IDEMPOTENT-SEPARATING EXTENSIONS OF INVERSE SEMIGROUPS

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Extensions of semigroups have been studied from two points of view; ideal extensions and Schreier extension. In this paper another type of extension is considered for the class of inverse semigroups. The main result (Theorem 2) is stated in the form of the classical treatment of Schreier extensions (see e.g. [7]). The motivation for the definition of idempotent-separating extension comes primarily from G. B. Preston's concept of a normal set of subsets of a semigroup [6]. The characterization of such extensions is applied to give another description of bisimple inverse ω-semigroups, which were first described by N. R. Reilly [8]. The main tool used in the proof of Theorem 2 is Preston's characterization of congruences on an inverse semigroup [5]. For the standard terminology used, the reader is referred to [1].

1. Definitions and preliminaries

Let A and B be inverse semigroups. A pair (S, f) where S is an inverse semigroup containing A as a subsemigroup and f is a homomorphism of S onto B such that $f^{-1}(E(B)) = A$ (where for a semigroup T, $E(T) = \{e \in T : e^2 = e\}$) is said to be an *extension* of A by B.

If moreover f has the property that $f(e) = f(g)$ implies $e = g$ for $e, g \in E(A)$ then (S, f) is said to be an *idempotent-separating extension* of A by B.

Let A and B be inverse semigroups and let (S, f) be an idempotent separating extension of A by B; by a *transversal* of B in S we mean a mapping g from B into S such that

(i) $f(g(b)) = b$ for all $b \in B$.

(ii) $g(e)$ is the unique idempotent in $f^{-1}(e)$ for all $e \in E(B)$.

(Sometimes $\{g(b) : b \in B\}$ will be itself called a transversal of B in S.)

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Note that a transversal always exists; however it is not usually a homomorphism.

2. The construction

Let A and B be inverse semigroups and let $\bar{f}$ be a homomorphism from A onto $E(B)$ such that $\bar{f}$ is one to one on $E(A)$. (The existence of such an $\bar{f}$ implies that A is a semilattice of groups, see e.g. [5]). Suppose that with each $b \in B$ there are associated mappings u and v of A into A denoted by $a \to a^b$ and $a \to b^a$ respectively and with each pair $(b, c) \in B \times B$ there is associated an element $b^c \in \bar{f}^{-1}(c^{-1}b^{-1}bc)$; such that:

(p1) u and v are endomorphisms of A and for all $a \in A$; $b, c \in B$
If $a \in \bar{f}^{-1}(b^{-1}b)$ then $a^c \in \bar{f}^{-1}(c^{-1}b^{-1}bc)$ and
If $a \in \bar{f}^{-1}(b^{-1}b)$ then $c^a \in \bar{f}^{-1}(cb^{-1}bc^{-1})$

(p2) If $a \in \bar{f}^{-1}(b^{-1}b)$ then $(b^a)^{bc} \cdot b^c = b^c \cdot a^c$ for all $c \in B$.

(p3) $a^{bc} \cdot b^c = b^c \cdot (a^b)^c$ for all $a \in A$; $c \in B$.

(p4) $(bc)^d (b^c)^d = b^{cd} c^d$ for $b, c, d \in B$.

(p5) (i) if $a \in \bar{f}^{-1}(bb^{-1})$ then $a^{bb^{-1}} = a$
(ii) if $a \in \bar{f}^{-1}(bb^{-1})$ then $(b^a)^b = a$
(iii) $(bb^{-1})^b = e$ where $e \in E(A) \cap \bar{f}^{-1}(b^{-1}b)$
(iv) if $b^2 = b \in B$, $a \in A$, $e \in E(A) \cap \bar{f}^{-1}(b)$ then $a^b = ea$
(v) $(bb^{-1})^{cc^{-1}} = eg$ where $e \in E(A) \cap \bar{f}^{-1}(bb^{-1})$
and $g \in E(A) \cap \bar{f}^{-1}(cc^{-1})$.

Let $S^* = \{(b, a) : b \in B, a \in \bar{f}^{-1}(b^{-1}b)\}$ and define an operation $\circ$ on S^* by

$$(b, a) \circ (c, a') = (bc, b^c a^c a').$$

Moreover, define a mapping f from S^* to B by $f(b, a) = b$.

Theorem 1. *(S^*, f) is an idempotent separating extension of A by B. Moreover $f|A = \bar{f}$.*

Proof. (1) S^* is closed under $\circ$:
If $(b, a), (c, a') \in S^*$ then

$$\begin{aligned}\bar{f}(b^c a^c a') &= \bar{f}(b^c)\bar{f}(a^c)\bar{f}(a') \\ &= c^{-1}b^{-1}bc \cdot c^{-1}b^{-1}bc \cdot c^{-1}c = (c^{-1}b^{-1}bc)^2 = c^{-1}b^{-1}bc\end{aligned}$$

hence $(b, a) \cdot (c, a') \in S^*$.

(2) Associativity:

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$$
\begin{aligned}
[(b, a) \circ (c, a')] \circ (d, a'') &= (bc, b^c a^c a') \circ (d, a'') \\
&= (bc \cdot d, (bc)^d (b^c a^c a')^d a'') \\
&= (b \cdot cd, (bc)^d (b^c)^d (a^c)^d a'^d a'') && \text{by (p1)} \\
&= (b \cdot cd, b^{cd} c^d (a^c)^d (a'^d) a'') && \text{by (p4)} \\
&= (b \cdot cd, b^{cd} a^{cd} c^d a'^d a'') && \text{by (p3)} \\
&= (b, a) \circ (cd, c^d a'^d a'') \\
&= (b, a) \circ [(c, a') \circ (d, a'')].
\end{aligned}
$$

(3) S^* is an inverse semigroup:

(i) S^* is regular:

Let $(b, a) \in S^*$, then

$$(b, a)(b^{-1}, x)(b, a) = (bb^{-1}, b^{b^{-1}} a^{b^{-1}} x)(b, a) = (b, (bb^{-1})^b (b^{b^{-1}} a^{b^{-1}} x)^b a).$$

Let $x = (a^{b^{-1}})^{-1} (b^{b^{-1}})^{-1}$; then, since

$$a \in \bar{f}^{-1}(b^{-1}b),\ a^{b^{-1}} \in \bar{f}^{-1}(bb^{-1}bb^{-1}) = \bar{f}^{-1}(bb^{-1})$$

and

$$b^{b^{-1}} \in \bar{f}^{-1}(bb^{-1}bb^{-1}) = \bar{f}^{-1}(bb^{-1})$$

$\therefore$ $x \in \bar{f}^{-1}(bb^{-1})$, thus $(b^{-1}, x) \in S^*$ and for this element x we have

$$(b, a)(b^{-1}, x)(b, a) = (b, (bb^{-1})^b (b^{b^{-1}} a^{b^{-1}} (a^{b^{-1}})^{-1} (b^{b^{-1}})^{-1})^b a) = (b, a)$$

by (p1 and p5).

Similarly, $(b^{-1}, x)(b, a)(b^{-1}, x) = (b^{-1}, x)$.

(ii) The idempotents of S^* commute:

Let $(b, a) \in E(S^*)$ then $(b, a)^2 = (b^2, b^b a^b a) = (b, a)$ thus $b^2 = b$ and $b^b a^b a = (bb^{-1})^b a^b a = e \cdot eaa = a^2$ where $e \in E(A) \cap \bar{f}^{-1}(b)$ hence $a^2 = a$. Therefore $E(S^*) = \{(b, a) : b^2 = b \text{ and } a^2 = a \in \bar{f}^{-1}(b)\}$. Now, let (b, a) and $(c, a') \in E(S^*)$ then

$$(b, a)(c, a') = (bc, b^c a^c a')$$

and

$$(c, a')(b, a) = (cb, c^b a'^b a).$$

Now, $bc = cb$ since $b, c \in E(B)$ and $b^c a^c a' = b^c a' a a' = b^c a' a$ since $a, a' \in E(A)$ and (p5iv) also, $c^b a'^b a = c^b a a' a = c^b a' a = b^c a' a$ by (p5v).

(4) (S, f) is an extension of A by B:

Clearly f is a homomorphism of S onto B. Let $A^* = \{(bb^{-1}, a)\} \subseteqq S^*$. Then,

$$(bb^{-1}, a) \circ (cc^{-1}, a') = (bb^{-1}cc^{-1}, (bb^{-1})^{cc^{-1}} a^{cc^{-1}} a') = (bb^{-1}cc^{-1}, eggaa')$$

by (p5(v) and (iv)) where $e \in E(A) \cap \bar{f}^{-1}(bb^{-1})$ and

$$g \in E(A) \cap \bar{f}^{-1}(cc^{-1}) = (bb^{-1}cc^{-1}, aa')$$

since $a \in \bar{f}^{-1}(bb^{-1})$ and $a' \in \bar{f}^{-1}(cc^{-1})$ and since A is a semilattice of groups the idempotents are in the center of A (see e.g. [1]).

Thus $A \cong A^*$; clearly $f^{-1}(E(B)) \cong A$. Moreover, if $e \in E(B)$ then $(e, a) \in A^*$ implies that $\bar{f}(a) = e$ thus $f((e, a)) = e = \bar{f}(a)$ and $f|A = \bar{f}$. Proving the theorem.

3. The structure theorem

THEOREM 2. *Let A and B be inverse semigroups and (S, f) an idempotent separating extension of A by B. Then with each $b \in B$ there are associated mappings $a \to a^b$ and $a \to b^a$ of A into A and with each pair $(b, c) \in B \times B$ there is associated an element $b^c \in f^{-1}(c^{-1}b^{-1}bc) \subseteqq A$; satisfying* (p1)—(p5) *such that if S^* is as in section 2, then $S \cong S^*$.*

PROOF. Fix a transversal g of B in S and denote $g(b)$ by s_b. Now, $s_b s_c \in f^{-1}(bc) = s_{bc}N_e$ (by Theorem 1 of [5]) say $s_b s_c = s_{bc}b^c$ where $b^c \in N_e = f^{-1}(e)$. Clearly b^c is uniquely defined (up to choice of the transversal) since $s_{bc}a = s_{bc}a'$ implies that $s_{bc}^{-1}s_{bc}a = s_{bc}^{-1}s_{bc}a'$ and since $s_{bc}^{-1}s_{bc}$ is the right unit of s_{bc} and e is the right unit of s_{bc} we have $a = a'$. Now, let $x \in s_b N_e$, $y \in s_c N_g$, say, $x = s_b a$; $y = s_c a'$ and let $h = s_c s_c^{-1}$ then $ah \in N_e N_h \subseteqq N_{eh} = N_{he}$ and $ah = h(ah) = ha$. Hence

$$xy = s_b a s_c a' = s_b a h s_c a' = s_b h a s_c a' = s_b s_c a^c a' = s_{bc} b^c a^c a'.$$

Now, $b^c = s_{bc}^{-1} s_b s_c$ thus, $b^c \in f^{-1}(c^{-1}b^{-1}bc)$.

(p1) Let

$$a_1, a_2 \in A; a_1 \in N_e, s_b s_b^{-1} = g.$$

Then,

$$s_b s_b^{-1} a_1 \in N_g N_e \subseteqq N_{ge} = N_{eg};$$

thus

$$s_b s_b^{-1} a_1 = s_b s_b^{-1} a_1 s_b s_b^{-1}.$$

Hence,

$$\begin{aligned}(a_1 a_2)^b &= s_b^{-1} a_1 a_2 s_b = s_b^{-1} s_b s_b^{-1} a_1 a_2 s_b \\ &= s_b^{-1} s_b s_b^{-1} a_1 s_b s_b^{-1} a_2 s_b = s_b^{-1} a_1 s_b s_b^{-1} a_2 s_b = a_1^b a_2^b.\end{aligned}$$

Similarly for b^a.

Clearly if $a \in f^{-1}(b^{-1}b)$ then $a^c \in f^{-1}(c^{-1}b^{-1}bc)$, $c^a \in f^{-1}(cb^{-1}bc^{-1})$.

(p2) If $a \in f^{-1}(b^{-1}b)$ then for all $c \in B$;

$$\begin{aligned}(b^a)^{bc} \cdot b^c &= s_{bc}^{-1} b^a s_{bc} \cdot b^c = s_{bc}^{-1} s_b a s_b^{-1} s_{bc} b^c = s_{bc}^{-1} s_b a s_b^{-1} s_b s_c = s_{bc}^{-1} s_b a s_c \\ &= (s_b s_c \cdot (b^c)^{-1})^{-1} s_b a s_c = b^c s_c^{-1} s_b^{-1} s_b a s_c = b^c s_c^{-1} a s_c = b^c a^c.\end{aligned}$$

(p3) Similar to the above.

(p4) $s_b \cdot s_c s_d = s_b \cdot s_{cd} c^d = s_{b \cdot cd} b^{cd} c^d$. Now $s_b s_c \cdot s_d = s_{bc} b^c s_d = s_{bc \cdot d} a'$ where $a' = (s_{bc \cdot d})^{-1} s_{bc} b^c s_d$. But, $s_{bcd}(bc)^d = s_{bc} s_d$ thus, $s_{bcd} = s_{bc} s_d [(bc)^d]^{-1}$ or $(s_{bcd})^{-1} = (bc)^d s_d^{-1} s_{bc}^{-1}$. Hence, $a' = (bc)^d \cdot s_d^{-1} \cdot s_{bc}^{-1} \cdot s_{bc} \cdot b^c \cdot s_d$ and since $s_{bc}^{-1} s_{bc}$ is a left unit for b^c, $a' = (bc)^d (b^c)^d$ proving (p4).

(p5) (i) Let $a \in f^{-1}(bb^{-1})$ then $a^{bb^{-1}} = s_{bb^{-1}}^{-1} a s_{bb^{-1}} = a$ (by (ii) of definition of transversal).

(ii) $(b^a)^b = s_b^{-1} b^a s_b = s_b^{-1} s_b a s_b^{-1} s_b = a \in f^{-1}(bb^{-1})$.

(iii) $(bb^{-1})^b = s_{bb^{-1}b}^{-1} s_{bb^{-1}} s_b = s_b^{-1} s_b s_{bb^{-1}} = s_b^{-1} s_b$ since $s_{bb^{-1}} \in E(A)$ and A is a semilattice of groups thus $(bb^{-1})^b \in E(A) \cap f^{-1}(b^{-1}b)$.

(iv) Let $b^2 = b$, then

$$a^b = s_b^{-1} a s_b = s_b^{-1} s_b a = s_b a$$

where $s_b \in E(A) \cap f^{-1}(b)$.

(v) $(bb^{-1})^{cc^{-1}} = s_{bb^{-1}cc^{-1}}^{-1} s_{bb^{-1}} s_{cc^{-1}} = s_{bb^{-1}} s_{cc^{-1}} s_{bb^{-1}} s_{cc^{-1}} = s_{bb^{-1}} s_{cc^{-1}}$.

Now, let α be the mapping from S to S^* defined by $\alpha(x) = (b, a)$ if $f(x) = b$ and $s_b^{-1} x = a$. Clearly α is well-defined and one to one; given $(b, a) \in S^*$, let $x = s_b a$ then $\alpha(x) = (b, a)$ thus α is onto. That α is a homomorphism comes from the discussion at the beginning of the proof. Thus $S \cong S^*$ proving the theorem.

4. Application to bisimple ω-semigroups

In [4], W. D. Munn and N. R. Reilly have shown that in any bisimple ω-semigroup S, $\mathscr{H}$ is a congruence and $S/\mathscr{H} \cong B$ where B is the bicyclic semigroup and $\mathscr{H}$ is Green's equivalence. Let G be a group and α an endomorphism of G and let Y be the chain of non-negative integers with multiplication $m \cdot n = \min(m, n)$ for all $m, n \in Y$, then Y is a semilattice. Let $A = A(G, Y, \alpha)$ be the semigroup constructed as follows:

Take A to be the union of $\aleph_0$ disjoint copies of G, indexed by elements of Y. If $m \leqq n$ in the semilattice ordering of Y define $\phi_{m,n}$ a mapping from G_m to G_n by $g\phi_{m,n} = g\alpha^{n-m}$ where we are considering α as a homomorphism from G_k to G_i for all k and $i \in Y$ and α^0 is the identity mapping on G_i for all $i \in Y$.

Now suppose $m < n < k$ for $m, n, k \in Y$, then if $g \in G_m$,

$$g\phi_{m,n}\phi_{n,k} = (g\alpha^{n-m})\alpha^{k-n} = g\alpha^{k-m} = g\phi_{m,k}$$

thus $\phi_{m,n}\phi_{n,k} = \phi_{m,k}$. Now define an operation on A by

$$g \circ h = (g\phi_{m,k})(h\phi_{n,k}) \text{ where } g \in G_m,\ h \in G_n \text{ and } k = \max(m, n).$$

By [1], Theorem 4.11, A is an inverse semigroup which is a union of groups.

Let B be the bicyclic semigroup, i.e.

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$$B = \{(m, n) : m, n \in Y\}$$

with multiplication defined by

$$(k, 1) \cdot (m, n) = (k+m-1 \cdot m, 1+n-1 \cdot m)$$

where $+$ is the addition of integers and $\cdot$ is the multiplication of the semilattice Y.

Let (S, f) be an idempotent separating extension of $A = A(G, Y, \alpha)$ by B.

LEMMA 1. (Munn [3]; Lallement [2]): *If ρ is an idempotent-separating congruence on a regular semigroup S, then $\rho \leqq H$.*

LEMMA 2. *Let S be a regular semigroup and f an idempotent-separating homomorphism of S onto B, then $(f(x), f(y)) \in \mathscr{H}$ if and only if $(x, y) \in \mathscr{H}$ for all $x, y \in S$.*

PROOF. Clearly if $(x, y) \in \mathscr{H}$ then $(f(x), f(y)) \in \mathscr{H}$. Suppose then, that $(f(x), f(y)) \in \mathscr{H}$; thus there is a $b \in B$ such that $f(x) = f(y) \cdot b$, say $b = f(u)$; then $f(x) = f(yu)$ and by Lemma 1, $(x, yu) \in \mathscr{H}$; in particular $(x, yu) \in \mathscr{R}$ thus there is a v such that $x = yuv$, hence $x \in yS$. Similarly, $y \in xS$ and $(x, y) \in \mathscr{R}$. A dual argument yields $(x, y) \in \mathscr{L}$.

COROLLARY 1. *Under the hypothesis of Lemma 2, $(f(x), f(y)) \in \mathscr{D}$ if and only if $(x, y) \in \mathscr{D}$.*

PROOF. We saw that if $(f(x), f(y)) \in \mathscr{R}$ then $(x, y) \in \mathscr{R}$ and similarly if $(f(x), f(y)) \in \mathscr{L}$ then $(x, y) \in \mathscr{L}$. Let $(f(x), f(y)) \in \mathscr{D}$ and let $b = f(u)$ be such that $(f(x), b) \in \mathscr{R}$ and $(b, f(y)) \in \mathscr{L}$. Thus $(x, u) \in \mathscr{R}$ and $(u, y) \in \mathscr{L}$; hence $(x, y) \in \mathscr{D}$.

Since B is a bisimple ω-semigroup with each $\mathscr{H}$-class consisting of a swingle element, we have

COROLLARY 2. *An idempotent-separating extension (S, f) of $A(G, Y, \alpha)$ by B is a bisimple ω-semigroup and the congruence induced by f on S is precisely $\mathscr{H}$.*

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