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ON MÖBIUS FUNCTIONS AND A PROBLEM IN COMBINATORIAL NUMBER THEORY

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1. **Introduction.** After the publication of the important paper by Rota [9] on Möbius functions a large number of papers have appeared in which the ideas are applied or generalized in various directions, the papers by Crapo [3], Smith [10] and Tainiter [11] are some of them. The theory of Möbius functions is now recognized as a valuable tool in combinatorial and arithmetical research.

It is the purpose of the present note to prove a valuable property of Möbius functions and then to apply this to generalize the method in [5] to construct detecting sets of vectors. We recall that a set of vectors $v_1, v_2, \ldots, v_n$ was said to be detecting if all the sums $\sum_1^n \epsilon_i v_i$ ($\epsilon_i = 0, 1, \ldots, k-1$) are different. The result depends on the function $h_k(x)$, which is defined as the maximum number h for which there exist integers d_i ($i=1, \ldots, h$) in the interval $1 \le d_i \le x$ such that the sums $\sum_1^h \epsilon_i d_i$ ($\epsilon_i = 0, 1, \ldots, k-1$) are different.

The problem to estimate $h_2(x)$ from above has been studied by Erdös and Moser (cf. [4]). The conjecture of Erdös in [4] that $h_2(2^k) \ge k+2$ for sufficiently large k has been studied by Conway and Guy [2].

2. **Möbius functions.** Let P be a finite partially ordered set. The Möbius function $\mu(x, y)$ of P is defined for x and y in P such that

$$\mu(x, x) = 1 \tag{2.1}$$

$$\mu(x, y) = 0 \quad \text{if } x \not\le y \tag{2.2}$$

$$\mu(x, y) = -\sum_{z;\, x \le z < y} \mu(x, z) \quad \text{if } x < y. \tag{2.3}$$

By duality [9, p. 345] is

$$\mu(x, y) = -\sum_{z;\, x < z \le y} \mu(z, y) \quad \text{if } x < y. \tag{2.4}$$

Observe that the function $\mu(x, y)$ is integervalued. When P is the Boolean algebra of all subsets of a finite set is

$$\mu(x, y) = (-1)^{n(y)-n(x)} \quad \text{if } x \subset y, \tag{2.5}$$

where $n(x)$ is the cardinality of x. A similar formula holds for the lattice associated with a convex polytope (cf. [7]).

We shall prove the following theorem.

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THEOREM 1. *Let P be a finite partially ordered set with* 0 *and a unique last element* 1. *Let $\mu(x, y)$ be the Möbius function of P. Put $m=\sum_{x\in P} |\mu(x, 1)|$. m is then an even integer. Let n be an arbitrary integer in the interval $0\le n\le m/2$. Then there exists a function $f(x)=0$ or 1 on P such that*

$$\sum_{x;\,0<x\le 1} f(x)\mu(x, 1) = -n \operatorname{sign} \mu(0, 1), \tag{2.6}$$

where sign $a=1$ *if* $a\ge 0$ *and* sign $a=-1$ *if* $a<0$.

Proof. We shall first prove another related result. Let e be an arbitrary integer in the interval $0\le e\le m$. We shall then prove the existence of a function $g(x)=1$ or -1 such that $g(0)=1$ and

$$\sum_{x;\,0\le x\le 1} g(x)\mu(x, 1) = e \operatorname{sign} \mu(0, 1). \tag{2.7}$$

Let Y be an arbitrary subset of P such that $y\in Y$ and $y<z$ (in P) implies $z\in Y$. Then Y is partially ordered by $<$ and the Möbius function of Y is the restriction to Y of $\mu(x, y)$. Put

$$m_Y = \sum_{y\in Y} |\mu(y, 1)|.$$

We shall prove by induction on the number of elements in Y that

$$\sum_{y\in Y} g(y)\mu(y, 1) = -m_Y,\ -m_Y+2, \ldots, \text{ or } m_Y \tag{2.8}$$

for a suitable function $g(y)=1$ or -1 on Y. This is true when the cardinality of Y is $|Y|=1$, in which case $Y=\{1\}$ and $\mu(1, 1)=1$.

Assume that $|Y|>1$. Let c denote a minimal element in Y and put $Z=Y-\{c\}$. By the inductive assumption it follows that we can find $g(y)=1$ or -1 on Z such that

$$\sum_{y\in Z} g(y)\mu(y, 1) = \text{any of } -m_Z,\ -m_Z+2, \ldots, \text{ or } m_Z.$$

It follows that the sum (2.8) equals any of the integers $-m_Z\pm\mu(c, 1)$, $-m_Z\pm\mu(c, 1)+2, \ldots$, or $m_Z\pm\mu(c, 1)$ if $g(c)=\pm 1$. Since $m_Y=m_Z+|\mu(c, 1)|$ and $|\mu(c, 1)|\le mY$ by (2.4) and the triangle inequality, it follows that (2.8) is true for a suitable function $g(y)=1$ or -1 on Y. In the special case when $Y=P$ is 0 one of the possible values by (2.4) and m must be even.

We apply the preceding result to $Y=P-\{0\}$. Put $g(0)=\operatorname{sign}\mu(0, 1)$. Since $|\mu(0, 1)|\le m_Y$ by (2.4), it follows that for any even e in $0\le e\le m$ we can find $g(y)=1$ or -1 on Y such that the value of the sum in (2.7) is e. We multiply the equality by sign $\mu(0, 1)$ and (2.7) follows for the function $g(x)$ sign $\mu(0, 1)=G(x)$.

If we subtract (2.7) from $\sum_{y\in P}\mu(y, 1)=0$ and divide by 2, we obtain (2.6) with $f(y)=\frac{1}{2}(1-G(y))$ and $f(0)=0$ since $G(0)=1$.

3. **Detecting sets.** A proof of the following lemma can be found in [6]. For the definition of semilattices (cf. [1, p. 24]).

LEMMA. *Let P be a finite semilattice with Möbius function $\mu(x, y)$. Let $a, b \in P$ and $b \not\leq a$. Let $f(x)$ be defined for all $x \leq a \wedge b$ with values in a commutative ring with unit. Then we have*

$$\sum_{x;\, x \leq b} f(x \wedge a)\mu(x, b) = 0.$$

The lemma in [5, p. 481] is a special case when P is a subsemilattice of a Boolean algebra. The value of the Möbius function can be found by (2.5) in this case.

We shall now prove our main result.

THEOREM 2. *Let P be a finite semilattice with $m+1$ elements. The product in P is $a \wedge b$ and P is partially ordered such that $a \leq b$ if and only if $a = a \wedge b$. The first element in P is θ. Put $m_y = \sum_{x;\, x \leq y} |\mu(x, y)|$. Then there exists a detecting set containing $\sum_{y > \theta} h_k(m_y/2)$ vectors of dimension m with all components* 0 *or* 1.

Proof. Let $x_0 = \theta, x_1, \ldots, x_m$ be an enumeration of P such that $x_i < x_j$ holds only if $i < j$. We shall write m_i instead of m_y if $y = x_i$.

Consider a particular i in the interval $1 \leq i \leq m$. Let $d_{i1}, \ldots, d_{ih}$, where $h = h_k(m_i/2)$, be a detecting sequence of integers with $1 \leq d_{ij} \leq m_i/2$ fo $j = $ r$1, \ldots, h$. By Theorem 1 we can find a function $f_{ij}(x) = 0$ or 1 on P such that

$$\sum_{x;\, \theta < x \leq x_i} f_{ij}(x)\mu(x, x_i) = -d_{ij} \operatorname{sign} \mu(\theta, x_i). \tag{3.1}$$

Then we have by the lemma

$$\sum_{v=1}^{m} f_{ij}(x_v \wedge x_i)\mu(x_v, x_r) = 0 \quad \text{if } i < r. \tag{3.2}$$

We shall prove that the set of all vectors

$$v_{ij} = (f_{ij}(x_1 \wedge x_i), \ldots, f_{ij}(x_m \wedge x_i)), \tag{3.3}$$

where $j = 1, \ldots, h_k(m_i/2)$ and $i = 1, \ldots, m$, is a detecting set. In order to prove this assume that

$$\sum_{i,j} e_{ij} v_{ij} = \mathbf{0}, \quad (e_{ij} = -k, \ldots, 0, \ldots, \text{ or } k), \tag{3.4}$$

where $1 \leq i \leq m$ and $1 \leq j \leq h_k(m_i/2)$. We shall prove that all $e_{ij} = 0$. If this is not true let r be the last i such that $e_{ij} \neq 0$ for some j. We multiply the vth component on both members of (3.4) by $-\mu(x_v, x_r) \operatorname{sign} \mu(\theta, x_r)$ and take the sum for $v = 1, \ldots, m$. Then we obtain by (3.2) and (3.3)

$$\sum_{j=1}^{h} e_{rj} d_{rj} = 0,$$

where $h = h_k(m_r/2)$. From the fact that the sequence d_{rj} ($j = 1, \ldots, h$) is detecting it follows that $e_{rj} = 0$ for $j = 1, \ldots, h$ in contradiction to the assumption that $e_{rj} \neq 0$ for some j. Hence all $e_{ij} = 0$ and we have proved that the set of all vectors v_{ij} defined in

(3.3) is a detecting set. The cardinality of the set is easily determined and the theorem is proved.

EXAMPLES. It seems to be a difficult problem to find the best possible estimate for given m. For certain classes of semilattices it is possible to find the best estimates. Consider e.g. the class of complexes in Boolean algebras. By the method in [8] it can be proved that the best possible choice was already made in [5].

If we apply the detecting sequences of Conway and Guy [2] one can improve the estimate $F_2(m) \geq A(m)$ in [5] to $F_2(m) \geq A(m)+m-C$ for a constant C, but this is a real improvement only if m is very large ($m \geq 2^{21}$).

If we apply Theorem 2 to a suitable semilattice it is possible to improve the estimate $F_2(m) \geq A(m)$ even for moderate m. We give an example when $m=10$. Let P be the lattice of the integers 1, 2, 3, 5, 6, 7, 10, 14, 21, 35, 210 ordered by divisibility ($x \leq y$ if x divides y). The value of $m_y/2$ for $y > \theta$ is 1, 1, 1, 2, 1, 2, 2, 2, 2, 4 respectively. Since $h_2(1)=1$, $h_2(2)=2$ and $h_2(7) \geq 4$ (the sequence 3, 5, 6, 7 is detecting), we obtain a detecting set of cardinality 18, which is an improvement since $A(10)=17$ (cf. [5, p. 481]).

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