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CONGRUENCES FOR RANKS OF PARTITIONS

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Abstract

Ranks of partitions play an important role in the theory of partitions. They provide combinatorial interpretations for Ramanujan's famous congruences for partition functions. We establish a family of congruences modulo powers of 5 for ranks of partitions.

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1. Introduction

A partition of a positive integer n is a sequence of nonincreasing positive integers whose sum equals n. Let $p(n)$ denote the number of partitions of n. Ramanujan found and proved the three famous congruences:

$$p(5n+4) \equiv 0 \pmod 5, \tag{1.1}$$

$$p(7n+5) \equiv 0 \pmod 7, \tag{1.2}$$

$$p(11n+6) \equiv 0 \pmod{11}.$$

Dyson [9] defined the rank of a partition to be the largest part minus the number of parts and conjectured that ranks of partitions provided combinatorial interpretations to Ramanujan's congruences (1.1) and (1.2). Atkin and Swinnerton-Dyer [3] proved these conjectures. Namely,

$$N(m,5,5n+4) = \frac{p(5n+4)}{5}, \quad 0 \le m \le 4, \tag{1.3}$$

$$N(j,7,7n+5) = \frac{p(7n+5)}{7}, \quad 0 \le j \le 6,$$

where $N(m,k,n)$ denotes the number of partitions of n with rank congruent to m modulo k. They also obtained the generating functions for every rank difference $N(b,\ell,\ell n+d) - N(c,\ell,\ell n+d)$ with $\ell = 5, 7$ and $0 \le b,c,d \le \ell$.

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Inspired by the works of Atkin and Swinnerton-Dyer, many authors studied properties of ranks of partitions. For equalities between $N(t,k,n)$ and $N(s,k,n)$, see [1, 10, 12, 17]. For example, Andrews *et. al.* [1] found that

$$\begin{aligned}N(0,4;2n) - N(2,4;2n) &= (-1)^n(N(0,8;2n) - N(4,8;2n)),\\ N(0,4;2n+1) - N(2,4;2n+1) &= (-1)^n(N(0,8;2n+1) + N(1,8;2n+1)\\ &\quad - 2N(3,8;2n+1) - N(4,8;2n+1)).\end{aligned}$$

In [2], Andrews and Lewis made conjectures on inequalities between ranks of partitions modulo 3. Bringmann [4] first proved these conjectures: for $n \geq 0$,

$$\begin{aligned}N(0,3,3n) &< N(1,3,3n) \quad (n \neq 1,3,7), \qquad (1.4)\\ N(0,3,3n+1) &> N(1,3,3n+1),\\ N(0,3,3n+2) &< N(1,3,3n+2).\end{aligned}$$

When $n = 1,3,7$, we have equality in (1.4). Bringmann's proof relies on the circle method to obtain asymptotic results on ranks of partitions modulo 3. For more studies on the asymptotic behaviour of ranks of partitions, see [5, 6]. Later, Chan and the author provided refinements of these inequalities by using elementary q-series manipulation (see [7, Corollary 1.7]).

More recently, Chen *et al.* [8] studied congruences for ranks of partitions. Let

$$a_f(n) := N(0,2,n) - N(1,2,n).$$

They proved that, for all $\alpha \geq 3$ and all $n \geq 0$,

$$a_f(5^\alpha n + \delta_\alpha) + a_f(5^{\alpha-2}n + \delta_{\alpha-2}) \equiv 0 \pmod{5^{\lfloor \alpha/2 \rfloor}},$$

where δ_α satisfies $0 < \delta_\alpha < 5^\alpha$ and $24\delta_\alpha \equiv 1 \pmod{5^\alpha}$. In this paper, we establish the following congruences.

THEOREM 1.1. *For integers $\alpha \geq 0$, let*

$$\lambda_\alpha := \begin{cases} \dfrac{23 \cdot 5^\alpha - 19}{24} & \text{if } \alpha \text{ is odd,}\\[2mm] \dfrac{19 \cdot (5^\alpha - 1)}{24} & \text{if } \alpha \text{ is even.}\end{cases}$$

Then,

$$N(1,10,5^{\alpha+1}n + 5\lambda_\alpha + 4) \equiv N(5,10,5^{\alpha+1}n + 5\lambda_\alpha + 4) \pmod{5^{\lfloor (\alpha+1)/2 \rfloor}}.$$

REMARK 1.2. From the proof of Proposition 3.1,

$$N(0,10,5n+4) - N(4,10,5n+4) = N(1,10,5n+4) - N(5,10,5n+4). \qquad (1.5)$$

Thus,

$$N(0,10,5^{\alpha+1}n + 5\lambda_\alpha + 4) \equiv N(4,10,5^{\alpha+1}n + 5\lambda_\alpha + 4) \pmod{5^{\lfloor (\alpha+1)/2 \rfloor}}.$$

We prove Theorem 1.1 by arguments similar to those in [14, 16, 18]. We first establish some identities between modular functions on $\Gamma_0(10)$ in Section 2. Then we prove Theorem 1.1 in Section 3.

2. Preliminaries

Recall the Dedekind eta-function given by

$$\eta(\tau) := q^{1/24}(q)_\infty.$$

In the above equation and for the rest of this article, we use the notation

$$(x)_\infty := (x;q)_\infty := \prod_{k=0}^{\infty}(1-xq^k),$$

where $q = e^{2\pi i\tau}$ with $\mathrm{Im}(\tau) > 0$. We also need

$$\rho := \rho(\tau) := \frac{\eta^2(2\tau)\eta^4(5\tau)}{\eta^4(\tau)\eta^2(10\tau)},$$
$$t := t(\tau) := \frac{\eta^2(5\tau)\eta^2(10\tau)}{\eta^2(\tau)\eta^2(2\tau)},$$
$$M := M(\tau) := \frac{\eta^2(2\tau)}{\eta^2(50\tau)},$$
$$K := K(\tau) = \frac{\eta(25\tau)}{\eta(\tau)}.$$

By the criteria for the modularity of eta-products [15, Theorem 4.7], ρ, t are modular functions on $\Gamma_0(10)$ and M, K are on $\Gamma_0(50)$ and $\Gamma_0(25)$, respectively.

For $g(\tau) := \sum_{n=-\infty}^{\infty} a_g(n)q^n$, the operator U_k is defined by

$$U_k(g)(\tau) := \frac{1}{k}\sum_{\lambda=0}^{k-1} g\Big(\frac{\tau+\lambda}{k}\Big).$$

One can easily verify that

$$U_k(f(q^k)g)(\tau) = f(q)\sum_{n=-\infty}^{\infty} a_g(kn)q^n.$$

We need the following fundamental lemma.

Lemma 2.1 (See [18, Lemma 2.3]). *Let*

$$a_0(t) = -t,$$
$$a_1(t) = -t(2\cdot 5 + 5^2 t),$$
$$a_2(t) = -t(11\cdot 5 + 2\cdot 5^3 t + 5^4 t^2),$$
$$a_3(t) = -t(28\cdot 5 + 11\cdot 5^3 t + 2\cdot 5^5 t^2 + 5^6 t^3),$$
$$a_4(t) = -t(7\cdot 5^2 + 28\cdot 5^3 t + 11\cdot 5^5 t^2 + 2\cdot 5^7 t^3 + 5^8 t^4).$$

Then, for $u : \mathbb{H} \to \mathbb{C}$ *and* $j \in \mathbb{Z}$,

$$U_5(ut^j) = -\sum_{l=0}^{4} a_l(t) U_5(ut^{j+l-5}). \tag{2.1}$$

LEMMA 2.2. *Let* $U^{(0,j)}(f) := U_5(K \cdot \rho^j \cdot f)$, $U^{(1,j)}(f) := U_5(M \cdot \rho^j \cdot f)$. *Then we have four groups of identities.*

Group I

$$\begin{aligned}
U^{(0,0)}(1) &= 5\rho t \\
U^{(0,0)}(t^{-1}) &= \rho \\
U^{(0,0)}(t^{-2}) &= 1 - 4 \cdot 5t - 5^3t^2 + \rho(-4 + 5^2t) \\
U^{(0,0)}(t^{-3}) &= -6 + 7 \cdot 5^2t + 2 \cdot 5^4t^2 + 5^5t^3 + \rho(5 - 4 \cdot 5^2t) \\
U^{(0,0)}(t^{-4}) &= 13 - 18 \cdot 5^2t - 2 \cdot 5^4t^2 + 2 \cdot 5^5t^3 + 5^7t^4 \\
&\quad + \rho(2 \cdot 5^2 - 5^4t - 2 \cdot 5^5t^2 - 2 \cdot 5^6t^3).
\end{aligned}$$

Group II

$$\begin{aligned}
U^{(0,1)}(1) &= 4t - 9 \cdot 5^2t^2 - 9 \cdot 5^4t^3 - 31 \cdot 5^5t^4 - 7 \cdot 5^7t^5 - 5^9t^6 \\
&\quad + \rho(29 \cdot 5t + 57 \cdot 5^3t^2 + 37 \cdot 5^5t^3 + 63 \cdot 5^6t^4 + 11 \cdot 5^8t^5 + 5^{10}t^6) \\
U^{(0,1)}(t^{-1}) &= -5t + \rho(1 + 5^2t) \\
U^{(0,1)}(t^{-2}) &= -4 \cdot 5t - 5^3t^2 + \rho(1 + 5^2t) \\
U^{(0,1)}(t^{-3}) &= 1 + 18 \cdot 5t + 7 \cdot 5^3t^2 + 5^5t^3 + \rho(-2 \cdot 5 - 3 \cdot 5^2t - 5^4t^2) \\
U^{(0,1)}(t^{-4}) &= -2 \cdot 5 + 7 \cdot 5^2t + 4 \cdot 5^4t^2 + 5^6t^3 + 5^7t^4 + \rho(9 \cdot 5 - 2 \cdot 5^3t - 5^6t^3).
\end{aligned}$$

Group III

$$\begin{aligned}
U^{(1,0)}(1) &= -1 \\
U^{(1,0)}(t^{-1}) &= 3 \cdot 5 + 4 \cdot 5^2t + \rho(t^{-1} - 5^2) \\
U^{(1,0)}(t^{-2}) &= -12 \cdot 5 - 4 \cdot 5^3t - 5^4t^2 + \rho(-4t^{-1} + 4 \cdot 5^2) \\
U^{(1,0)}(t^{-3}) &= t^{-1} - 5^5t^2 - 5^6t^3 + \rho(2 \cdot 5^3 + 5^5t + 5^6t^2) \\
U^{(1,0)}(t^{-4}) &= -2 \cdot 5t^{-1} + 54 \cdot 5^2 + 18 \cdot 5^4t + 2 \cdot 5^6t^2 - 5^8t^4 \\
&\quad + \rho(18 \cdot 5t^{-1} - 38 \cdot 5^3 - 2 \cdot 5^6t - 2 \cdot 5^7t^2).
\end{aligned}$$

Group IV

$$\begin{aligned}
U^{(1,1)}(1) &= -4 - 5^2t - 5^4t^2 + \rho(11 \cdot 5 + 5^4t + 5^5t^2) \\
U^{(1,1)}(t^{-1}) &= 2 \cdot 5 + 4 \cdot 5^2t + \rho(t^{-1} - 4 \cdot 5) \\
U^{(1,1)}(t^{-2}) &= -5^3t - 5^4t^2 + 5^3\rho t \\
U^{(1,1)}(t^{-3}) &= -33 \cdot 5 - 11 \cdot 5^3t - 5^5t^2 - 5^6t^3 + \rho(-11t^{-1} + 11 \cdot 5^2 + 5^5t^2)
\end{aligned}$$

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$$U^{(1,1)}(t^{-4}) = t^{-1} + 33 \cdot 5^2 + 11 \cdot 5^4 t - 5^5 t^2 - 5^7 t^3 - 5^8 t^4 \\ + \rho(11 \cdot 5t^{-1} - 9 \cdot 5^3 + 5^5 t + 5^6 t^2 + 5^7 t^3).$$

SKETCH OF PROOF. The equations in Groups I–IV are identities between modular functions on $\Gamma_0(10)$. One can automatically prove them by the MAPLE package ETA [11]. For example, the Maple commands for verifying the second identity in Group II are provided at https://github.com/dongpanghu/code5.

We call a map $d : \mathbb{Z} \times \mathbb{Z} \longrightarrow \mathbb{Z}$ a discrete array if for each i, the map $d(i, -) : \mathbb{Z} \longrightarrow \mathbb{Z}$ given by $j \mapsto d(i, j)$ has finite support.

LEMMA 2.3. *There exist discrete arrays $a_{i,j}, b_{i,j}$ with $0 \le i, j \le 1$ such that for $k \in \mathbb{Z}$,*

$$U^{(i,j)}(t^k) = \sum_{n=\lceil (k-s_{i,j})/5 \rceil}^{\infty} a_{i,j}(k,n)t^n + \rho\Big(\sum_{n=\lceil (k-s_{i,j})/5 \rceil}^{\infty} b_{i,j}(k,n)t^n\Big), \tag{2.2}$$

where

$$s_{i,j} = \begin{cases} -1 & \text{when } (i,j) = (0,0), \\ -1 & \text{when } (i,j) = (0,1), \\ 4 & \text{when } (i,j) = (1,0), \\ 4 & \text{when } (i,j) = (1,1). \end{cases}$$

Moreover, the values of $a_{i,j}(k,n)$ and $b_{i,j}(k,n)$ for $-4 \le k \le 0$ are given in Groups I–IV of Lemma 2.2 and, for other k, $a_{i,j}(k,n), b_{i,j}(k,n)$, satisfy the recurrence relation in [18, (2.17)]*:*

$$\begin{aligned} m(k,n) = {} & (7 \cdot 5^2 m(k-1,n-1) + 28 \cdot 5^3 m(k-1,n-2) + 11 \cdot 5^5 m(k-1,n-3) \\ & + 2 \cdot 5^7 m(k-1,n-4) + 5^8 m(k-1,n-5)) + (28 \cdot 5m(k-2,n-1) \\ & + 11 \cdot 5^3 m(k-2,n-2) + 2 \cdot 5^5 m(k-2,n-3) + 5^6 m(k-2,n-4)) \\ & + (11 \cdot 5m(k-3,n-1) + 2 \cdot 5^3 m(k-3,n-2) + 5^4 m(k-3,n-3)) \\ & + (2 \cdot 5m(k-4,n-1) + 5^2 m(k-4,n-2)) + m(k-5,n-1). \end{aligned} \tag{2.3}$$

PROOF. We verify that the result holds for $-4 \le k \le 0$ by Groups I–IV in Lemma 2.2. Then we apply (2.1) to prove Lemma 2.3 by induction on k. □

Denote the 5-adic order of n by $\pi(n)$ and set $\pi(0) = +\infty$.

LEMMA 2.4 [18, Lemma 2.8]. *Let $g(k,n)$ be integers which satisfy the recurrence relation (2.3). Suppose there exists an integer l and a constant γ such that, for $l \le k \le l+4$,*

$$\pi(g(k,n)) \ge \Big\lfloor \frac{5n-k+\gamma}{3} \Big\rfloor. \tag{2.4}$$

Then (2.4) holds for any $k \in \mathbb{Z}$.

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Lemma 2.5. *Recall that $a_{i,j}, b_{i,j}$ are given in Lemma 2.3. For $n, k \in \mathbb{Z}$,*

$$\pi(a_{0,0}(k,n)) \geq \left\lfloor \frac{5n-k-2}{3} \right\rfloor, \quad \pi(b_{0,0}(k,n)) \geq \left\lfloor \frac{5n-k}{3} \right\rfloor,$$
$$\pi(a_{0,1}(k,n)) \geq \left\lfloor \frac{5n-k-3}{3} \right\rfloor, \quad \pi(b_{0,1}(k,n)) \geq \left\lfloor \frac{5n-k}{3} \right\rfloor,$$
$$\pi(a_{1,0}(k,n)) \geq \left\lfloor \frac{5n-k+2}{3} \right\rfloor, \quad \pi(b_{1,0}(k,n)) \geq \left\lfloor \frac{5n-k+5}{3} \right\rfloor,$$
$$\pi(a_{1,1}(k,n)) \geq \left\lfloor \frac{5n-k+2}{3} \right\rfloor, \quad \pi(b_{1,1}(k,n)) \geq \left\lfloor \frac{5n-k+4}{3} \right\rfloor.$$

Proof. For $-4 \leq k \leq 0$, the inequalities in Lemma 2.5 can be verified by using Groups I–IV in Lemma 2.2. Then the results follow from Lemma 2.4 immediately. □

3. Proof of Theorem 1.1

We first need the following generating function.

Proposition 3.1. *We have*

$$\sum_{n=0}^{\infty}(N(1,10,5n+4)-N(5,10,5n+4))q^n = \frac{(q^{10};q^{10})_\infty^2}{(q;q)_\infty}. \tag{3.1}$$

Proof. We deduce from [13, (1.14)] that

$$\sum_{n=0}^{\infty}(N(0,10,5n+4)+N(1,10,5n+4)-N(4,10,5n+4)-N(5,10,5n+4))q^n$$
$$= \frac{2(q^{10};q^{10})_\infty^2}{(q;q)_\infty}. \tag{3.2}$$

Note that

$$N(0,5,n) = N(0,10,n) + N(5,10,n)$$

and

$$N(1,5,n) = N(1,10,n) + N(4,10,n),$$

which together with (1.3) give

$$N(0,10,5n+4) - N(4,10,5n+4) = N(1,10,5n+4) - N(5,10,5n+4). \tag{3.3}$$

Equation (3.1) follows immediately from (3.2) and (3.3). □

Define

$$\sum_{n=0}^{\infty} e(n)q^n := \frac{(q^{10};q^{10})_\infty^2}{(q;q)_\infty}.$$

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Then Proposition 3.1 implies

$$N(1, 10, 5n + 4) - N(5, 10, 5n + 4) = e(n). \tag{3.4}$$

Let $L_0 := 1$ and, for $\alpha \geq 1$,

$$L_{2\alpha-1} := \frac{(q^5; q^5)_\infty}{(q^2; q^2)^2_\infty} \sum_{n=0}^{\infty} e(5^{2\alpha-1}n + \lambda_{2\alpha-1})q^{n+1},$$
$$L_{2\alpha} := \frac{(q; q)_\infty}{(q^{10}; q^{10})^2_\infty} \sum_{n=0}^{\infty} e(5^{2\alpha}n + \lambda_{2\alpha})q^n.$$

LEMMA 3.2. *For all* $\alpha \geq 0$,

$$L_{2\alpha+1} = U^{(0,0)}(L_{2\alpha}), \tag{3.5}$$

$$L_{2\alpha+2} = U^{(1,0)}(L_{2\alpha+1}). \tag{3.6}$$

PROOF. For any $\alpha \geq 0$,

$$\begin{aligned} U^{(0,0)}(L_{2\alpha}) &= U_5\Big(\frac{q(q^{25}; q^{25})_\infty}{(q; q)_\infty} \cdot \frac{(q; q)_\infty}{(q^{10}; q^{10})^2_\infty} \sum_{n=0}^{\infty} e(5^{2\alpha}n + \lambda_{2\alpha})q^n\Big) \\ &= \frac{(q^5; q^5)_\infty}{(q^2; q^2)^2_\infty} \cdot U_5\Big(\sum_{n=0}^{\infty} e(5^{2\alpha}n + \lambda_{2\alpha})q^{n+1}\Big) \\ &= \frac{(q^5; q^5)_\infty}{(q^2; q^2)^2_\infty} \cdot \sum_{n=0}^{\infty} e(5^{2\alpha}(5n + 4) + \lambda_{2\alpha})q^{n+1} \\ &= \frac{(q^5; q^5)_\infty}{(q^2; q^2)^2_\infty} \cdot \sum_{n=0}^{\infty} e(5^{2\alpha+1}n + \lambda_{2\alpha+1})q^{n+1} \\ &= L_{2\alpha+1}. \end{aligned}$$

Similarly,

$$\begin{aligned} U^{(1,0)}(L_{2\alpha+1}) &= U_5\Big(\frac{(q^2; q^2)^2_\infty}{q^4(q^{50}; q^{50})^2_\infty} \cdot \frac{(q^5; q^5)_\infty}{(q^2; q^2)^2_\infty} \sum_{n=0}^{\infty} e(5^{2\alpha+1}n + \lambda_{2\alpha+1})q^{n+1}\Big) \\ &= \frac{(q; q)_\infty}{(q^{10}; q^{10})^2_\infty} \cdot U_5\Big(\sum_{n=0}^{\infty} e(5^{2\alpha+1}n + \lambda_{2\alpha+1})q^{n-3}\Big) \\ &= \frac{(q; q)_\infty}{(q^{10}; q^{10})^2_\infty} \cdot \sum_{n=0}^{\infty} e(5^{2\alpha+1}(5n + 3) + \lambda_{2\alpha+1})q^n \\ &= \frac{(q; q)_\infty}{(q^{10}; q^{10})^2_\infty} \cdot \sum_{n=0}^{\infty} e(5^{2\alpha+2}n + \lambda_{2\alpha+2})q^n \\ &= L_{2\alpha+2}. \end{aligned}$$

□

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THEOREM 3.3. *There exists discrete arrays c, d such that for $\alpha \geq 1$,*

$$L_\alpha = \sum_{n \geq \delta_\alpha} c(\alpha, n) t^n + \rho\Big(\sum_{n \geq \delta_\alpha} d(\alpha, n) t^n\Big),$$

where

$$\delta_\alpha = \begin{cases} 1 & \text{if } \alpha \text{ is odd,} \\ 0 & \text{otherwise.} \end{cases}$$

Moreover,

$$\pi(c(\alpha, n)) \geq \begin{cases} \dfrac{\alpha - 1}{2} + \left\lfloor \dfrac{5n - 2}{3} \right\rfloor & \text{if } \alpha \text{ is odd,} \\ \dfrac{\alpha}{2} + \left\lfloor \dfrac{5n + 1}{3} \right\rfloor & \text{otherwise} \end{cases} \tag{3.7}$$

and

$$\pi(d(\alpha, n)) \geq \begin{cases} \dfrac{\alpha - 1}{2} + \left\lfloor \dfrac{5n}{3} \right\rfloor & \text{if } \alpha \text{ is odd,} \\ \dfrac{\alpha}{2} + \left\lfloor \dfrac{5n + 3}{3} \right\rfloor & \text{otherwise.} \end{cases} \tag{3.8}$$

PROOF. Let $c(1, k) = 0$ for $k \geq 1$ and $d(1, 1) = 5, d(1, k) = 0$ for $k \geq 2$. For $\alpha \geq 1$, define

$$\begin{aligned} c(2\alpha, n) &= \sum_{k \geq 1} [c(2\alpha - 1, k) a_{1,0}(k, n) + d(2\alpha - 1, k) a_{1,1}(k, n)], \\ d(2\alpha, n) &= \sum_{k \geq 1} [c(2\alpha - 1, k) b_{1,0}(k, n) + d(2\alpha - 1, k) b_{1,1}(k, n)], \\ c(2\alpha + 1, n) &= \sum_{k \geq 0} [c(2\alpha, k) a_{0,0}(k, n) + d(2\alpha, k) a_{0,1}(k, n)], \\ d(2\alpha + 1, n) &= \sum_{k \geq 0} [c(2\alpha, k) b_{0,0}(k, n) + d(2\alpha, k) b_{0,1}(k, n)]. \end{aligned}$$

From Lemma 2.2, Group I and (3.5),

$$L_1 = U^{(0,0)}(L_0) = U^{(0,0)}(1) = \rho\Big(\sum_{n=1}^{1} d(1, n) t^n\Big),$$

which gives $\pi(c(1, n)) \geq \lfloor (5n - 2)/3 \rfloor$ and $\pi(d(1, n)) \geq \lfloor 5n/3 \rfloor$. From (3.6),

$$\begin{aligned} L_2 &= U^{(1,0)}(L_1) = U_5(M \cdot L_1) = \sum_{n=1}^{1} d(1, n) U_5(M \rho t^n) = \sum_{n=1}^{1} d(1, n) U^{(1,1)}(t^n) \\ &= \sum_{n=1}^{1} d(1, n)\Big(\sum_{k \geq \lceil (n-4)/5 \rceil} a_{1,1}(n, k) t^k + \rho\Big(\sum_{k \geq \lceil (n-4)/5 \rceil} b_{1,1}(n, k) t^k\Big)\Big) \qquad \text{(by (2.2))} \end{aligned}$$

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$$
\begin{aligned}
&= \sum_{n\geq 0}\Big[\sum_{k=1}^{1} d(1,k)a_{1,1}(k,n)\Big]t^n + \rho\Big\{\sum_{n\geq 0}\Big[\sum_{k=1}^{1} d(1,k)b_{1,1}(k,n)\Big]t^n\Big\}\\
&= \sum_{n\geq 0} c(2,n)t^n + \rho\Big(\sum_{n\geq 0} d(2,n)t^n\Big).
\end{aligned}
$$

From Lemma 2.5,

$$
\begin{aligned}
\pi(c(2,n)) &= \pi(d(1,1)) + \pi(a_{1,1}(1,n)) \geq 1 + \Big\lfloor \frac{5n+1}{3} \Big\rfloor,\\
\pi(d(2,n)) &= \pi(d(1,1)) + \pi(b_{1,1}(1,n)) \geq 1 + \Big\lfloor \frac{5n+3}{3} \Big\rfloor.
\end{aligned}
$$

Thus, the result holds for L_α when $\alpha = 1, 2$. We proceed by induction. Suppose that the result holds for $L_{2\alpha}$. Then, applying (2.2) and (3.5),

$$
\begin{aligned}
L_{2\alpha+1} &= U^{(0,0)}(L_{2\alpha}) = U_5(K\cdot L_{2\alpha})\\
&= \sum_{n\geq 0} c(2\alpha,n)U_5(Kt^n) + \sum_{n\geq 0} d(2\alpha,n)U_5(K\rho t^n)\\
&= \sum_{n\geq 0} c(2\alpha,n)U^{(0,0)}(t^n) + \sum_{n\geq 0} d(2\alpha,n)U^{(0,1)}(t^n)\\
&= \sum_{n\geq 0} c(2\alpha,n)\Big(\sum_{k=\lceil (n+1)/5\rceil}^{\infty} a_{0,0}(n,k)t^k + \rho\Big(\sum_{k=\lceil (n+1)/5\rceil}^{\infty} b_{0,0}(n,k)t^k\Big)\Big)\\
&\quad + \sum_{n\geq 0} d(2\alpha,n)\Big(\sum_{k=\lceil (n+1/5\rceil}^{\infty} a_{0,1}(n,k)t^k + \rho\Big(\sum_{k=\lceil (n+1)/5\rceil}^{\infty} b_{0,1}(n,k)t^k\Big)\Big)\\
&= \sum_{n\geq 1}\Big[\sum_{k=0}^{\infty}(c(2\alpha,k)a_{0,0}(k,n) + d(2\alpha,k)a_{0,1}(k,n))\Big]t^n\\
&\quad + \rho\Big\{\sum_{n\geq 1}\Big[\sum_{k=0}^{\infty}(c(2\alpha,k)b_{0,0}(k,n) + d(2\alpha,k)b_{0,1}(k,n))\Big]t^n\Big\}\\
&= \sum_{n\geq 1} c(2\alpha+1,n)t^n + \rho\Big(\sum_{n\geq 1} d(2\alpha+1,n)t^n\Big).
\end{aligned}
$$

Moreover, using Lemma 2.5, (3.7) and (3.8), we find that

$$
\begin{aligned}
\pi(c(2\alpha+1,n)) &\geq \min_{k\geq 0}\{\pi(c(2\alpha,k)) + \pi(a_{0,0}(k,n)), \pi(d(2\alpha,k)) + \pi(a_{0,1}(k,n))\}\\
&\geq \alpha + \Big\lfloor \frac{5n-2}{3} \Big\rfloor
\end{aligned}
$$

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and

$$\pi(d(2\alpha+1,n)) \geq \min_{k\geq 0}\{\pi(c(2\alpha,k))+\pi(b_{0,0}(k,n)),\pi(d(2\alpha,k))+\pi(b_{0,1}(k,n))\}$$
$$\geq \alpha+\left\lfloor\frac{5n}{3}\right\rfloor.$$

Next, we consider $L_{2\alpha+2}$. Using (2.2) and (3.6),

$$\begin{aligned}
L_{2\alpha+2} &= U^{(1,0)}(L_{2\alpha+1}) = U_5(M\cdot L_{2\alpha+1})\\
&= \sum_{n\geq 1} c(2\alpha+1,n)U_5(Mt^n) + \sum_{n\geq 1} d(2\alpha+1,n)U_5(M\rho t^n)\\
&= \sum_{n\geq 1} c(2\alpha+1,n)U^{(1,0)}(t^n) + \sum_{n\geq 1} d(2\alpha+1,n)U^{(1,1)}(t^n)\\
&= \sum_{n\geq 1} c(2\alpha+1,n)\Big(\sum_{k=\lceil (n-4)/5\rceil}^{\infty} a_{1,0}(n,k)t^k + \rho\Big(\sum_{k=\lceil (n-4)/5\rceil}^{\infty} b_{1,0}(n,k)t^k\Big)\Big)\\
&\quad + \sum_{n\geq 1} d(2\alpha+1,n)\Big(\sum_{k=\lceil (n-4)/5\rceil}^{\infty} a_{1,1}(n,k)t^k + \rho\Big(\sum_{k=\lceil (n-4)/5\rceil}^{\infty} b_{1,1}(n,k)t^k\Big)\Big)\\
&= \sum_{n\geq 0}\Big[\sum_{k=1}^{\infty}(c(2\alpha+1,k)a_{1,0}(k,n)+d(2\alpha+1,k)a_{1,1}(k,n))\Big]t^n\\
&\quad + \rho\Big\{\sum_{n\geq 0}\Big[\sum_{k=1}^{\infty}(c(2\alpha+1,k)b_{1,0}(k,n)+d(2\alpha+1,k)b_{1,1}(k,n))\Big]t^n\Big\}\\
&= \sum_{n\geq 0} c(2\alpha+2,n)t^n + \rho\Big(\sum_{n\geq 0} d(2\alpha+2,n)t^n\Big).
\end{aligned}$$

Then again by Lemma 2.5, (3.7) and (3.8),

$$\pi(c(2\alpha+2,n)) \geq \min_{k\geq 1}\{\pi(c(2\alpha+1,k))+\pi(a_{1,0}(k,n)),\pi(d(2\alpha+1,k))+\pi(a_{1,1}(k,n))\}$$
$$\geq \alpha+1+\left\lfloor\frac{5n+1}{3}\right\rfloor$$

and

$$\pi(d(2\alpha+2,n)) \geq \min_{k\geq 1}\{\pi(c(2\alpha+1,k))+\pi(b_{1,0}(k,n)),\pi(d(2\alpha+1,k))+\pi(b_{1,1}(k,n))\}$$
$$\geq \alpha+1+\left\lfloor\frac{5n+3}{3}\right\rfloor.$$

Thus, the result holds for $L_{2\alpha+2}$. This proves Theorem 3.3 by induction. □

COROLLARY 3.4. *For $\alpha \geq 1$,*

$$e(5^{\alpha}n+\lambda_{\alpha}) \equiv 0 \pmod{5^{\lfloor(\alpha+1)/2\rfloor}}.$$

PROOF. The result follows immediately from Lemma 3.2 and Theorem 3.3. □

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Note that (3.4) together with Corollary 3.4 implies (1.5). Thus, the proof of Theorem 1.1 is complete.

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