Acknowledgements:

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108.07 De Moivre's theorem via difference equations

An alternative proof of De Moivre's theorem to the usual methods of *mathematical induction* and *exponential form* (e.g. [1]) is given that is based on the solution of *linear difference equations* [2]. The derivation is surprisingly straightforward and emerges from two trigonometric identities. It should be stressed that this derivation is not claimed to be in any way superior to traditional approaches, only an interesting different approach to achieve the same result. We also see that the difference equation/recurrence formula gives a useful way of expressing $\cos n\theta$ and $\sin n\theta$ as sums of powers of $\cos\theta$ and $\sin\theta$, as well as the reverse process of expressing powers of $\cos\theta$ and $\sin\theta$ as a sum of multiple angles of these functions.

First, we consider the well-known *addition of cosines* identity [3]

$$\cos(\alpha + \beta) + \cos(\alpha - \beta) \equiv 2\cos\alpha\cos\beta$$

with $\alpha = (n + 1)\theta$ and $\beta = \theta$, to give

$$\cos(n + 2)\theta + \cos n\theta \equiv 2\cos(n + 1)\theta\cos\theta$$

$$\text{or} \qquad c_{n+2} + c_n = 2c_1c_{n+1}, \qquad \text{where} \quad c_k = \cos k\theta$$

$$\text{i.e.} \qquad c_{n+2} - 2c_1c_{n+1} + c_n = 0 \qquad (1)$$

$n = 0, 1, 2, 3, \ldots$, with initial conditions $c_0 = 1$, $c_1 = \cos\theta$.

Notice that (1) is a linear, constant-coefficient difference equation as c_1 is a fixed value for any particular θ. Following the procedure for solving such equations [2], the *auxiliary equation* for (1) is

$$m^2 - 2c_1m + 1 = 0$$

giving

$$m = c_1 \pm \sqrt{c_1^2 - 1} = c_1 \pm is_1,$$

where $i = \sqrt{-1}$ and $s_1 = \sin\theta$.

Hence the general solution [2] of the difference equation (1) is

$$c_n = A(c_1 + is_1)^n + B(c_1 - is_1)^n, \quad (A \text{ and } B \text{ arbitrary constants}).$$

Using the initial conditions, $c_0 = 1$, $c_1 = \cos\theta$ we have $A = B = \frac{1}{2}$ and so

$$c_n = \tfrac{1}{2}(c_1 + is_1)^n + \tfrac{1}{2}(c_1 - is_1)^n = \text{Re}\left\{(c_1 + is_1)^n\right\}$$

where the second term is the conjugate of the first.

Next, we consider the well-known *addition of sines* identity [3]

$$\sin(\alpha + \beta) + \sin(\alpha - \beta) \equiv 2\sin\alpha\cos\beta$$

again with $\alpha = (n + 1)\theta$ and $\beta = \theta$, yielding

$$\sin(n + 2)\theta + \sin n\theta \equiv 2\sin(n + 1)\theta\cos\theta$$

$$\text{or} \qquad s_{n+2} + s_n = 2c_1 s_{n+1}, \text{ where } s_k = \sin k\theta$$

$$\text{i.e.} \qquad s_{n+2} - 2c_1 s_{n+1} + s_n = 0. \tag{2}$$

We notice that (1) and (2) are essentially the same, except that (2) has the initial conditions, $s_0 = 0$ and $s_1 = \sin\theta$. Therefore the general solution of (2) is

$$s_n = C(c_1 + is_1)^n + D(c_1 - is_1)^n \qquad (C \text{ and } D \text{ arbitrary constants})$$

and using the initial conditions, we have $C = -\frac{1}{2}i$ and $D = \frac{1}{2}i$, giving

$$s_n = -\tfrac{1}{2}i(c_1 + is_1)^n + \tfrac{1}{2}i(c_1 - is_1)^n$$

$$\Rightarrow is_n = \tfrac{1}{2}(c_1 + is_1)^n - \tfrac{1}{2}(c_1 - is_1)^n, \text{ (where the terms are conjugates)}$$

$$\text{i.e.} \qquad s_n = \text{Im}\left\{(c_1 + is_1)^n\right\}.$$

Hence, adding the results for c_n and s_n gives the result for De Moivre's theorem

$$c_n + is_n = (c_1 + is_1)^n \qquad (n = 0, 1, 2, 3, \dots)$$

$$\text{or,} \qquad \cos n\theta + i\sin n\theta = (\cos\theta + i\sin\theta)^n.$$

It is easily seen that this result also applies to negative integer values of n by considering its complex conjugate, as shown by the more traditional methods [1].

Expressing multiple-angle functions as powers and vice versa

We should note here that, for obtaining trigonometric identities, equations (1) and (2) provide useful recurrence formulae for the expansions of $\cos n\theta$ and $\sin n\theta$ in terms of powers of $\cos\theta$ and $\sin\theta$. For illustration, take the successive values of $n = 0$ and 1 in (1) to give

$c_2 = 2c_1^2 - c_0 \Rightarrow \cos 2\theta = 2\cos^2\theta - 1$

$c_3 = 2c_1c_2 - c_1 = 2c_1(2c_1^2 - c_0) - c_1 = 4c_1^3 - 3c_1 \Rightarrow \cos 3\theta = 4\cos^3\theta - 3\cos\theta$

and so on.

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Similarly, for (2) we have

$s_2 = 2c_1s_1 - s_0 = 2c_1s_1 \Rightarrow \sin 2\theta = 2\cos\theta\sin\theta$

$s_3 = 2c_1s_2 - s_1 = 2c_1(2c_1s_1) - s_1 = 4(1 - s_1^2)s_1 - s_1 \Rightarrow \sin 3\theta = 3\sin\theta - 4\sin^3\theta$

and so on.

To ensure that the recurrence relations (1) and (2) hold true for all integers $n \geqslant 1$ we may use mathematical induction, i.e. assume (1) is true for some value $n = k$ and so

$$c_{k+2} - 2c_1c_{k+1} + c_k = 0.$$

Multiplying throughout by c_1 then gives

$$c_1c_{k+2} - 2c_1^2c_{k+1} + c_1c_k = 0$$

and using again the *addition of two cosines* identities $c_1c_{k+2} = \frac{1}{2}(c_{k+3} + c_{k+1})$ and $c_1c_{k+1} = \frac{1}{2}(c_{k+2} + c_k)$ this becomes

$$\tfrac{1}{2}(c_{k+3} + c_{k+1}) - 2c_1 \times \tfrac{1}{2}(c_{k+2} + c_k) + c_1c_k = 0$$

$$\text{giving } c_{k+3} - 2c_1c_{k+2} + c_{k+1} = 0.$$

Hence true for $n = k + 1$ if true for $n = k$, and we have shown it to be true for $n = 1$ above and thus true for 2, 3, 4, … . We see that a similar proof may be applied to (2) verifying its validity also.

These formulae are also particularly useful for the reverse problem of expressing $\cos^n\theta$ and $\sin^n\theta$ in terms of cosines and sines of multiple angles, which can be of value in the integration of these functions.

Rearranging the recurrence formula (1) as

$$2c_1c_{n+1} = c_{n+2} + c_n \tag{3}$$

and starting with $n = 0$ gives the familiar result

$$2c_1^2 = c_2 + 1 \Rightarrow c_1^2 = \tfrac{1}{2}(c_2 + 1) \qquad \text{or} \qquad \cos^2\theta = \tfrac{1}{2}(\cos 2\theta + 1).$$

Multiplying this by c_1 gives $2c_1^3 = c_1c_2 + c_1$ and using the recurrence relation (3) with $n = 1$ gives

$$2c_1^3 = \tfrac{1}{2}(c_3 + c_1) + c_1 = \tfrac{1}{2}(c_3 + 3c_1)$$

$$\Rightarrow c_1^3 = \tfrac{1}{4}(c_3 + 3c_1) \text{ or } \cos^3\theta = \tfrac{1}{4}(\cos 3\theta + 3\cos\theta).$$

Repeating the procedure by multiplying the last result by c_1 we have

$$4c_1^4 = c_1c_3 + 3c_1^2$$

and using (3) on both terms on the right-hand side ($n = 2$ and $n = 0$) gives

$$4c_1^4 = \tfrac{1}{2}(c_4 + c_2) + \tfrac{3}{2}(c_2 + 1) = \tfrac{1}{2}(c_4 + 4c_2 + 3)$$

$$\Rightarrow c_1^4 = \tfrac{1}{8}(c_4 + 4c_2 + 3) \text{ or } \cos^4\theta = \tfrac{1}{8}(\cos 4\theta + 4\cos 2\theta + 3).$$

Repeated multiplication by c_1 and use of the recurrence relation (3) easily generates the higher order expansions for $c_1^n = \cos^n\theta$.

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We see that the expansions for $\sin^n\theta$ can be obtained from those of $\cos^n\theta$ by substitution of $\frac{1}{2}\pi - \theta$ for θ. However, they may also be obtained by rearranging the recurrence formula (2) as

$$2c_1 s_{n+1} = s_{n+2} + s_n \tag{4}$$

and putting $n = 0$ gives the familiar result $2c_1 s_1 = s_2$ or $2\cos\theta\sin\theta = \sin 2\theta$. Multiplying this by c_1 gives

$$2c_1^2 s_1 = c_1 s_2 \Rightarrow 2\left(1 - s_1^2\right)s_1 = \tfrac{1}{2}\left(s_3 + s_1\right), \text{ using (4)}$$
$$\Rightarrow s_1^3 = \tfrac{1}{4}\left(3s_1 - s_3\right) \text{ or } \sin^3\theta = \tfrac{1}{4}(3\sin\theta - \sin 3\theta).$$

We note here that, because of the c_1 term appearing in the recurrence formula (4), only the *odd* powers of $s_1 = \sin\theta$ can be expressed in terms of multiple angles of sine only. If we multiply the above result by c_1 again, we have

$$\begin{aligned} 4c_1 s_1^3 &= 3c_1 s_1 - c_1 s_3 \\ &= \tfrac{3}{2}s_2 - \tfrac{1}{2}\left(s_4 + s_2\right), \text{ using (4)}, \end{aligned}$$
$$\Rightarrow c_1 s_1^3 = \tfrac{1}{8}\left(2s_2 - s_4\right) \text{ or } \cos\theta\sin^3\theta = \tfrac{1}{8}(2\sin 2\theta - \sin 4\theta).$$

Continuing the procedure of successive multiplication by c_1 and using $c_1^2 = 1 - s_1^2$ along with the recurrence formula (4) will generate the higher order expansions for s_1^n when n is odd. For example, applying this to the last relation gives

$$\left(1 - s_1^2\right)s_1^3 = \tfrac{1}{8}\left(2c_1 s_2 - c_1 s_4\right) = \tfrac{1}{8}\left[s_3 + s_1 - \tfrac{1}{2}\left(s_5 + s_3\right)\right]$$
$$\Rightarrow s_1^5 = s_1^3 - \tfrac{1}{16}\left(2s_1 + s_3 - s_5\right) = \tfrac{1}{4}\left(3s_1 - s_3\right) - \tfrac{1}{16}\left(2s_1 + s_3 - s_5\right)$$
$$= \tfrac{1}{16}\left(s_5 - 5s_3 + 10s_1\right)$$
$$\text{or } \sin^5\theta = \tfrac{1}{16}\left(\sin 5\theta - 5\sin 3\theta + 10\sin\theta\right).$$

However, for n even, we may express s_1^n in terms of only *cosines* of multiple angles by integration of $c_1 s_1^{n-1}$, i.e. $\int\cos\theta\sin^{n-1}\theta\, d\theta = \dfrac{\sin^n\theta}{n} = \dfrac{s_1^n}{n}$. Performing this on the expansion for $c_1 s_1^3$ above gives

$$\tfrac{1}{4}s_1^4 = \tfrac{1}{8}\left(-c_2 + \tfrac{1}{4}c_4\right) + E \quad (E \text{ an arbitrary constant})$$

and $\theta = 0$ gives $E = \frac{3}{32}$, hence we have $\sin^4\theta = \frac{1}{8}(\cos 4\theta - 4\cos 2\theta + 3)$.

Summing up, the derivation of De Moivre's theorem result by difference equations is seen to be an interesting alternative, not necessarily superior, approach to existing methods. Through recurrence formulae, it enables straightforward expansions of $\sin n\theta$ and $\cos n\theta$ in terms of powers of $\sin\theta$ and $\cos\theta$, as well as the reverse process of expressing $\sin^n\theta$ and $\cos^n\theta$ in terms of multiple-angle sines and cosines.

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108.08 Cone and Integral

Introduction

The process of deriving an equation by finding the value of a quantity in two different ways and then equating those values with each other has been there in mathematics since antiquity. In this Note, we employ the same technique to evaluate an integral by finding the volume of a right circular cone in two different ways.

Description of the cone

Let us consider a right circular cone that has a circular base of radius R in the horizontal xy-plane. The centre of the circle lies at the origin, O, of coordinates. The apex, D, of the cone lies on the vertical z-axis at the point with (x, y, z) coordinates $(0, 0, H)$, where $H > 0$ is the cone's height. Let A be the point of intersection of the cone with the positive x-axis, B be any point on the surface of the cone above the xy-plane and C be a point on the z-axis having the same z coordinate as B. In terms of spherical polar coordinates r, θ, ϕ, we take θ to be the angle measured in space from the z-axis and r to be the distance from O. The diagram is as follows:

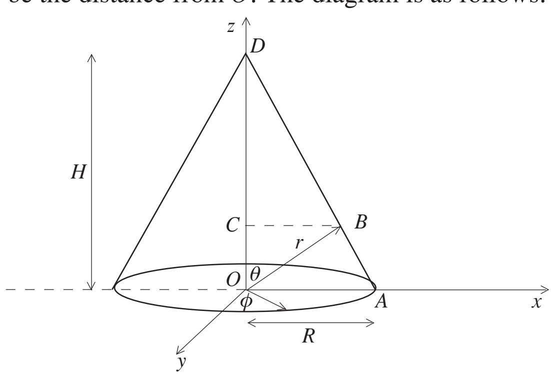

FIGURE 1