^{j'} + \xi^{j+j'})/((1 + \xi^j + \xi^{j'})^2)$, we have $\alpha_{j,j'} = \alpha_{j',j}$. Factoring our ξ^{2j} (resp. $\xi^{2j'}$) from the numerator and denominator gives $\alpha_{k-j,j'-j} = \alpha_{k-j,j'-j} = \alpha_{k-j',j-j'}$.

We now seek pairs (l, q) for which $\alpha_{l,q} = \alpha_{j,j'}$. We know from the above that we have at least three such pairs. To show that these are the only ones, we write $\alpha_{j,j'} = c + id$ (so $(c, d) \neq (0, 0)$) and represent points on the unit circle, with -1 removed, in the form $z = (1 - t^2)/(1 + t^2) + i(2t/(1 + t^2))$ and $w = (1 - s^2)/(1 + s^2) + i(2s/(1 + s^2))$, with $t, s \in \mathbb{R}$. We set

$$\alpha_{z,w} = \frac{z + w + zw}{(1 + z + w)^2}.$$

The real and imaginary parts of $\alpha_{z,w} - (c + id)$ vanish if and only if $P_{(c,d)}(s, t) = Q_{(c,d)}(s, t) = 0$, where

$$\begin{aligned}
P_{(c,d)}(s, t) &= (cs^4 + 4ds^3 + s^4 - 6cs^2 - 4ds + 2s^2 + c + 1)t^4 \\
&\quad + 4(s^2 + 1)(ds^2 - 2cs - d + s)t^3 - 2(3cs^4 - s^4 + 10cs^2 + 8ds - c + 1)t^2 \\
&\quad - 4(s^2 + 1)(ds^2 + 2cs + 3d - s)t \\
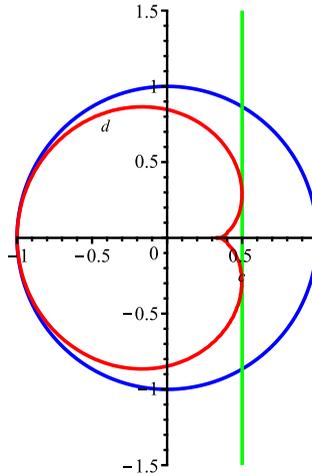&\quad + cs^4 - 4ds^3 + s^4 + 2cs^2 - 12ds - 2s^2 + 9c - 3,
\end{aligned}$$

Figure 3. Red curve is the discriminant of $R_{(c,d)}$, the blue curve is the unit circle and the green lines are the values of $\alpha_{p,w}$ and $\alpha_{2p,w}$ when $k = 3p$.

$$Q_{(c,d)}(s,t) = -(ds^4 - 4cs^3 - 6ds^2 + 4cs + d)t^4 + 4(s^2 + 1)(cs^2 + 2ds - c + 1)t^3$$
$$+ 2(3ds^4 + 10ds^2 + 2s^3 - 8cs - d + 2s)t^2$$
$$- 4(s^2 + 1)(cs^2 - 2ds + 3c - 1)t - ds^4 - 4cs^3 - 2ds^2$$
$$+ 4s^3 - 12cs - 9d + 4s.$$

Observe that $P_{(c,d)}$ and $Q_{(c,d)}$ are symmetric polynomials. Their resultant with respect to t vanishes if and only if $s^2 = 3$ or $R_{(c,d)}(s) = 0$, with

$$R_{(c,d)}(s) = (c^2 + d^2 - 1)^2 s^6 - (9c^4 + 18c^2 d^2 + 9d^4 + 2c^2 + 2d^2 + 8c - 3)s^4$$
$$+ (27c^4 + 54c^2 d^2 + 27d^4 + 18c^2 + 18d^2 - 16c + 3)s^2$$
$$- 27c^4 - 54c^2 d^2 - 27d^4 + 18c^2 + 18d^2 - 8c + 1.$$

We have $s^2 = 3$ if and only if $k = 3p$ and $w = \xi^p$ or $w = \xi^{2p}$, and this is excluded from the hypotheses. The component $R_{(c,d)}$ of the resultant is a cubic polynomial in s^2 provided $c^2 + d^2 - 1 \neq 0$, i.e. $|\alpha_{j,j'}| \neq 1$. Suppose that $|\alpha_{j,j'}| \neq 1$. Then the discriminant of $R_{(c,d)}$ vanishes if and only if $d = 0$ (i.e. $\alpha_{j,j'}$ is real and this is treated in (4) below) or

$$\delta_R = 27c^4 + 54c^2 d^2 + 27d^4 - 18c^2 - 18d^2 + 8c - 1 = 0.$$

For (c, d) in the interior region bounded by the curve $\delta_R = 0$ (see figure 3), $R_{(c,d)}$ has a unique solution in s^2. As we know that there are at least three distinct solution pairs to the problem, it follows that (c, d) must be in the exterior region (R^+) bounded by the curve $\delta_R = 0$.

Observe that $s = 0$ is a root of $R_{(c,d)}$ if and only if $\delta_R = 0$. Therefore, the roots of $R_{(c,d)}$ in s^2 do not change sign in (R^+). Choosing any point in that region, we

find that they are all positive. It follows that $R_{(c,d)}$ has six roots $\pm s_1, \pm s_2, \pm s_3$. These correspond to six points on the unit circle w_1, w_2, w_3 and $\overline{w}_1, \overline{w}_2, \overline{w}_3$.

For each root of $R_{(c,d)}$ we show, by considering the subresultant (see e.g. [**18**]) of $P_{(c,d)}$ and $Q_{(c,d)}$ that $P_{(c,d)}(\pm s_i, t)$ and $Q_{(c,d)}(\pm s_i, t)$ have only one common root. As $P_{(c,d)}$ and $Q_{(c,d)}$ are symmetric polynomials, that common root is a root of $R_{(c,d)}$. Interchanging w_i with $\overline{w}_i$ if necessary, we can set $w_1 = \xi^j$, $w_1 = \xi^{k-j}$, $w_1 = \xi^{k-j'}$, Then the solutions of $\alpha_{z,w} - c - id = 0$ are exactly, up to permutation of z and w, $(\xi^j, \xi^{j'})$, $(\xi^{k-j}, \xi^{j'-j})$, $(\xi^{k-j'}, \xi^{j-j'})$.

We turn now to the case when $|\alpha_{j,j'}| = 1$, i.e. $c^2 + d^2 - 1 = 0$. This occurs if and only if $2 \mid k$ and $j' = j + k/2$ or $j' = k/2$. Suppose that $2 \mid k$ and $j' = j + k/2$. Then $\alpha_{j,j+k/2} = -\xi^{2j}$, and $\alpha_{l,l+k/2} = \alpha_{j,j+k/2}$ if, and only if, $l = j$ or $l = j + k/2$. In both cases, we get only the pair $(j, j + k/2)$. Now $\alpha_{k/2,l} = -\xi^{-2l}$, so $\alpha_{k/2,l} = \alpha_{j,j+k/2}$ if and only if $l = k - j$ or $l = k/2 - j$. This shows that $\alpha_{j,j+k/2} = \alpha_{k-j,k/2} = \alpha_{k/2-j,k/2}$ and the equality $\alpha_{j,j+k/2} = \alpha_{l,q}$ holds only for these three pairs.

(4) We can write $\alpha_{j,j'}$ in the form

$$\alpha_{j,j'} = \frac{6(1 + \Re(\xi^j) + \Re(\xi^{j'}) + \Re(\xi^{j'-j})) + \xi^{j-2j'} + \xi^{j+j'} + \xi^{j'-2j}}{|1 + \xi^j + \xi^{j'}|^2}.$$

Therefore, $\alpha_{j,j'}$ is real if and only if $\zeta = \xi^{j-2j'} + \xi^{j+j'} + \xi^{j'-2j}$ is real. Setting $\theta = j - 2j'$ and $\phi = j' - 2j$, we get $\Im(\zeta) = \sin(2\pi\theta/k) + \sin(2\pi\phi/k) - \sin((2\pi(\theta+\phi))/k)$. Then

$$\Im(\zeta) = 0 \iff \sin\left(\frac{2\pi\theta}{k}\right) + \sin\left(\frac{2\pi\phi}{k}\right) = \sin\left(\frac{2\pi(\theta+\phi)}{k}\right)$$
$$\iff 2\sin\left(\frac{\pi(\theta+\phi)}{k}\right)\cos\left(\frac{\pi(\theta-\phi)}{k}\right)$$
$$= 2\sin\left(\frac{\pi(\theta+\phi)}{k}\right)\cos\left(\frac{\pi(\theta+\phi)}{k}\right).$$

Now, $\sin((\pi(\theta+\phi))/k) = 0$ when $j' = k - j$ and $\cos((\pi(\theta-\phi))/k) = \cos((\pi(\theta+\phi)/k)$ when $j' = 2j$ or $j = 2j'$. Clearly, $\alpha_{j,k-j} = 1/(1 + \xi^j + \xi^{k-j}) = \alpha_{j,2j}$.

(5) As $a_{31}a_{33} \neq 0$, we can write $\Omega_{j,j'}^k = a_{31}a_{33}(1 - \alpha_{j,j'}\alpha)$, so $\Omega_{j,j'}^k = 0$ if and only if $\alpha_{j,j'} = 1/\alpha$ and the statement follows by (4).

(6) When α is real, by (4), we have $\Omega_{j_0,k-j_0} = 0$ when $\xi_0^j + \xi^{k-j_0} + 1 = \alpha$. It follows that $\xi_0^j = \eta_1$ with $\eta_1 = (\alpha - 1)/2 \pm i\sqrt{1 - ((\alpha-1)/2)^2}$. Suppose that there exist a j for which $\Delta_j^k = 0$, equivalently, $\xi^{2j} + (2(\alpha-2)/(\alpha-4))\xi^j + 1 = 0$. Then $\xi^j + \xi^{k-j} = -((2\alpha-2)/(\alpha-4))$ which gives $\xi_j = \eta_2$ with $\eta_2 = -((\alpha-2)/(\alpha-4)) \pm i\sqrt{1 - ((\alpha-2)/(\alpha-4))^2}$. We have $\eta_1 = \eta_2$ if and only if $\alpha = 0$ or 3. Then $k = 3p$ and $j_0 = p$ or $2p$, which is excluded from our hypotheses, so $\eta_1 \neq \eta_2$.

The complex number η_1 (resp. η_2) is a kth-root of unity if and only if α is a root of the polynomial $P_1(\alpha)$ (resp. $P_2(\alpha)$) of degree k obtained by taking the numerator of $\Re(\eta_1^k) - 1$ (resp. $\Re(\eta_2^k) - 1$). But if P_1 and P_2 have one common root, all the other roots must also be common. (This follows from the fact that the map $\cos\theta = -(\alpha-2)/(\alpha-4)$ is a bijection for $\alpha \in (-\infty, 3]$ and $\cos\theta = (\alpha-1)/2$

is also a bijection for $\alpha \in (-1, 3]$.) As P_1 and P_2 are distinct polynomials, it follows that they have no common roots. Consequently, $\Delta_j^k \neq 0$ for all j.

The argument for showing that $\Omega_{j,k-j}^k \neq 0$ when $\Delta_{j0}^k = 0$ is the same as above.

(7) We have, with the hypothesis, $\mathcal{D}_j \cdot \mathcal{D}_{j'} = \dim_{\mathbb{C}} \mathcal{O}_2 / \langle xy, x^2 + y^2 \rangle = 4$. $\qquad\square$

In view of proposition 4.21(7), we give in the rest of this section $\mathcal{D}_j \cdot \mathcal{D}_{j'}$ for the exceptional branches only, i.e. when one or both branches have a singularity more degenerate than A_1 or one or both of their components are tangential (which is equivalent to $\mathcal{D}_j \cdot \mathcal{D}_{j'} > 4$). We start with the case when all the branches have an A_1-singularity.

PROPOSITION 4.22. *Suppose that $a_{11} = a_{21} = a_{22} = 0$, $a_{31}a_{33} \neq 0$ and $\Delta_j^k \neq 0$ for all j. Any k-folding map-germ F_k satisfying the additional conditions in* (a) *or* (b) *is finitely $\mathcal{A}$-determined and is topologically equivalent to one of the following germs:*

$$\mathbf{U}_l^k \colon (x, y) \mapsto (x, y^k, x^2y + 2xy^2 + y^3 + y^{3l-5}), \quad l = 3, 4.$$

Every branch of the double-point curve has an A_1-singularity and the invariants associated to the germs in these strata are as in table 7.

(a) *If $3 \nmid k$ and $\Omega_{j,j'}^k \neq 0$ for all distinct pairs, then F_k is topologically equivalent to $\mathbf{U}_3^k$.*

(b) *Suppose that $k = 3p$ and $\Omega_{j,j'}^k \neq 0$ for all distinct pairs with $j, j' \neq \{p, 2p\}$.*
    If $a_{44} \neq 0$, then $\mathcal{D}_p \cdot \mathcal{D}_{2p} = 5$ and F_k is topologically equivalent to $\mathbf{U}_3^k$.

    If $a_{44} = 0$ and $CndUm_8 \neq 0$, then $\mathcal{D}_p \cdot \mathcal{D}_{2p} = 8$ and F_k is topologically equivalent to $\mathbf{U}_4^k$.

Proof. Each branch of the double-point curve consists of a transverse intersection of two regular curves. In (a) all of these curves are pairwise transverse.

In (b), $j^2\lambda_s(x, y) = x(a_{31}x + a_{32}\vartheta_{2s}y)$ for $s = p, 2p$. Observe that $a_{32} \neq 0$ as we supposed $\Delta_s^k \neq 0$, so the branches $\mathcal{D}_p$ and $\mathcal{D}_{2p}$ have one common line $x = 0$ in their tangent cone and their associated curves tangent to this line are parametrized by $t \mapsto (\gamma_s(t), t)$ with $\gamma_s(t) = -(a_{44}/(\vartheta_{2s}a_{32}))t^2 + O(3)$, so they have order of contact 2 when $a_{44} \neq 0$. Then $\mathcal{D}_p \cdot \mathcal{D}_{2p} = \dim_{\mathbb{C}} \mathcal{O}_2 / \langle xy, x^2 + y^3 \rangle = 5$.

When $a_{44} = 0$, the two curves are parametrized by $t \mapsto (\gamma_s(t), t)$ with

$$\gamma_s(t) = -\frac{a_{55}}{a_{32}}t^3 + \frac{2}{\vartheta_{2s}a_{32}^3}(a_{55}(a_{31}a_{55} - a_{54}a_{32}) + a_{77}a_{32}^2)t^5 + O(6),$$

and have contact order 5 when $(a_{55}(a_{31}a_{55} - a_{54}a_{32}) + a_{77}a_{32}^2 \neq 0$, i.e.$CndUm_8 \neq 0$. Then $\mathcal{D}_p \cdot \mathcal{D}_{2p} = \dim_{\mathbb{C}} \mathcal{O}_2 / \langle xy, x^2 + y^6 \rangle = 8$.

For the topological normal forms, we choose a_{31}, a_{32}, a_{33} real with $\alpha > 3$. By proposition 4.21(2), all Δ_j^k are non-zero. Also, by proposition 4.21(4) and (5) $\Omega_{j,k-j} = 0$ when $\xi^j + \xi^{k-j} + 1 = \alpha$, which cannot happen when $\alpha > 3$, so the $\Omega_{j,j'}^k$ in (a) and (b) are non-zero. As the strata are connected sets, we can choose $a_{31} = a_{32} = 1, a_{32} = 2$ so that the conditions on Δ_j^k and $\Omega_{j,j'}^k$ are satisfied.

The number of triple points is calculated as in proposition 4.16. If $k = 3p$, the germ $\mathbf{U}_3^k$ (resp. $\mathbf{U}_4^k$) has $T_{p,2p} = 3$ (resp. $T_{p,2p} = 6$), while all other indices j and j' give $T_{j,j'} = 2$. $\qquad \square$

PROPOSITION 4.23. *Suppose that* $a_{11} = a_{21} = a_{22} = 0$, $a_{31}a_{33} \neq 0$, $a_{32} \neq 0$ *and* $\Omega_{j_0,j_0'}^k = 0$ *for some pair* (j_0, j_0'), *with* $j_0, j_0' \notin \{p, 2p\}$ *when* $k = 3p$. *Then* $\mathcal{D}_s \cdot \mathcal{D}_q = 5$ *for* $(s,q) \in \{(j_0, j_0'), (k - j_0, j_0' - j_0), (k - j_0', j_0 - j_0')\}$ *if and only if* $CndVm_{5j_0,j_0'} \neq 0$. *For* $k = 3p$, *we have* $\mathcal{D}_p \cdot \mathcal{D}_{2p} = 5$ *if and only if* $a_{44} \neq 0$. *Any* k-*folding map-germ* F_k *satisfying the above conditions is finitely* $\mathcal{A}$-*determined and is topologically equivalent to*

$$\mathbf{V}_4^{k,j_0,j_0'} : (x,y) \mapsto \left(x, y^k, x^2 y + xy^2 + \alpha_{j_0,j_0'} y^3 + \left(1 - \frac{\beta_{j_0,j_0'}}{\alpha_{j_0,j_0'}^3}\right) x^3 y + y^4\right).$$

The invariants associated to germs in these strata are as in table 7.

Proof. When $a_{32} = 0$ and $k = 3p$, we have $\Delta_p^k = \Delta_{2p}^k = 0$. This case is dealt with in proposition 4.25. With the hypothesis and proposition 4.21(5), we have $\Omega_{j_0,j_0'} = \Omega_{k-j_0,j_0'-j_0} = \Omega_{k-j_0',j_0-j_0'} = 0$.

We need to consider the order of contact between the two tangential components of the pairs $(\mathcal{D}_s, \mathcal{D}_q)$ with $(s,q) \in \{(j_0, j_0'), (k - j_0, j_0' - j_0), (k - j_0', j_0 - j_0')\}$. Using the fact that if two quadratic equations $x^2 + a_i x + b_i = 0$, $i = 1, 2$, have one root in common, then the root is given by $x = -(b_1 - b_2)/(a_1 - a_2)$, we can get the initial terms of parametrizations of the tangential components of the above pairs. These are given by $t \mapsto (t, \gamma_l(t))$, $l = s, q$, with $\gamma_l(t) = -a_{33}/a_{32}(1 + \xi^s + \xi^q)t + \lambda_l t^2 + O(3), l = s, q$. A calculation shows that $\lambda_s - \lambda_q = 0$ if and only if

$$CndVm_{5s,q} = a_{32}^2 a_{44} \beta_{s,q} + a_{32}a_{33}(2a_{33}a_{42} - a_{32}a_{43})\alpha_{s,q} + a_{41}a_{33}^3 = 0,$$

with

$$\beta_{j,j'} = \frac{\xi^{j+3j'} + \xi^{3j+j'} + \xi^{3j'} + 2\xi^{2j+j'} + 2\xi^{j+2j'} + \xi^{3j} + 2\xi^{j+j'} + \xi^j + \xi^{j'}}{(1 + \xi^j + \xi^{j'})^4}. \quad (4.4)$$

Observe that $\beta_{j,j'} = \beta_{k-j,j'-j} = \beta_{k-j',j-j'}$, so $CndVm_{5s,q}$ has the same value for $(s,q) \in \{(j_0, j_0'), (k - j_0, j_0' - j_0), (k - j_0', j_0 - j_0')\}$. It follows that $\mathcal{D}_s \cdot \mathcal{D}_q = \dim_{\mathbb{C}} \mathcal{O}_2/\langle xy, x^2 + y^3 \rangle = 5$ if and only if $CndVm_{5j_0,j_0'} \neq 0$.

For the topological model, we take $a_{31} = a_{32} = 1$ and $a_{33} = \alpha_{j_0,j_0'}$ so $\Omega_{j_0,j_0'} = 0$. We also set $a_{44} = 1$, $a_{42} = a_{43} = 0$, then $CndVm_{5j_0,j_0'} = \beta_{j_0,j_0'} + a_{41}\alpha_{j_0,j_0'}^3$. We set $a_{41} = 1 - \beta_{j_0,j_0'}/\alpha_{j_0,j_0'}^3$, so that $CndVm_{5j_0,j_0'} \neq 0$. For calculating the triple points, we find that $T_{j,j'} = 3$ for all j and j' with $j \neq j'$. $\qquad \square$

We deal now with the case when one pair of branches of the double-point curve has a singularity more degenerate than A_1 (see proposition 4.21(1)).

REMARK 4.24. For $k = 3p$, the singularities $\mathbf{U}_4^k$ and $\mathbf{V}_4^{k,j,j'}$ have the same invariants, but the intersection numbers between their double-point branches shows that they are not topologically equivalent.

PROPOSITION 4.25. *Suppose that $a_{11} = a_{21} = a_{22} = 0$, $a_{31}a_{33} \neq 0$. The strata below are of codimension* 4, *and the invariants associated to germs in the strata are as in table 7.*

(1) *Suppose that $\Delta_{j_0}^k = \Delta_{k-j_0}^k = 0$, with $j_0 \notin \{p, 2p\}$ when $k = 3p$ (so $a_{32} \neq 0$). The branches $\mathcal{D}_{j_0}$ and $\mathcal{D}_{k-j_0}$ have an A_2-singularity if and only if $CndW A_2 \neq 0$. We have $\mathcal{D}_{j_0} \cdot \mathcal{D}_j = \mathcal{D}_{k-j_0} \cdot \mathcal{D}_j = 4$ for all distinct pairs. When $k = 3p$ and $a_{44} \neq 0$, we have $\mathcal{D}_p \cdot \mathcal{D}_{2p} = 5$. Any k-folding map-germ F_k satisfying the above conditions is $\mathcal{A}$-finitely determined and is topologically equivalent to*

$$\mathbf{W}_4^{k,j_0} \colon (x, y) \mapsto \left(x, y^k, x^2 y + xy^2 + \frac{(\xi^{j_0} + 1)^2}{4(\xi^{2j_0} + \xi^{j_0} + 1)} y^3 + 4x^2 y^2 + y^4 \right).$$

(2) *If $k = 3p$ and $j_0 = p$, then $\mathcal{D}_p$ and $\mathcal{D}_{2p}$ have an A_2-singularity if and only if $a_{44} \neq 0$. We have $\mathcal{D}_p \cdot \mathcal{D}_{2p} = 8$ if and only if $Cnd\widetilde{W} m_8 \neq 0$, and $\mathcal{D}_s \cdot \mathcal{D}_j = 5$ for $s = p, 2p$ and $j \neq s$. Any k-folding map-germ F_k satisfying the above conditions is $\mathcal{A}$-finitely determined and is topologically equivalent to*

$$\mathbf{W}_4^{3p,p} \colon (x, y) \mapsto (x, y^k, x^2 y + y^3 + y^4 + y^5).$$

Proof. The condition for an A_2-singularity of $\mathcal{D}_{j_0}$ and $\mathcal{D}_{k-j_0}$ follows by analysing the 3-jets of, respectively, λ_{j_0} and λ_{k-j_0}. Observe that when $2 \mid k$, $\xi^{j_0} = -1$ is a solution of $\Delta_{j_0}^k = \Delta_{k-j_0}^k = 0$ if and only if $a_{31}a_{33} = 0$. As we are assuming $a_{31}a_{33} \neq 0$, we have $j_0 \neq k/2$ in (1) when $2 \mid k$.

With the hypothesis in (1), $\mathcal{D}_{j_0} \cdot \mathcal{D}_j = \mathcal{D}_{k-j_0} \cdot \mathcal{D}_j = \dim_{\mathbb{C}} \mathcal{O}_2 / \langle x^2 + y^3, x^2 - y^2 \rangle = 4$ for any distinct pairs.

For (2), we have $\mathcal{D}_p \cdot \mathcal{D}_{2p} = \dim_{\mathbb{C}} \mathcal{O}_2 / \langle x^2 + y^3, y^4 \rangle = 8$ when $Cnd\widetilde{W} m_8 \neq 0$.

For the topological normal forms, we use the fact that $\Delta_{j_0}^k = 0$ if and only if $\alpha = a_{32}^2 / (a_{31}a_{33}) = 4(\xi^{2j_0} + \xi^{j_0} + 1)/(\xi^{j_0} + 1)^2$ (see the proof of proposition 4.21) and set $a_{31} = a_{32} = 1$, so $a_{33} = (\xi^{j_0} + 1)^2 / 4(\xi^{2j_0} + \xi^{j_0} + 1)$. We have $T_{j,j'} = 2$ for all j and j' with $j \neq j'$.   □

PROPOSITION 4.26. *Suppose that $a_{11} = a_{21} = a_{22} = 0$, $a_{31} = 0$, $a_{33}a_{32} \neq 0$. All the branches of the double-point curve share only one line in their tangent cones except when $k = 3p$ where the branches $\mathcal{D}_p$ and $\mathcal{D}_{2p}$ have the same tangent cone. Any k-folding map-germ F_k satisfying the additional conditions in* (a), (b), (c) *or* (d) *is finitely $\mathcal{A}$-determined and is topologically equivalent to*

$$\mathbf{X}_4^k \colon (x, y) \mapsto (x, y^k, xy^2 + y^3 + x^3 y + y^4).$$

The stratum is of codimension 4, *and the invariants associated to the germs in the strata above are as in table 7.*

(a) *If $2 \nmid k$, $3 \nmid k$, all of the branches $\mathcal{D}_j$ of the double-point curve have an A_1-singularity and $\mathcal{D}_j \cdot \mathcal{D}_{j'} = 5$, $j \neq j'$, when $a_{41} \neq 0$.*

(b) *If $k = 2p$, $3 \nmid k$, the branches $\mathcal{D}_j$, $j \neq p$, behave as in* (a). *The branch $\mathcal{D}_p$ has an A_2-singularity if and only if $a_{41} \neq 0$. We also have $\mathcal{D}_j \cdot \mathcal{D}_p = 5$, $j \neq p$.*

(c) *If $k = 3p$, $2 \nmid k$, the branches $\mathcal{D}_j$ behave as in* (a) *except for $\mathcal{D}_p$ and $\mathcal{D}_{2p}$ where $\mathcal{D}_p \cdot \mathcal{D}_{2p} = 6$ when $a_{41}a_{44} \neq 0$.*

(d) If $k = 6p$, when $a_{41}a_{44} \neq 0$, the exceptional branches of the double-point curve behave as in (b) and (c), the remaining branches behave as in (a).

Proof. We have $j^2\lambda_j(x,y) = y(\vartheta_{2j}a_{32}x + \vartheta_{3j}a_{33}y)$. For (a), $\vartheta_{2j} \neq 0$ for all j and $\vartheta_{2j}/\vartheta_{2j} \neq \vartheta_{3j'}/\vartheta_{2j'}$ for $j \neq j'$, so all of the branches of the double-point curve have an A_1-singularity and one common line $y = 0$ in their tangent cone. The component of $\mathcal{D}_j$ with tangent $y = 0$ can be parametrized by $t \mapsto (t, \gamma_j(t))$ with by $\gamma_j(t) = -a_{41}/((1 + \xi^j)a_{32})t^2 + O(3)$. Therefore, when $a_{41} \neq 0$, we have $\mathcal{D}_j \cdot \mathcal{D}_{j'} = \dim_{\mathbb{C}} \mathcal{O}_2/\langle xy, y^2 + x^3 \rangle = 5$.

The remaining parts of the proof follow similarly and are omitted. $\qquad\square$

PROPOSITION 4.27. *Suppose that $a_{11} = a_{21} = a_{22} = 0$, $a_{33} = 0$, $a_{31}a_{32} \neq 0$. All the branches of the double-point curve share only one line in their tangent cones. Any k-folding map-germ F_k satisfying the additional conditions in (a) or (b) is finitely $\mathcal{A}$-determined and is topologically equivalent to*

$$\mathbf{Y}_4^k\colon (x,y) \mapsto (x, y^k, -xy^2 + x^2y + y^4 + y^5).$$

The stratum is of codimension 4, and the invariants associated to germs in the stratum are as in table 7.

(a) If $2 \nmid k$, all of the branches of the double-point curve have an A_1-singularity and $\mathcal{D}_j \cdot \mathcal{D}_{j'} = 5$, $j \neq j'$, when $a_{44} \neq 0$.

(b) If $k = 2p$, the branch $\mathcal{D}_p$ has an A_3-singularity if and only if $CndY A_3 \neq 0$. Then $\mathcal{D}_p \cdot \mathcal{D}_j = 6$, $j \neq p$, and $\mathcal{D}_j \cdot \mathcal{D}_{p+j} = 6$, $1 \leqslant j < p$, if and only if $CndY m_{6j} \neq 0$. The other pairs of branches behave as in (a).

Proof. We have $j^2\lambda_j(x,y) = x(a_{31}x + a_{32}\vartheta_{2j}y)$ so when $2 \nmid k$, all of the branches of the double-point curve have an A_1-singularity as $\vartheta_{2j} \neq 0$ for all j. All the branches have only $x = 0$ as a common line in their tangent cone. The components of the branches which are tangent to $x = 0$ are parametrized by $t \mapsto (\gamma_j(t), t)$, with $\gamma_j(t) = -a_{44}(1 + \xi^{2j})/a_{32}t^2 + O(3)$. When $a_{44} \neq 0$, $\mathcal{D}_j \cdot \mathcal{D}_{j'} = \dim_{\mathbb{C}} \mathcal{O}_2/\langle xy, x^2 + y^3 \rangle = 5$.

When $k = 2p$, we have $\vartheta_{2p} = \vartheta_{4p} = 0$, so considering the 4-jet of λ_p, we find that the branch $\mathcal{D}_p$ has an A_3-singularity if and only if $CndY A_3 \neq 0$. When this is the case, $\mathcal{D}_p \cdot \mathcal{D}_j = \dim_{\mathbb{C}} \mathcal{O}_2/\langle xy, x^2 + y^4 \rangle = 6$, for $j \neq p$.

All pairs of branches $\mathcal{D}_j$, $\mathcal{D}_{p+j}$, $1 \leqslant j < p$ have tangential component parametrized by $t \mapsto (\gamma_s(t), t)$, with $\gamma_s(t) = -a_{44}(1 + \xi_j^2)/a_{32}t^2 + \alpha_s t^3 + O(4)$, $s = j, p + j$, with $\alpha_1 - \alpha_2 \neq 0$ if and only if $CndY m_{6j} \neq 0$. When this is the case, $\mathcal{D}_j \cdot \mathcal{D}_{p+j} = \dim_{\mathbb{C}} \mathcal{O}_2/\langle xy, x^2 + y^4 \rangle = 6$, for $1 \leqslant j < p$. The other pairs of branches behave as when $2 \nmid k$. $\qquad\square$

REMARK 4.28. The stratification $\mathcal{S}_k$ is determined, for each $k \geqslant 4$, by the conditions in tables 2, 4 and 6 ($\mathcal{S}_2$ follows from the results in [8] and $\mathcal{S}_3$ is indicated in table 1). One can easily check that the strata are smooth manifolds (they are also algebraic sets so in particular, $\mathcal{S}_k$ admits a sub-stratification which is Whitney (a) and (b) regular).

Germs in the following strata have pairwise the same Milnor number of their double-point curves:

Table 5: $\mathbf{Q}_4^k$ and $\tilde{\mathbf{Q}}_4^k$ for $12 \mid k$,
Table 7: $\mathbf{U}_4^k$ and $\mathbf{V}_3^{k,j,j'}$ for $3 \mid k$,
Table 7: $\mathbf{X}_4^k$ and $\mathbf{Y}_4^k$ for $2 \nmid k$ and $3 \nmid k$.

A simple calculation shows that all the Milnor numbers of the double-point curves in tables 3, 5 and 7, except those of the above pairs, are pairwise distinct. Therefore, removing the exceptional pairs, each stratum is precisely the connected component of the jets for which the Milnor number of the double-point curve is constant.

5. Generic singularities of k-folding map-germs

We define, by varying the plane $\pi \in \mathrm{Graff}(2,3)$, the family of Whitney k-folds $\Omega^k : \mathbb{C}^3 \times \mathrm{Graff}(2,3) \to \mathbb{C}^3$, given by $\Omega^k(p,\pi) = \omega_\pi^k(p)$, with ω_π^k as in definition 2.2.

Given a complex surface M in $\mathbb{C}^3$, we call the restriction of Ω^k to M the family of k-folding maps on M and denote it by $\mathcal{F}_k$. We have $\mathcal{F}_k(p,\pi) = F_k^\pi(p) = \omega_\pi^k(p)$ for all $p \in M$ and $\pi \in \mathrm{Graff}(2,3)$.

Recall that a property of surfaces is said to be *generic* if it is satisfied in a residual set of embeddings of the surfaces to $\mathbb{C}^3$. The image of a surface M by an embedding in the residual set is then called generic, or simply that M is generic. When k is large, the $\mathcal{A}$-singularities of F_k^π may have high $\mathcal{A}_e$-codimensions (for the cases in this paper, this means high modality). However, they do occur on generic surfaces. To make sense of this, we follow a similar approach to that in [**5**] and proceed as follows.

As we are interested here in the local singularities of k-folding maps, we consider the setting in remark 2.3(4) at a point $p_0 \in M$ and choose a suitable system of coordinates so that $\pi_0 : y = 0$ and $F_k = F_k^{\pi_0}(x,y) = (x,y^k,f(x,y))$ for (x,y) in a small enough neighbourhood U of the origin. A plane $\pi = \overline{(d,v)}$ near π_0 is obtained by applying a translation T_π followed by an orthogonal transformation $R_\pi \in U(3)$ to π_0. We choose T_π and R_π as follows. The translation T_π takes the origin to the point of intersection of π with the y-axis (the point exists because π_0 is orthogonal to the y-axis and π is close to π_0). The transformation R_π near the identity (and is taken to be the identity if v is parallel to v_0) takes $v/\|v\|$ to $v_0 = (0,1,0)$ and fixes the line through the origin orthogonal to v_0 and v. By varying the planes π in a neighbourhood V of π_0 in $\mathrm{Graff}(2,3)$, we get the family of k-folding maps given by $\mathcal{F}_k : U \times V \to \mathbb{C}^3$, with

$$\mathcal{F}_k((x,y),\pi) = (R_\pi \circ T_\pi) \circ \omega_k \circ (R_\pi \circ T_\pi)^{-1}(\phi(x,y)),$$

with $\phi(x,y) = (x,y,f(x,y))$. (We choose R_π and T_π to depend analytically on π.) The $\mathcal{A}$-type of the singularity of F_k^π at a given point in U is the same as that of the germ of

$$\tilde{\mathcal{F}}_k((x,y),\pi) = \tilde{F}_\pi^k(x,y) = (R_\pi \circ T_\pi)^{-1} \circ \mathcal{F}_k((x,y),\pi) = \omega_k \circ (R_\pi \circ T_\pi)^{-1}(\phi(x,y))$$

at that point. At any point $p = (x,y) \in U$, there exist a bi-holomorphic map-germ $K : (\mathbb{C}^2,0) \to (\mathbb{C}^2,(x,y))$ such that

$$(R_\pi \circ T_\pi)^{-1}(\phi(K(X,Y)) = (\bar{x} + X, \bar{y} + Y, \bar{z} + g_p(X,Y)),$$

for some germ of a holomorphic function g_p, where $(\bar{x}, \bar{y}, \bar{z}) = (R_\pi \circ T_\pi)^{-1}(\phi(x,y))$. Composing $\tilde{F}_\pi^k$ with K gives the germ $(\tilde{F}_\pi^k)_p$ of $\tilde{F}_\pi^k$ at p

$$(\tilde{F}_\pi^k)_p(X,Y) = (\bar{x} + X, (\bar{y} + Y)^k, \bar{z} + g_p(X,Y))$$

$$\sim_\mathcal{A} (X, (\bar{y} + Y)^k, g_p(X,Y)).$$

Observe that g_p depends on the choice of R_π, T_π and of the coordinates system, but the $\mathcal{A}$-class of the resulting germs $(\tilde{F}_\pi^k)_p$ is independent of these choices.

Clearly, a necessary condition for $(\tilde{F}_k^\pi)_p$ to be singular at the origin is $\bar{y} = 0$, equivalently, $p \in \pi$. We define the family of maps $\Phi : U \times V \to \mathbb{C} \times J^l(2,1)$ given by

$$\Phi(p, \pi) = (\langle v, p \rangle - d, j^l g_p(0,0)).$$

The map Φ plays a similar role in the Monge–Taylor map in [5]. Here we include the first component to capture the planes through a given point $p \in M$ which can give rise to singular k-folding map-germs at p.

We stratify $\mathbb{C} \times J^l(2,1)$ by $0 \times \mathcal{S}_k$, together with $\mathbb{C} \times J^l(2,1) \setminus 0 \times J^l(2,1)$. Following standard transversality arguments (see e.g. [5]), one can show that for a residual set of local embeddings of M in $\mathbb{C}^3$, the map Φ is transverse to the strata in $\mathbb{C} \times J^l(2,1)$. As the domain of the family Φ is of dimension 5, for a generic local embedding of M in $\mathbb{C}^3$, Φ intersects a stratum $0 \times X$ only when X has codimension $\leqslant 4$ in $J^l(2,1)$. This means that the only singularities of k-folding maps that can occur on a generic surface are those belonging to the strata listed in tables 1, 2, 4 and 6. Furthermore, Φ is transverse to these strata. Therefore, the locus of points where the l-jets of the k-folding map-germs belong to a stratum X of codimension 3 (resp. 4) is a regular curve on M (resp. isolated points on this curve). The above discussion also clarifies why the singularities of F_k^π occur generically even though they may have high $\mathcal{A}_e$-codimensions: they belong to strata of low codimension. (The strata can be viewed as a gluing of non $\mathcal{A}$-equivalent orbits which depends on a finite set of moduli.)

It is worth observing that the above discussion is valid for smooth surfaces in $\mathbb{R}^3$.

6. k-Folding map-germs on surfaces in $\mathbb{R}^3$

We consider in this section the geometry of k-folding maps-germs on smooth (i.e. regular and of class C^∞) surfaces in $\mathbb{R}^3$.

6.1. Robust features on surfaces

Robust features on a surface in $\mathbb{R}^3$ is a terminology introduced by Ian Porteous to indicate special characteristic features that can be traced when the surface evolves. What is sought after in applications are robust features which are represented by curves or points on the surface as these form a 'skeletal structure' of the surface (open regions bounded by robust curves are also robust features). They play an important role in computer vision and shape recognition (see e.g. [27]) as they can be used to distinguish two shapes (surfaces) from each other and, in some cases, reconstruct the surface.

We consider here the *parabolic, ridge, sub-parabolic and flecnodal curves* on a smooth surface M in $\mathbb{R}^3$ and special points on these curves (see [**14**] for references on work on these curves from a singularity theory point of view). These are robust features on M. We recall briefly what they are.

The parabolic curve is the locus of points where the Gaussian curvature vanishes. It is captured by the contact of the surface M with planes: it is the locus of points where the height function along a normal direction to M has an $A_{\geqslant 2}$-singularity. The parabolic curve is regular on a generic surface, and the height function has an A_2-singularity at its points except at isolated ones, called *Cusps of Gauss*, where it has an A_3-singularity.

The ridge is the locus of points on M where a principal curvature is extremal along its associated lines of principal curvature. It is also the locus of points on M which correspond to singular points on its focal set. The ridge is captured by the contact of M with spheres: it is the locus of points on M where the distance squared function from a given point in $\mathbb{R}^3$ (the point belong to the focal set) has an $A_{\geqslant 3}$-singularity. Away from umbilic points (i.e. points where the principal curvatures coincide), the ridge is a regular curve on a generic surface and the distance squared function has an A_3-singularity at its points except at isolated points where it has an A_4-singularity. At umbilics, the ridge consist of one regular curve or a transverse intersection of three regular curves. (Umbilics and A_4-points are used as *seed points* for drawing ridges on a given shape, see [**27**].)

The *sub-parabolic curve* is the locus of points on M corresponding to parabolic points of its focal set. It is the locus of geodesic inflections of the lines of principal curvature; it is also the locus of points along which a principal curvature is extremal along the other lines of curvature. It is captured by the singularities of the 2-folding map on M: it is the locus of points where some map F_2^π has an $S_{\geqslant 2}$-singularity (or one which is adjacent to an S_2-singularity). The singularities of F_2 and their geometric characterization on a generic surface parametrized in Monge form $z = f(x, y)$ at a given non-umbilic point, with f as in (3.1), are as in table 8. At umbilics, the sub-parabolic curve consist of one regular curve or a transverse intersection of three regular curves. Observe that the ridge is also captured by the singularities of F_2^π (see table 8).

The *flecnodal curve* is the locus of geodesic inflections of the asymptotic curves. It is captured by the contact of the surface M with lines: it is the locus of points where the orthogonal projection of the surface has a singularity of type swallowtail or worse. We recall briefly some results on these projections as they are needed for interpreting the singularities of k-folding maps. The orthogonal projection P_v of M along the direction $v \in S^2$ to the plane $T_v S^2$ is given by $P_v(p) = p - (p \cdot v)v$, with $p \in M$. This can be represented locally by a map-germ from the plane to the plane. Varying v yields the family of orthogonal projections $P: M \times S^2 \to TS^2$ given by $P(p, v) = (v, P_v(p))$.

A transversality theorem asserts that for an open and dense (i.e.generic) set of embeddings $\phi: U \to \mathbb{R}^3$, the surface $M = \phi(U)$ has the following property: for any $v \in S^2$ the map-germ P_v has only local singularities $\mathcal{A}$-equivalent to one in table 9 at any point on M. By translating $p_0 \in M$ to the origin and taking M locally at p_0 in Monge form, an open subset of M is parametrized by $\phi(x, y) = (x, y, f(x, y))$, with f having no constant nor linear terms. Projecting along the direction $v = (0, 1, 0)$

Table 8. *Geometric characterization of the singularities of the folding-map F_2*

Name	Algebraic conditions and geometric meaning
Crosscap	$a_{21} \neq 0$
$B_1 = S_1$	$a_{21} = 0,\ a_{31} \neq 0,\ a_{33} \neq 0$
	General smooth point of focal set
B_2	$a_{21} = 0, a_{31} \neq 0, a_{33} = 0, CndNA_3 = 4a_{31}a_{55} - a_{43}^2 \neq 0$
	General cusp point of focal set corresponding to a point on the ridge curve
B_3	$a_{21} = 0, a_{31} \neq 0, a_{33} = 0, CndNA_3 = 0,$
	$CndNA_5 = 8a_{31}^3 a_{77} - 4a_{65}a_{43}a_{31}^2 + 2a_{53}a_{43}^2 a_{31} - a_{41}^2 a_{43}^3 \neq 0$
	(Cusp) point of focal set in closure of parabolic curve on symmetry set
S_2	$a_{21} = 0,\ a_{31} = 0,\ a_{33} \neq 0,\ a_{41} \neq 0$
	Parabolic smooth point of focal set corresponding to a point on the sub-parabolic curve
S_3	$a_{21} = 0,\ a_{31} = 0,\ a_{33} \neq 0,\ a_{41} = 0,\ a_{51} \neq 0$
	Cusp of Gauss at smooth point of focal set
C_3	$a_{21} = 0,\ a_{31} = 0,\ a_{33} = 0,\ a_{41} \neq 0,\ a_{43} \neq 0$
	Intersection point of cuspidal-edge and parabolic curve on focal set

gives $P_v(x, y) = (x, f(x, y))$, which is singular at the origin. Table 9 shows the conditions on the coefficients of f for the map-germ P_v to have a singularity at the origin of $\mathcal{A}_e$-codimension $\leqslant 2$, where $d_e(G, \mathcal{A})$ denotes the $\mathcal{A}_e$-codimension of G (see e.g. [**14**]).

The projection P_v has a fold singularity at p if, and only if, v is a non-asymptotic tangent direction to M at p. The singularity at p is of type cusp or worse if, and only if, v is an asymptotic direction at p. For a generic surface M, the closure of the set of points where P_v has a swallowtail (resp. lips/beaks) singularity is a precisely the flecnodal (resp. parabolic) curve. The flecnodal curve meets tangentially the parabolic curve at the cusps of the Gauss map, which are the gulls singularities of P_v [**2**]. We call a point where this happens a *gulls-point* of M.

The goose (resp. butterfly) singularities of P_v appear at special points on the parabolic (resp. flecnodal) curve. We call a point on M where these singularities occur a *goose-point* (resp. *butterfly-point*) on M.

The above robust features are defined in terms of the principal curvatures κ_1 and κ_2. We can suppose $\kappa_1 < \kappa_2$ (away from umbilic curves) and give different colours to the robust features associated to each principal curvature. For instance, we can have a blue ridge (associated to κ_1) and a red ridge (associated to κ_2).

6.2. The geometry of *k*-folding map-germs

We obtain here the robust features determined by the singularities of k-folding map-germs, for $k \geqslant 3$. We start with the case where p is not an umbilic point and treat the $k = 3$ case separately.

Table 9. $\mathcal{A}_e$-Codimension $\leqslant 2$ singularities of map-germs $G : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$

Name	Normal form	$d_e(G, \mathcal{A})$	Algebraic conditions on f in (3.1) for the singularities of $G(x,y) = (x, f(x,y))$
Fold	(x, y^2)	0	$a_{22} \neq 0$
Cusp	$(x, xy + y^3)$	0	$a_{22} = 0$, $a_{21} \neq 0$, $a_{33} \neq 0$
Swallowtail	$(x, xy + y^4)$	1	$a_{22} = 0$, $a_{33} = 0$, $a_{21} \neq 0$, $a_{44} \neq 0$
Lips/beaks	$(x, y^3 \pm x^2 y)$	1	$a_{22} = 0$, $a_{21} = 0$, $a_{33} \neq 0$, $a_{32}^2 - 3a_{31}a_{33} \neq 0$
Goose	$(x, y^3 + x^3 y)$	2	$a_{22} = 0$, $a_{21} = 0$, $a_{32}^2 - 3a_{31}a_{33} = 0$, $a_{33} \neq 0$, $27a_{41}a_{33}^3 - 18a_{42}a_{32}a_{33}^2 + 9a_{43}a_{32}^2 a_{33} - 4a_{44}a_{32}^3 \neq 0$
Butterfly	$(x, xy + y^5 \pm y^7)$	2	$a_{22} = 0$, $a_{33} = 0$, $a_{44} = 0$, $a_{21} \neq 0$, $a_{55} \neq 0$, $(8a_{55}a_{77} - 5a_{66}^2)a_{21}^2 + 2a_{55}(a_{32}a_{66} - 20a_{43}a_{55})a_{21} + 35a_{32}^2 a_{55}^2 \neq 0$
Gulls	$(x, xy^2 + y^4 + y^5)$	2	$a_{22} = 0$, $a_{21} = 0$, $a_{33} = 0$, $a_{32} \neq 0$, $a_{44} \neq 0$, $a_{55}a_{32}^2 - 2a_{43}a_{44}a_{32} + 4a_{31}a_{44}^2 \neq 0$

THEOREM 6.1. *Let M be a generic surface in $\mathbb{R}^3$, p a non-umbilical point on M and π a plane through p and orthogonal to v. The map-germ F_3^π at p is an immersion if and only if v is not tangent to M at p. It has an S_1-singularity if and only if $v \in T_p M$ but is neither a principal nor an asymptotic direction. We have the following when v is a principal or an asymptotic direction at p.*

(1) *If v is a principal direction at p, then the possible singularities of F_3^π at p and their geometric characterizations are as follows:*

S_3: p is not on the sub-parabolic curve associated to v.
S_5: p is a generic point on the sub-parabolic curve associated to v.
S_7: p is an S_3-point on the sub-parabolic curve associated to v.

(2) *Suppose that v is an asymptotic and that p is a hyperbolic point. Then F_3^π has a singularity at p of type $H_k, k = 2, 3, 4$. The H_3-singularities occur on a regular curve on M. We call its closure the H_3-curve. The H_4-singularities occur at isolated points on this curve.*

(3) *If p is a parabolic point and v is the unique asymptotic direction at p which is also a principal direction, then the possible topological classes of F_3^π at p and their geometric characterizations are as follows:*

X_4: p is a generic point on the parabolic curve.
$\mathbf{U}_4^3$: p is an A_2^*-point (see [10, 11]); it is on the closure of the H_3-curve.

$\mathbf{X}_4^3$: *p is on the sub-parabolic curve associated to v; the principal map with value v at p has a beaks singularity at p* [**7**].

$\mathbf{W}_4^{3,1}$: *p is on the sub-parabolic curve associated to the other principal direction $v^\perp$; it is on the closure of the H_3-curve; the frame map has a cross-cap singularity at p* [**7**].

Proof. The proof follows by considering the defining equations and the open conditions of the strata of 3-folding map-germs in theorem 4.4 and the geometric interpretation of the algebraic conditions in table 1 given in tables 8 and 9. □

PROPOSITION 6.2. *The H_3-curve of a generic surface is a regular curve and meets the parabolic curve tangentially at A_2^* and $\mathbf{W}_4^{3,1}$-points.*

Proof. The regularity of the H_3-curve follows by a transversality argument. We compute the 1-jets of the parabolic and H_3-curves and find that they are tangential at their points of intersection. This is expected as the H_3-singularities of F_3^π occur when v is an asymptotic direction, so the H_3-curve lies in the closure of the hyperbolic region of the surface. □

We turn now to the case $k \geqslant 4$, still assuming p not to be an umbilical point. We recall that some topological classes in tables 2, 4 and 6 come with divisibility conditions on k. For example, the $\mathbf{N}_3^k$ class requires $2 \mid k$ (see remarks 6.4).

THEOREM 6.3. *Let $k \geqslant 4$ and let M be a generic surface in $\mathbb{R}^3$, p a non-umbilical point on M and π a plane through p and orthogonal to v. The map-germ F_k^π at p is an immersion if and only if v is not tangent to M at p. It has a singularity which is topologically equivalent to M_1^k if and only if $v \in T_pM$ but is neither a principal nor an asymptotic direction at p. We have the following when v is a principal or an asymptotic direction at p.*

(1) *If v is a principal direction at p, then F_k^π belongs to a stratum in table 2. The possible topological classes of F_k^π and their geometric interpretations are as follows:*

$\mathbf{M}_2^k$: *p is not on the sub-parabolic curve associated to v.*
$\mathbf{M}_3^k$: *p is a generic point on the sub-parabolic curve associated to v.*
$\mathbf{M}_4^k$: *p is an S_3-point on the sub-parabolic curve associated to v.*
$\mathbf{N}_3^k$: *p is a generic point on the ridge curve associated to v.*
$\mathbf{N}_4^k$: *p is a B_3-point on the ridge curve associated to v.*
$\mathbf{O}_4^k$: *p is a C_3-point (intersection point of the ridge and sub-parabolic curves associated to v).*

(2) *If p is a hyperbolic point and v is an asymptotic direction at p, then F_k^π belongs to a stratum in table 4. The possible topological classes of F_k^π and their geometric interpretations are as follows:*

$\mathbf{P}_2^k$: *p is not on the flecnodal curve associated to v, and is not on the H_3-curve associated to v when $3 \mid k$.*
$\mathbf{P}_3^k$: *p is a generic point on the H_3-curve associated to v.*

$\mathbf{P}_4^k$: *p is an H_4-point on the H_3-curve associated to v.*
$\mathbf{Q}_3^k$: *p is a generic point on the flecnodal curve associated to v.*
$\mathbf{Q}_4^k$: *p is a point of intersection of the flecnodal and H_3-curves associated to
       v.*
$\widetilde{\mathbf{Q}}_4^k$: *p is a special point on the flecnodal curve associated to v.*
$\mathbf{R}_4^k$: *p is a butterfly-point on the flecnodal curve associated to v.*

(3) *If p is a parabolic point and v is an asymptotic (and a principal) direction at
    p, then F_k^π belongs to a stratum in table 6. The possible topological classes of
    F_k^π and their geometric interpretations are as follows:*

$\mathbf{U}_3^k$: *p is a generic point on the parabolic curve.*
$\mathbf{U}_4^k$: *p is an A_2^*-point.*
$\mathbf{V}_4^{k,j,j'}$: *p is a special point on the parabolic curve.*
$\mathbf{W}_4^{k,j}$: *p is a special point on the parabolic curve.*
$\mathbf{W}_4^{3q,q}$: *p is on the sub-parabolic curve associated to $v^\perp$; it is on the closure
       of the H_3-curve associated to v; the frame map has a cross-cap
       singularity at p [7].*
$\mathbf{X}_4^k$: *p is on the intersection of the parabolic and sub-parabolic curves
       associated to v; the principal map with value v at p has a beaks singularity
       at p [7].*
$\mathbf{Y}_4^k$: *p is a cusp of Gauss point and a gulls-point.*

Proof. With the setting as in the proof of theorem 6.1, the results follow by
interpreting the conditions in tables 2, 4 and 6 using tables 8 and 9.   □

REMARKS 6.4. 1. The singularities in branches 1 and 2 are associated to principal
directions, those in branch 3 to asymptotic directions at hyperbolic points and those
in branch 4 to asymptotic directions at parabolic points.

2. Theorems 6.1 and 6.3 show clearly that k-folding maps capture the robust
features obtained by 2-folding maps and by the contact of the surface with lines,
planes and spheres, giving thus new geometric characterizations of these features
and a unified approach to study them. We also obtain a new 1-dimensional robust
feature (the H_3-curve) and several 0-dimensional ones: the H_4, $\mathbf{Q}_4^k$, $\widetilde{\mathbf{Q}}_4^k$, $\mathbf{V}_4^{k,j,j'}$ and
$\mathbf{W}_4^{k,j}$ points.

3. The H_3-curve is captured by k-folding maps when k is divisible by 3. The
sub-parabolic and flecnodal curves are captured by k-folding maps for any k while
the ridge curve is captured by k-folding maps when k is even. Regarding the ridge,
it is the locus of points where the surface has more infinitesimal symmetry with
respect to planes [8]. The map-germs $(x,y) \mapsto (x, y^{2p}, f(x,y))$ identify the pair of
points (x,y) and $(x,-y)$, which explains why all the F_{2p} folding maps capture the
ridge curve.

4. The condition $\Delta_j^k = 0$ can be satisfied for any k, j when the coefficients a_{sl}
are real, so $\mathbf{W}_4^{k,j}$-points can occur on surfaces in $\mathbb{R}^3$ for all k, j. For the $\mathbf{V}_4^{k,j,j'}$
topological class, it follows from proposition 4.21 that only $\mathbf{V}_4^{k,j,k-j}$, $\mathbf{V}_4^{k,j,2j}$ and
$\mathbf{V}_4^{k,k-j,2j}$-points can occur on surfaces in $\mathbb{R}^3$.

5. Theorem 6.1 gives a new geometric interpretation for the A_2^*-points in [**10**, **11**].

6. Observe that the open conditions (those involving expressions $\neq 0$) for the singularities of the k-folding map in tables 1, 2, 4 and 6 and their associated ones in tables 8 and 9 are not always identical. For the 0-dimensional robust features, the open conditions in both tables are satisfied on a generic surface. For the 1-dimensional robust features, this means that some special points in one setting are not special in the other. For example, the 3-folding map does not distinguish between a C_3-point and a generic point on the sub-parabolic curve.

We consider now the situation at umbilic points. For a generic surface M, these occur at isolated points in its elliptic region, and every direction in the tangent plane of M at such points can be considered a principal direction. We take M locally in Monge form $z = f(x, y)$, consider the origin to be an umbilic point and write $f(x, y) = \kappa/2(x^2 + y^2) + C(x, y) + O_4(x, y)$ where C is a homogeneous cubic form in x, y. We can take $C(x, y)$ to be the real part of $z^3 + \beta z^2\overline{z}$, with $z = x + iy$ and $\beta = s + it$ (see e.g. [**8**]). Then, $C = (1 + s)x^3 - tx^2y + (s - 3)xy^2 - ty^3$.

THEOREM 6.5. *Let $k \geqslant 3$ and let M be a generic smooth surface in $\mathbb{R}^3$, p an umbilic point on M and π a plane through p and orthogonal to $v \in T_pM$.*

1. *If $2 \nmid k$, then for almost all directions v in T_pM the singularity of F_k^π at p is of type S_3 when $k = 3$ and of type $\mathbf{M}_2^k$ when $k \geqslant 4$. There are three directions (resp. one direction) where the singularity is of type S_5 when $k = 3$ and of type $\mathbf{M}_3^k$ when $k \geqslant 4$ if β is inside (resp. outside) the outer hypocycloid $\beta = -3(2e^{2i\theta} + e^{-4i\theta})$ in figure 4.*

2. *If $2 \mid k$, then for almost all directions v in T_pM the singularity of F_k^π at p is of type $\mathbf{M}_2^k$. There are three directions (resp. one direction) where the singularity is of type $\mathbf{M}_3^k$ if β is inside (resp. outside) the hypocycloid $\beta = -3(2e^{2i\theta} + e^{-4i\theta})$. There are also three directions (resp. one direction) where the singularity is of type $\mathbf{N}_3^k$ when β is inside (resp. outside) the inner hypocycloid $\beta = 2e^{2i\theta} + e^{-4i\theta}$ in figure 4.*

Proof. We take $v = (\cos(\theta), \sin(\theta), 0)$, $\theta \in [0, 2\pi]$ and consider the rotation

$$R = \begin{pmatrix} \sin(\theta) & \cos(\theta) & 0 \\ -\cos(\theta) & \sin(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

which takes the direction $(0, 1, 0)$ to v. Then,

$$F_k^\pi \circ R^{-1}(x, y) = (x\sin(\theta) - y\cos(\theta), (x\cos(\theta) + y\sin(\theta))^k, f(x, y)).$$

Changes of coordinates in the source give

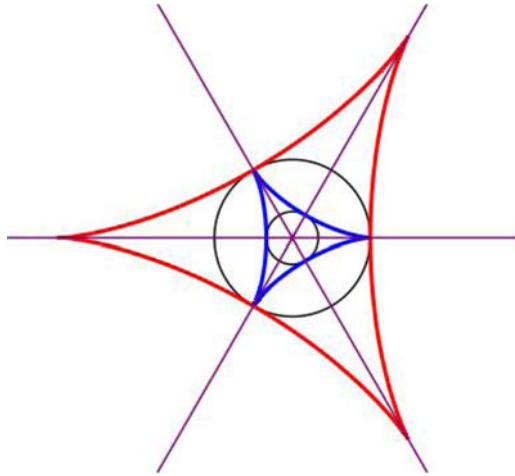$$F_k^\pi \circ R^{-1}(X, Y) = (X, Y^k, f(X\sin(\theta) + Y\cos(\theta), -X\cos(\theta) + Y\sin(\theta))).$$

Figure 4. Partition of the space of cubic forms.

We denote by $\bar{a}_{lj}$ the coefficient of $X^{l-j}Y^j$ in the Taylor expansion of $f(X\sin(\theta) + Y\cos(\theta), -X\cos(\theta) + Y\sin(\theta))$. The proof follows then considering the conditions for the singularities of F_k^π in tables 1 and 2.

(1) We have

$$\bar{a}_{31} = (s-3)\cos(\theta)^3 - t\cos(\theta)^2\sin(\theta) + (9+s)\sin(\theta)^2\cos(\theta) - t\sin(\theta)^3.$$

When $2\nmid k$ and $\bar{a}_{31} \neq 0$, the singularity of F_k^π at the origin is of type S_3 when $k = 3$ or of type M_2^k when $k \geqslant 4$.

The coefficient $\bar{a}_{31}$ is a cubic form in $\cos(\theta)$ and $\sin(\theta)$. Its discriminant is the hypocycloid $\beta = -3(2\mathrm{e}^{2\mathrm{i}\theta} + \mathrm{e}^{-4\mathrm{i}\theta})$. The cubic has three roots for β inside the hypocycloid and one root when it is outside. For v corresponding to one of these roots, the singularity of F_k^π is of type S_5 when $k = 3$ and of type $\mathbf{M}_3^k$ when $k \geqslant 4$, provided $\bar{a}_{41} \neq 0$. The condition $\bar{a}_{41} \neq 0$ is satisfied at umbilic points on generic surfaces.

(2) We have

$$\bar{a}_{33} = (1+s)\cos(\theta)^3 - t\cos(\theta)^2\sin(\theta) + (s-3)\cos(\theta)\sin(\theta)^2 - t\sin(\theta)^3.$$

When $2 \mid k$ (so $k \geqslant 4$) and $\bar{a}_{33} \neq 0$, the singularity of F_k^π at the origin is of type $\mathbf{M}_2^k$. We also get the $\mathbf{M}_3^k$ singularities as in (1) when $\bar{a}_{31} = 0$.

The coefficient $\bar{a}_{33}$ is also a cubic form in $\cos(\theta)$ and $\sin(\theta)$. Its discriminant is the hypocycloid $\beta = 2\mathrm{e}^{2\mathrm{i}\theta} + \mathrm{e}^{-4\mathrm{i}\theta}$. The cubic has three roots for β inside the hypocycloid and one root when it is outside. For v corresponding to one of these roots, the singularity of F_k^π is of type $\mathbf{N}_3^k$ if $\bar{a}_{31} \neq 0$.

We have $\bar{a}_{33} = \bar{a}_{31} = 0$ if and only if β is on one of the tangent lines $t(3s^2 - t^2) = 0$ to the hypocycloids at their cusp points, see figure 4. (On these lines, the singularity is of type $\mathbf{O}_4^k$ or more degenerate. This singularity does not occur at umbilic points on a generic surface.) $\qquad\square$

REMARK 6.6. 1. Theorem 6.5 is merely another interpretation of the results in [**8**, **32**] when using the geometric characterizations of the singularities of k-folding maps in theorems 6.1 and 6.3. We know from [**5**, **8**, **32**] that there are one or three ridge curves and one or three sub-parabolic curves at umbilic points on a generic surface. These curves meet transversally and change colour at the umbilic point.

2. Figure 4 is first obtained in [**8**] when considering 2-folding maps. In that case both hypocycloids are present, whereas when $k \geqslant 3$ only one of them is present when k is odd (both are present when k is even). Also in [**8**] is considered the circle $|\beta| = 3$ which corresponds to the Monge–Taylor map failing to be transverse to the umbilics stratum. The circle $|\beta| = 1$ is also exceptional and corresponds to two ridges through the umbilic being tangential, see [**6**]. As these conditions are geometric, the circles $|\beta| = 1$ and $|\beta| = 3$ can also be considered exceptional for k-folding maps and are added to figure 4.

3. Umbilic points on a generic surface occur at elliptic points, that is why we do not get flecnodal curves or H_3-curves through such points.

Acknowledgements

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