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PERFECT POWERS THAT ARE SUMS OF TWO POWERS OF FIBONACCI NUMBERS

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Abstract

In this paper, we consider the Diophantine equations

$$F_n^q \pm F_m^q = y^p$$

with positive integers $q, p \geq 2$ and $\gcd(F_n, F_m) = 1$, where F_k is a Fibonacci number. We obtain results for $q = 2$ or q an odd prime with $q \equiv 3 \pmod 4$, $3 < q < 1087$, and complete solutions for $q = 3$.

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1. Introduction

The Fibonacci numbers are the sequence of numbers $(F_n)_{n\geq 0}$ defined by the linear recurrence equation

$$F_{n+1} = F_n + F_{n-1}, \quad F_0 = 0, \quad F_1 = 1.$$

The Lucas numbers are the sequence of numbers $(L_n)_{n\geq 0}$ defined by the linear recurrence equation

$$L_{n+1} = L_n + L_{n-1}, \quad L_0 = 2, \quad L_1 = 1.$$

Finding all perfect powers in the Fibonacci sequence was a fascinating long-standing conjecture. In 2006, this problem was completely solved by Bugeaud *et al.* [6], who innovatively combined the modular approach with linear forms in logarithms. Also, Bugeaud *et al.* [3] found all the integer solutions to $F_n \pm 1 = y^p, p \geq 2$. Luca and Patel [11] consider the generalisation $F_n \pm F_m = y^p, p \geq 2$.

In this paper, we consider the Diophantine equation

$$F_n^q \pm F_m^q = y^p \quad \text{with } \gcd(F_n, F_m) = 1 \text{ and } q, p \geq 2.$$

We obtain the following results.

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THEOREM 1.1. *All solutions of the Diophantine equation*

$$F_n^2 + F_m^2 = y^p \quad \text{with } \gcd(F_n, F_m) = 1, p \geq 2, \tag{1.1}$$

in integers (n, m, y, p) *with* $n \not\equiv m \pmod 2$*,* $n > m \geq 0$ *and* $y > 0$ *are of the form* $(1, 0, 1, k)$*, with integer* $k \geq 2$*. The integer solutions of the Diophantine equation*

$$F_n^2 - F_m^2 = y^p \quad \text{with } \gcd(F_n, F_m) = 1, p \geq 2, \tag{1.2}$$

in integers (n, m, y, p) *with* $n \equiv m \pmod 2$ *and* $n > m \geq 0$ *and* $y > 0$ *are*

$$(n, m, y, p) = (2, 0, 1, k), (4, 2, 2, 3), (7, 5, 12, 2),$$

with integer $k \geq 2$*.*

THEOREM 1.2. *Let* q *be an odd prime. All solutions of the Diophantine equation*

$$F_n^q \pm F_m^q = y^p \quad \text{with } \gcd(F_n, F_m) = 1, p \geq 2$$

in integers (n, m, y, q, p) *with* $n \equiv m \pmod 2$*,* $n > m \geq 0$ *and* $y > 0$ *are* $(2, 0, 1, k, l)$*, for* $q < 1087$ *and* $q \equiv 3 \pmod 4$*.*

THEOREM 1.3. *The Diophantine equation*

$$F_n^3 \pm F_m^3 = y^p \quad \text{with } \gcd(F_n, F_m) = 1, p \geq 3$$

has only the integer solutions $(n, m, y, p) = (1, 0, 1, k), (2, 0, 1, k)$*, with* $n > m \geq 0$ *and* $y > 0$*.*

We organise this paper as follows. In Section 2, we recall and prove some results that will be useful for the proofs of Theorems 1.1–1.3. These proofs follow in Section 3. Divisibility properties of Fibonacci and Lucas numbers play a key role in the proofs.

2. Preliminaries

The Binet formulas for F_n and L_n are

$$F_n = \frac{1}{\sqrt{5}}(\alpha^n - \beta^n), \quad L_n = \alpha^n + \beta^n, \quad n \in \mathbb{Z},$$

where

$$\alpha = \frac{1 + \sqrt{5}}{2}, \quad \beta = \frac{1 - \sqrt{5}}{2}.$$

From the Binet formulas, we can obtain the useful formulas

$$F_{2n} = F_n L_n, \quad L_{3n} = L_n(L_n^2 + 3(-1)^{n+1}),$$

and Catalan's identity

$$F_n^2 - F_{n+r}F_{n-r} = (-1)^{n-r}F_r^2.$$

The following result can be obtained from [5, 6].

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LEMMA 2.1. *If* $F_n = 2^s y^b$, *for some integers* $n \geq 1, y \geq 1, b \geq 2$ *and* $s \geq 0$, *then we have* $n \in \{1, 2, 3, 6, 12\}$. *The solutions of the similar equation with* F_n *replaced by* L_n *have* $n \in \{1, 3, 6\}$.

The next result is well known and can also be proved using Binet's formulas (see also [11, Lemma 2.1]).

LEMMA 2.2. *Assume* $n \equiv m \pmod 2$. *Then*

$$F_n + F_m = \begin{cases} F_{(n+m)/2} L_{(n-m)/2} & \text{if } n \equiv m \pmod 4, \\ F_{(n-m)/2} L_{(n+m)/2} & \text{if } n \equiv m + 2 \pmod 4. \end{cases}$$

Similarly,

$$F_n - F_m = \begin{cases} F_{(n-m)/2} L_{(n+m)/2} & \text{if } n \equiv m \pmod 4, \\ F_{(n+m)/2} L_{(n-m)/2} & \text{if } n \equiv m + 2 \pmod 4. \end{cases}$$

The following result can be found in [12].

LEMMA 2.3. *Let* $n = 2^a n_1$ *and* $m = 2^b m_1$ *be positive integers with* n_1 *and* m_1 *odd integers and* a *and* b *nonnegative integers. If* $\gcd(n, m) = d$, *then*

(i) $\gcd(F_n, F_m) = F_d$;
(ii) $\gcd(F_n, L_m) = L_d$ *if* $a > b$ *and* 1 *or* 2 *otherwise.*

The following lemma is often set as an exercise in elementary number theory.

LEMMA 2.4. *Let* p *be an odd prime,* a, b, c, k *integers with* $\gcd(a, b) = 1$ *and* $k \geq 2$. *If*

$$a^p + b^p = c^k,$$

then $a + b = d^k$ *or* $p^{k-1} d^k$, *for some integer* d.

The following lemma can be obtained from [11].

LEMMA 2.5. *All solutions of the Diophantine equation*

$$F_n \pm F_m = y^p, \quad p \geq 2$$

in integers (n, m, y, p) *with* $n \equiv m \pmod 2$, $\gcd(n, m) = 1$ *or 2 and* $n > m$ *are given by*

$$F_2 + F_0 = 1, \quad F_4 + F_2 = 2^2, \quad F_6 + F_2 = 3^2,$$
$$F_2 - F_0 = 1, \quad F_3 - F_1 = 1, \quad F_5 - F_1 = 2^2, \quad F_7 - F_5 = 2^3, \quad F_{13} - F_{11} = 12^2.$$

When $p = 3$, the following lemma is a classical result. When $p \geq 17$ is a prime, it can be obtained from [10]. When $p = 4, 5, 7, 11, 13$, it can be obtained from the result of Bruin [2] and Dahmen [7].

LEMMA 2.6. *Let* p *be a prime. Suppose* (a, b, c) *is an integer solution of the Diophantine equation*

$$x^3 + y^3 = z^p, \quad p \geq 3$$

with $\gcd(a, b) = 1$, $abc \neq 0$ *and* $2 | ac$. *Then* $3 | c$ *and* $2 | a$ *but* $4 \nmid a$.

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The next lemma is proved by Darmon [8] and Darmon and Merel [9].

LEMMA 2.7. *Let $n \geq 4$ be an integer, and $p = 2$ or 3. Then there are no integer solutions of the equations*

$$x^n + y^n = z^p$$

with $\gcd(x, y) = 1$ *and* $xy \neq 0$.

One can obtain the next lemma from [4] and [5].

LEMMA 2.8. *Let q be an odd prime, with $q \equiv 3 \pmod 4$. Then the only nonnegative integer solutions (n, y, p) of the equations*

$$F_n = q^a y^p, \quad p \geq 2,\ a > 0,$$

and

$$L_n = q^a y^p, \quad p \geq 2,\ a > 0,$$

with $q < 1087$, are

$$F_0 = 0, \quad F_4 = 3, \quad F_{12} = 3^2 \times 4^2 = 3^2 \times 2^4$$

and

$$L_2 = 3, \quad L_4 = 7, \quad L_5 = 11, \quad L_8 = 47, \quad L_9 = 19 \times 2^2, \quad L_{11} = 199.$$

The next lemma can easily be obtained from the definition of the Fibonacci and Lucas sequences.

LEMMA 2.9. *The Fibonacci and Lucas sequences have the following divisibility properties:*

$$\begin{aligned}
2|F_n &\Leftrightarrow n \equiv 0 \pmod 3;\\
4|F_n &\Leftrightarrow n \equiv 0 \pmod 6;\\
3|F_n &\Leftrightarrow n \equiv 0 \pmod 4;\\
9|F_n &\Leftrightarrow n \equiv 0 \pmod{12};\\
2|L_n &\Leftrightarrow n \equiv 0 \pmod 3;\\
4|L_n &\Leftrightarrow n \equiv 3 \pmod 6;\\
3|L_n &\Leftrightarrow n \equiv 2 \pmod 4;\\
9|L_n &\Leftrightarrow n \equiv 6 \pmod{12}.
\end{aligned}$$

The residue of F_n modulo 9 depends on the residue of n modulo 12, as in the following table.

$n \pmod{12}$:	0	1	2	3	4	5	6	7	8	9	10	11
$F_n \pmod 9$:	0	1	1	2	3	5	8	4	3	7	1	8

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LEMMA 2.10. *Let $q > 3$ be an odd prime and $q \equiv 3 \pmod 4$. Then there are no positive integer solutions (n, y, p) of the equations*

$$F_n = 3^a q^b y^p, \quad p \geq 4, a \geq 2, b > 0$$

and

$$L_n = 3^a q^b y^p, \quad p \geq 4, a \geq 2, b > 0,$$

with $q < 1087$.

PROOF. First, we consider the equation $L_n = 3^a q^b y^p$. Since $a \geq 2$, Lemma 2.9 implies $n \equiv 6 \pmod{12}$. Let $n = 6k$. Then,

$$3^a q^b y^p = L_{6k} = L_{2k}(L_{2k}^2 + 3(-1)^{2k+1}) = L_{2k}(L_{2k}^2 - 3).$$

As $3|L_n$, we can see that $\gcd(L_{2k}, L_{2k}^2 - 3) = 3$. We therefore consider two cases.

Case (i). $L_{2k} = 3^{a-1} z^p$. Here $2k = 2$ by Lemma 2.8, so $3^a q^b y^p = L_{6k} = L_6 = 3^2 \times 2$. Therefore, there are no integer solutions.

Case (ii). $L_{2k}^2 - 3 = 3z^p$. This gives the Diophantine equation $3x^2 - 1 = z^p$ with $L_{2k} = 3x$, which has no integer solutions by [1, Theorem 1.1].

Next, we consider the equation $F_n = 3^a q^b y^p$. By Lemma 2.9, we have $12|n$ for $a \geq 2$. Put $n = 12k$. Then $3^a q^b y^p = F_{12k} = F_{6k} L_{6k}$. By Lemma 2.9, $2|F_{6k}, 2|L_{6k}$ and by Lemma 2.3, $\gcd(F_{6k}, L_{6k}) = 2$. Now there are three possibilities.

Case (1). $F_{6k} = 2^s w^p$ or $L_{6k} = 2^s w^p$ with $s = 1$ or $p - 1$. By Lemma 2.1, $6k = 6$ or 12 for the first and $6k = 6$ for the second. There are no integer solutions in any of these cases.

Case (2). $F_{6k} = 2^s \times 3^a w^p$ with $s = 1$ or $p - 1$. Here, $2^s \times 3^a w^p = F_{6k} = F_{3k} L_{3k}$. Using Lemmas 2.3 and 2.9, we see that $\gcd(F_{3k}, L_{3k}) = 2$. Thus, $F_{3k} = 2^t u^p$ or $L_{3k} = 2^t u^p$, for some integer u. We deduce that $3k = 3, 6, 12$ or $3, 6$ respectively, yielding no integer solutions.

Case (3). $L_{6k} = 2^s \times 3^a w^p$, with $s = 1$ or $p - 1$. By Lemma 2.8, we get $s = 1$. Hence, from $2 \times 3^a w^p = L_{6k} = L_{2k}(L_{2k}^2 - 3)$ and $\gcd(L_{2k}, L_{2k}^2 - 3) = 3$, we obtain either $L_{2k} = 3^{a-1} z^p$, or $L_{2k}^2 - 3 = 3z^p$. Neither possibility yields any integer solutions by an argument similar to that at the beginning of the proof.

This completes the proof of Lemma 2.10. □

3. Proofs of the main theorems

Let $\gcd(n, m) = d$, so that $\gcd(F_n, F_m) = F_d$ by Lemma 2.3. Thus $\gcd(F_n, F_m) = 1$ means $\gcd(n, m) = 1$ or 2. We assume $y > 0$ for the remainder of the proofs.

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3.1. Proof of Theorem 1.1. Under the congruence conditions on n and m,

$$y^p = F_n^2 \pm F_m^2 = F_{n+m}F_{n-m}$$

by Catalan's identity. Since $\gcd(F_n, F_m) = 1$, we get $\gcd(n, m) = 1$ or 2 and then $\gcd(n + m, n - m) = 1, 2$ or 4. Hence, by Lemma 2.3(i), $\gcd(F_{n+m}, F_{n-m}) = 1$ or 3 since $F_1 = F_2 = 1$, $F_4 = 3$. Therefore, we have one of the two following cases:

(i) $F_{n+m} = z^p, F_{n-m} = w^p, y = zw$;
(ii) $F_{n+m} = 3^s z^p, F_{n-m} = 3^{p-s} w^p, y = 3zw, s = 1$ or $p - 1$.

By Lemma 2.1, $n + m = 1, 2, 6$, or 12 in Case (i) and $(n, m, y, p) = (1, 0, 1, k)$ for Equation (1.1) and $(n, m, y, p) = (2, 0, 1, k), (4, 2, 2, 3), (7, 5, 12, 2)$ for Equation (1.2). By Lemma 2.8, $n + m = 4$ or 12 in Case (ii), which yields no integer solutions. Therefore, Theorem 1.1 is proved.

3.2. Proof of Theorem 1.2. Since $\gcd(F_n, F_m) = 1$, Lemma 2.4 implies the two cases:

(1) $F_n \pm F_m = z^p$;
(2) $F_n \pm F_m = q^{p-1} z^p$.

Case (1). $F_n \pm F_m = z^p$. Recall the condition $n \equiv m \pmod 2$. By Lemma 2.5,

$$F_2 + F_0 = 1, \quad F_4 + F_2 = 2^2, \quad F_6 + F_2 = 3^2,$$
$$F_2 - F_0 = 1, \quad F_3 - F_1 = 1, \quad F_5 - F_1 = 2^2, \quad F_7 - F_5 = 2^3, \quad F_{13} - F_{11} = 12^2.$$

This gives the potential solutions

$$F_2^q + F_0^q = y^p, \quad F_4^q + F_2^q = y^p, \quad F_6^q + F_2^q = y^p,$$
$$F_2^q - F_0^q = y^p, \quad F_3^q - F_1^q = y^p, \quad F_5^q - F_1^q = y^p, \quad F_7^q - F_5^q = y^p, \quad F_{13}^q - F_{11}^q = y^p,$$

that is

$$1^q + 0^q = y^p, \quad 3^q + 1^q = y^p, \quad 8^q + 1^q = y^p,$$
$$1^q - 0^q = y^p, \quad 2^q - 1^q = y^p, \quad 5^q - 1^q = y^p, \quad 13^q - 5^q = y^p, \quad 233^q - 89^q = y^p.$$

From $1^q \pm 0^q = 1^p$, we get the integer solutions $(n, m, y, q, p) = (2, 0, 1, k, l)$ for the two equations. By the well-known result on the Catalan equation (that the Catalan equation $x^p - y^q = 1$ only has the solution $3^2 - 2^3 = 1$), the only equations we need to treat are the last two, that is $13^q - 5^q = y^p$ and $233^q - 89^q = y^p$.

For the equation $13^q - 5^q = y^p$, because $13 - 5 = F_7 - F_5 = 2^3$, we obtain $p = 3$. Then, $13^q - 5^q = y^3$, that is $13^q + (-5)^q = y^3$ since q is an odd prime. However, the equation $x^3 + y^3 = z^3$ has no integer solutions with $xyz \neq 0$, so $q \neq 3$ and then $q \geq 5$. This is impossible by Lemma 2.7.

For the equation $233^q - 89^q = y^p$, because $233 - 89 = F_{13} - F_{11} = 12^2$, we obtain $p = 2$. Thus, we consider the equation $233^q - 89^q = y^2$, that is $233^q + (-89)^q = y^2$ as

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q is an odd prime. Since $3|12^2$, we get $q \neq 3$ and deduce $q \geq 5$. This is impossible by Lemma 2.7.

Case 2. $F_n \pm F_m = q^{p-1}z^p$. From Lemma 2.7 we can assume $p \geq 5$. By Lemma 2.2, $F_n \pm F_m = F_N L_M$, with $N = (n \pm m)/2$ and $M = (n \mp m)/2$. As $\gcd(n, m) = 1$ or 2, we get $\gcd(N, M) = 1$ or 2. Since $L_2 = 3$, by Lemma 2.3, $\gcd(F_N, L_M) = 1, 2$, or 3.

First, consider $\gcd(F_N, L_M) = 3$. We have $F_N = 3^t w^p$ or $L_M = 3^t w^p$ with $t = 1$ or $t \geq p - 2 \geq 3$. Hence, by Lemma 2.8, $N = 4, t = 1$ or $M = 2, t = 1$. If $N = 4 = (n + m)/2$ or $M = 2 = (n + m)/2$, it is easy to see that there are no solutions since $F_n \pm F_m = q^{p-1}z^p \geq 3 \geq 3^4 = 81$. Thus, $N = 4 = (n - m)/2$ or $M = 2 = (n - m)/2$, that is $n = m + 8$ or $n = m + 4$, and so $(n + m)/2 = m + 4$ or $m + 2$. Therefore, we must consider the following two cases.

(i) $F_{m+2} = 3^{p-2}z^p$ or $L_{m+4} = 3^{p-2}z^p$, for $q = 3$. Since $p - 2 \geq 3$, there are no integer solutions by Lemma 2.8.

(ii) $F_{m+2} = 3^{p-1}q^{p-1}z^p$ or $L_{m+4} = 3^{p-1}q^{p-1}z^p$, for $q > 3$. There are no integer solutions by Lemma 2.10.

Now, we consider $\gcd(F_N, L_M) = 1$ or 2. Then, $F_N = 2^s w^p$ or $L_M = 2^s w^p$ with $s = 0, 1$, or $p - 1 \geq 4$. Using Lemma 2.1, $N = 1, 2, 3, 6, 12$, or $M = 1, 3, 6$. Since $F_6 = 2^3, F_{12} = 2^2 \times 6^2 = 2^4 \times 3^2$ and $L_3 = 2^2, L_6 = 2 \times 3^2$, we only need to consider $N = 1, 2, 3$, or $M = 1$. Similarly, $N = (n - m)/2 = 1, 2, 3$, or $M = (n - m)/2 = 1$. For $N = (n - m)/2 = 1, 2$, or $M = (n - m)/2 = 1$, we have $F_1 = F_2 = L_1 = 1$, so $L_{m+1} = q^{p-1}z^p$, or $L_{m+2} = q^{p-1}z^p$, or $F_{m+1} = q^{p-1}z^p$, none of which yield any integer solutions by Lemma 2.8 as $p \geq 5$.

Finally, we only need to deal with the case $N = (n - m)/2 = 3$, that is, $n = m + 6$. From $F_2 = 2$, we have $s = 1$ and then $L_{m+3} = 2^{p-1}q^{p-1}z^p$. By Lemma 2.9, we have $m + 3 \equiv 3 \pmod 6$ since $4|L_{m+3}$, so $2|m$. Let $m + 3 = 3k, 2 \nmid k$. Then $L_{m+3} = L_k(L_k^2 + 3)$. If $3|L_{m+3}$, then $m + 3 \equiv 2 \pmod 4$ and thus $2 \nmid m$, which is a contradiction. Therefore, $3 \nmid L_{m+3}$ and $\gcd(L_k, L_k^2 + 3) = 1$. We deduce that $L_k = 2^{p-1}u^p, q^{p-1}v^p$ or $L_k^2 + 3 = w^p$. The first two equations have no integer solutions by Lemma 2.1 and Lemma 2.8. The last equation also has no integer solution from Nagell [13] since $p \geq 5$. This proves Theorem 1.2.

3.3. Proof of Theorem 1.3. By Theorem 1.2, we only need to treat the case $n \not\equiv m \pmod 2$ with $n > m$. If $m = 0$, then $n = 1$ since $\gcd(F_n, F_m) = 1$. Therefore, we assume $m \geq 1$, which gives $yF_nF_m \neq 0$ and $\gcd(F_n, F_m) = 1$. By Lemma 2.6, we have $3|y$ and, by Lemma 2.4, $F_n \pm F_m = 3^{p-1}z^p$. We deduce that $9|F_n \pm F_m = \pm(F_{n'} \pm F_{m'})$ with $2|n'$ and $2 \nmid m'$. On the other hand, $2|F_k$ but $4 \nmid F_k$ if and only if $k \equiv 3 \pmod 6$ by Lemma 2.9. Moreover, by Lemma 2.6, $m' \equiv 3 \pmod 6$. Put $n' = 2s$ and $m' = 6t + 3$. Then, by Lemma 2.9 and $3 \nmid F_n$, $3 \nmid F_m$, we see that $F_{n'} = F_{2s} \equiv 1, 8 \pmod 9$ and $F_{m'} = F_{6t+3} \equiv 2, 7 \pmod 9$. Thus, $F_{n'} \pm F_{m'} \not\equiv 0 \pmod 9$. This is impossible. Therefore, Theorem 1.3 is proved.

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