# The role of convexity in defining regular polyhedra

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*Introduction*

It is almost twenty years since Branko Grünbaum lamented that the 'original sin' in the theory of polyhedra is that from Euclid onwards "the writers failed to define what are the 'polyhedra' among which they are finding the 'regular' ones" ([1, p. 43]). Various definitions of 'regular' can be found in the literature with a condition of convexity often included (e.g. [2, p. 301], [3, p. 77], [4, p. 47], [5, p. 435], [6, p. 16]). The condition of convexity is usually cited to exclude regular self-intersecting polyhedra, i.e. the Kepler-Poinsot polyhedra, such as the 'great dodecahedron' consisting of twelve intersecting pentagonal faces shown in Figure 1 with one face shaded. Richeson also notes ([4, pp. 47-48]) that, for a particular definition of 'regular', convexity is needed to exclude the 'punched-in' icosahedron shown in Figure 2.

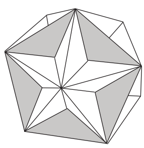

FIGURE 1

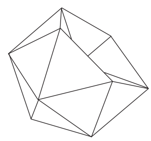

FIGURE 2

A condition of convexity is clearly stronger than that of non-self-intersection however, as shown by the existence of non-convex non-self-intersecting polyhedra. This raises the question of *why a condition of convexity is generally included for regular polyhedra rather than the lesser condition of non-self-intersection*. This Article looks at the question of whether non-convex non-self-intersecting regular polyhedra can exist for the various definitions of 'regular' found in the literature. It is found that, while most of the cases of possible polyhedra can be dealt with very quickly, there is one case that slips through the net and requires a longer analysis than might be expected. A proof settling the question of the necessity of a condition of convexity for this special case is given using elementary geometry.

*Initial definitions*

In light of the 'original sin', a polyhedron will be defined here, following Coxeter's treatment in *Regular polytopes* [6, p. 4], as simply connected* and consisting of 'a finite, connected set of plane polygons, such that every side of each polygon belongs to just one other polygon, with the proviso that the polygons surrounding each vertex form a single circuit (to exclude anomalies such as two pyramids with a common apex).' Also following Coxeter's initial definitions, the focus will be on polyhedra whose faces do not intersect each other, the faces being non-self-intersecting polygons.

* i.e. 'every simple closed curve drawn on the surface can be shrunk' [6, p. 9]

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Among the various possible definitions of a ‘regular polyhedron’, it can be shown (e.g. [3, p. 77-78] and [5. p. 446]) that if a *convex* polyhedron is bounded by equal regular polygons then the following further conditions are equivalent:

(a) the polyhedron has the same number of faces at each vertex,

(b) all dihedral angles are equal (the dihedral angle of an edge being the angle formed between two lines which meet on the edge and are both perpendicular to the edge, one line lying in one face and the other in the other face),

(c) the vertices of the polyhedron all lie on a sphere,

(d) for any two vertices of the polyhedron, there is a rigid motion of the figure taking one to the other.

It should be noted that (d) refers to pairs of vertices, not to the symmetry of the polyhedron as a whole. The condition captures the intuitive notion that any two vertices are ‘the same’ if the arrangement of the face angles that meet at both vertices are the same as well as the dihedral angles between two corresponding faces. Such vertices will be described as ‘congruent’ hereafter.

The role of convexity in each of these conditions is not the same, however. Considering each in turn:

(a) can be satisfied by a non-convex polyhedron, as witnessed by the ‘punched-in’ icosahedron in Figure 2 above; to exclude this figure from the regular polyhedra we need to add the condition of convexity to condition (a), as Richeson does in [4];

(b) implies convexity given that any non-convex polyhedron has a mixture of dihedral angles, some less than 180° and some greater;

(c) also implies convexity (recalling that all polygons and polyhedra being considered here are non-self-intersecting);

(d) is not so clear, it not being immediately obvious whether, like (a), it needs an additional condition of convexity in order to define the usual regular polyhedra, or whether, like (b) and (c), convexity follows automatically. If the former, i.e. we need to state convexity as an additional condition, then this could only be because of the possibility of a non-convex polyhedron bounded by equal regular polygons that satisfied (d). On the other hand, if it is possible to show that no such polyhedron could exist, the condition of convexity is implied by (d) and so need not be stated as an extra condition.

We note in passing that Coxeter's definition of a regular polyhedron contains convexity as an implicit condition saying that ‘a polyhedron is *regular* if its faces and vertex figures are all regular’ ([6, p. 16]) where the vertex figure for a vertex $O$ is the polygon whose vertices ‘are the mid-points of all the edges through $O$’ (ibid.). For non-self-intersecting polyhedra, the regularity of the vertex figures implies convexity given that each internal angle of the non-self-intersecting regular polygon is non-

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convex from which it follows that all dihedral angles are equal. The question of whether condition (d) needs an additional specification of convexity in order to define the regular polyhedra is equivalent to the question of whether there is any need for the condition of the regularity of vertex figures or whether congruence among them is enough.

*Simple exclusions*

Without a condition of convexity there are many possible vertices, such as those shown in Figure 3 where eight equilateral triangles meet at the first vertex and five squares at the second. Examining whether condition (d) automatically implies convexity requires us to consider such vertices to see whether they could be the common vertex of a non-convex ‘regular’ polyhedron.

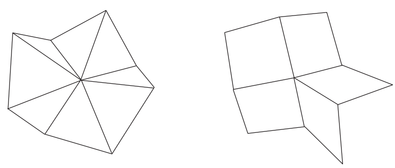

FIGURE 3

We note first that if three polygons meet at a vertex then the vertex is rigid, i.e. cannot be deformed, just as a two-dimensional triangle cannot be deformed. It follows that when creating a polyhedron with exactly three equilateral triangles, squares or regular pentagons meeting at each vertex, there is no ‘freedom’ in how the polyhedron is created and we get the tetrahedron, cube or dodecahedron respectively.

To examine further cases it is useful to observe that the definition of a polyhedron given above, in particular the condition of simple-connectedness, implies that Euler's polyhedra formula $V - E + F = 2$ holds for the number of vertices, edge and faces (whether or not the polyhedron is convex). One consequence of Euler's formula is Descartes' earlier observation that the sum of the angular deficits of a solid is 720°, where the angular deficit of a vertex is defined as 360° minus the sum of the angles of the face angles (not the dihedral angles) that meet at a vertex*.

Descartes' angular deficit formula shows that if all vertices of a polyhedron are congruent, i.e. condition (d) holds, then the sum of the face angles at any vertex must be less than 360°, namely that if the sum were greater than 360° then the vertex deficit at each would be negative and their sum could not equal 720°. This excludes a large number of possibilities for regular polygons based on condition (d) without having to assume convexity, for example, excluding having six or more equilateral triangles around a vertex or four or more regular $n$-sided polygons for $n \geqslant 4$.

* [5, p .449] gives a quick derivation of Descartes' observation from Euler's formula.

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*Four or five equilateral triangular faces*

This leaves the possibility of four or five equilateral triangles surrounding a vertex such as those around the 'rim' of the 'punched-in' icosahedron, where five equilateral triangles meet at a vertex. A polyhedron being composed solely of such non-convex vertex cannot be ruled out by Descartes' observation, as the angular defect of each is $360° - 5 \times 60° = 60°$ and so twelve such vertices might conceivably form a polyhedron. A polyhedron such as this cannot be excluded by Cauchy's rigidity theorem either as we are not assuming convexity. It may feel 'intuitively' obvious that a polyhedron could not be formed where every vertex was of this form and congruent to each other, e.g. due to the sides going 'in and out', but intuition is fallible. It can be noted for example that at least a few such congruent non-convex vertices can be placed together, as they are in around the rim of the punched-in icosahedron. The remainder of this Article shows that the intuition about the impossibility of forming a polyhedron from a set of such vertices is indeed correct although for five equilateral triangles it is not a simple one-line argument.

Before examining this argument, we note that a quick argument shows that the only figure in which four equilateral triangles surround each vertex is the octahedron. This is due to the fact that four equilateral triangles cannot form a non-convex vertex in the same way that five equilateral triangles can. Euler's formula can therefore be combined with Cauchy's rigidity theorem to argue for the uniqueness of the octahedron. This is illustrated in Figure 4 where we start with four equilateral triangles around a vertex but in such a way that the dihedral angles are unequal. This is continued to a second vertex, forming the figure shown. Euler's formula implies that a polyhedron with four equilateral triangles at each vertex must be formed of just eight equilateral triangles but it is apparent that the figure cannot be completed in this way. The only way to complete the figure is to start instead with a vertex surrounded by equal dihedral angles, leading to the octahedron.

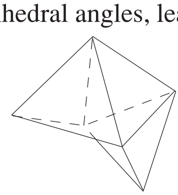

FIGURE 4

*Vertex patterns*

The remaining case where five equilateral triangles meet at a vertex will be considered in the remainder of this Article. We label the dihedral angles around a vertex $A$ to $E$, describing the vertex arrangement as $ABCDE$ where the order of the letters matters. A vertex with dihedral angles $ABCDE$ is therefore congruent to one with angles $BCDEA$, neither being congruent to a vertex described by $ACDBE$. If it turns out that two of the dihedral angles are the same, e.g. $A = E$, then this is indicated by saying that the vertex is of the form $ABCDA$.

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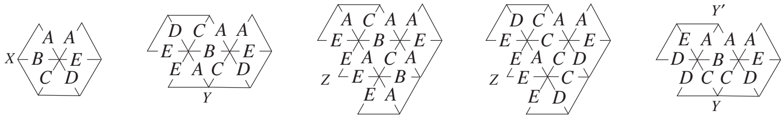
*Theorem* 1: If a polyhedron is composed of congruent vertices each consisting of five equilateral triangles then at each vertex either three adjacent dihedral angles are the same or two pairs of adjacent dihedral angles are the same, i.e. the vertices are of the form *AAADE* or *AABBC*.




(a) (b) (c) (d) (e)

FIGURE 5: Dihedral angles around vertices

*Proof*: Fig. 5 (a) shows a vertex with the dihedral angles $A$, $B$, $C$, $D$, $E$ going anti-clockwise around it. The diagram has one edge 'opened' to allow the vertex to be flattened onto a page leading to the two $A$s which represent the same edge. The dihedral angles around the vertex at the other end of the edge $B$, labelled $X$, can go either (i) anti-clockwise or (ii) clockwise (if we insist that they can only go in the same direction as for the original vertex, i.e. anti-clockwise, this only shortens the argument).

Case (i): If the angles around $X$ go anti-clockwise then, given that we already have $B$ in place, we get Figure 5 (b). We consider next another vertex at the centre of the bottom of Figure 5 (b), labelled $Y$, considering again the cases when the dihedral angles go ($\alpha$) anti-clockwise or ($\beta$) clockwise.

Case ($\alpha$): If the angles around $Y$ go anti-clockwise then $A$ follows $C$. This can only happen for a vertex arrangment *ABCDE* if either $D = A$, i.e. we actually have *ABCAE*, or $E = C$, i.e. we have *ABCDC*. These are of the same form and so we only need to consider one of them, e.g. *ABCAE*. If we redraw the diagram based on this and extend it around the vertex labelled $Y$ we get Figure 5 (c). The vertex labelled $Z$ on the left-hand side (this appears in three places in the diagram due to the 'opening' of edges), shows that there are two adjacent dihedral angles both $E$. But $A$ appears on either side of $E$ in the pattern currently being considered, *ABCAE*, and so $E = A$ and the angles must be of the form *ABCAA*, i.e. the first of the two possible arrangements in the Theorem.

Case ($\beta$): If the angles around $Y$ go clockwise then we have $C$ following $A$. This means that we must either have $B = C$, i.e. *ACCDE*, or $B = A$, i.e. *AACDE*. These are again of the same form and so choosing the first and again redrawing the diagram and completing the angles around $Y$ gives Figure 5 (d). Again we have a vertex at $Z$ with adjacent $E$s, leading to either $D = E$, i.e. *ACCEE*, or $E = A$, i.e. *ACCDA* as stated in the Theorem.

Case (ii): If we let *ABCDE* go clockwise around $X$ we get Figure 5 (e). The vertex in the middle of the top of the diagram labelled $Y'$ has an $A$ next to an $A$ and so we have either $B = A$, i.e. *AACDE*, or $E = A$, i.e. *ABCDA*. Considering also the vertex at the middle of the bottom we have $C$ next to $C$ and so for the $AACDE$ case we must have either $C = A$, i.e. $AAADE$ or $D = C$, i.e. $AACCE$, with similar results if we choose the $ABCDA$ case.

In each of these cases we have ended up with a set of dihedral angles of the form either $AAADE$ or $AABBC$.

It should be noted that it is not possible to continue this line of argument further to show that all dihedral angles must be equal, i.e. to a vertex pattern of $AAAAA$, as Figure 6 illustrates a case where a full icosahedron net has been filled with vertices with pattern $AABBC$ (with the dotted lines indicating which vertices of the net are connected). An argument can be given that the vertex pattern $AAADE$ must lead to $AAAAD$, but this can go no further as an icosahedron net can also be filled with $AAAAD$ and so an argument along the lines of the next section is needed in either case.

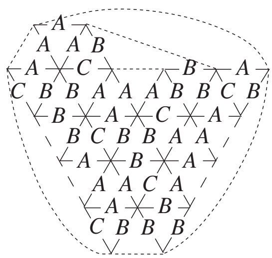



FIGURE 6

*Spherical pentagons*

We show instead that the angles must all be equal by considering a spherical pentagon drawn on the surface of a sphere centred at one of the vertices with a radius less than the length of the shortest edge coming from that vertex, illustrated in Figure 7. Given that each of the faces of the original polyhedron is an equilateral triangle, the spherical pentagon will also be equilateral, albeit with arcs as sides rather than the sides of the triangles.

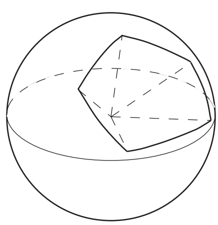


FIGURE 7

The geometric fact needed is that the angles at the base of an isosceles triangle are equal, valid in spherical geometry just as it is for planar geometry. We use this to show the following Theorem.

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*Theorem* 2: If two adjacent angles in a planar or spherical equilateral pentagon are equal then the two angles adjacent to those angles are also equal to each other.

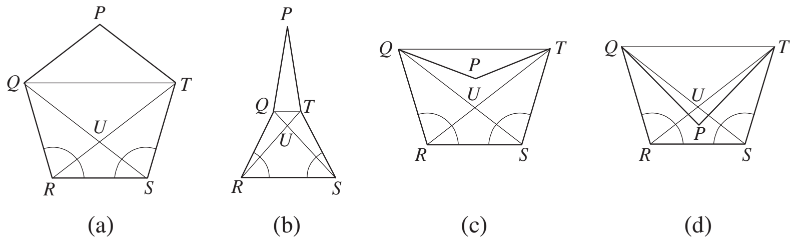


FIGURE 8: Two adjacent equal angles in an equilateral pentagon

*Proof*: The four cases to consider are shown in Figure 8 with similar reasoning holding in each.

As the pentagon is equilateral, $PQ = QR = RS = ST = TP$, and we can suppose that $\angle QRS = \angle RST$. Let $U$ be the intersection of $QS$ and $TR$. Then $\Delta QRS \cong \Delta RST$ and both are are isosceles so $QS = RT$ and $\angle RQS = \angle RSQ = \angle SRT = \angle RTS$. $\Delta RUS$ is therefore isosceles with $UR = US$ and so $UQ = UT$. $\Delta QUT$ is therefore isosceles with $\angle TQU = \angle QTU$. $\Delta PQT$ is also isosceles with $\angle PQT = \angle PTQ$.

For the cases shown in Figures 8 (a) and (b):

$$\angle PQR = \angle RQS + \angle TQU + \angle PQT = \angle RTS + \angle QTU + \angle PTQ = \angle PTS$$

and the cases for Figures 8 (c) and (d) just need the slight modification that:

$$\angle PQR = \angle RQS + \angle TQU - \angle PQT = \angle RTS + \angle QTU - \angle PTQ = \angle PTS.$$

*Regularity established*

We show lastly:

*Theorem* 3: If each vertex of a polyhedron is composed of five equilateral triangles, then the dihedral angles around each vertex are all equal to each other.

*Proof*: Figure 9 shows the two cases *AAADE* or *AABBC* reached at the end of Theorem 1.

In Figure 9 (a) we have $\angle PQR = \angle QRS = \angle RST$. By Theorem 2 $\angle PQR = \angle PTS$ and $\angle QPT = \angle RST$ and so all angles are equal

In Figure 9 (b) we have $\angle QPT = \angle PRQ$ and $\angle QRS = \angle RST$. By Theorem 2 $\angle PTS = \angle QRS$ and so $\angle QPS = \angle QRS$ and so all angles are equal.

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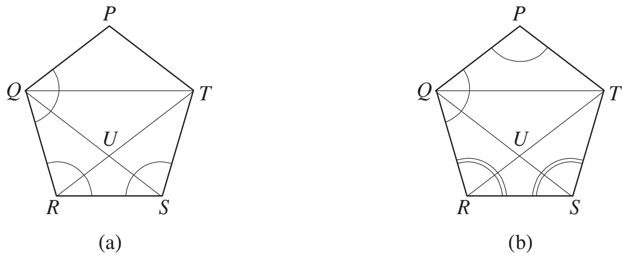


FIGURE 9: *AAADE* and *AABBC* angles in a pentagon

It has been shown that if a polyhedron has five equilateral triangles around each vertex and all vertices are congruent then each dihedral angle is the same, i.e. we have the icosahedron. This completes the list of the five regular non-intersecting polyhedra, i.e. the Platonic solids.

*Conclusion*

To summarise, among the possible conditions (a)-(d) of regularity considered above, an additional condition of convexity is only needed with (a). Convexity follows immediately for (b) and (c), and eventually for (d) too, though the latter takes more work than might be expected. To impose a condition of convexity solely to avoid the argument given here would seem to imply the possibility of a non-convex polyhedron which satisfied one of (b)-(d) but it has been shown that there is no such figure.

We note finally that the above discussion justifies the argument that Euclid makes at the end of the *Elements* for there being only five regular solids, the argument being based on there being at most five triangles, three squares or three pentagons around a vertex. There can be more than this number of triangles, squares or regular pentagons, or even higher-sided regular polygons, around a vertex but such vertices cannot lead to a regular non-self-intersecting polyhedron by any of the standard definitions of regular.

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