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PONTRYAGIN DUALITY FOR VARIETIES OVER p-ADIC FIELDS

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Abstract We define cohomological complexes of locally compact abelian groups associated with varieties over p-adic fields and prove a duality theorem under some assumption. Our duality takes the form of Pontryagin duality between locally compact motivic cohomology groups.

Key words and phrases: motivic cohomology; duality; local fields

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1. Introduction

Let K be a finite extension of $\mathbb{Q}_p$. Let $\mathcal{O}_K$ be the ring of integers in K, and let $\mathcal{X}$ be a regular, proper and flat scheme over $\mathcal{O}_K$ of dimension d. We denote by $\mathcal{X}_K$ its generic fibre and by $i:\mathcal{X}_s \to \mathcal{X}$ its special fibre. It is a classical result that for any integer $m>0$, we have a perfect duality of motivic cohomology with finite coefficients

$$H^i_{\mathrm{et}}(\mathcal{X}_K,\mathbb{Z}/m(n)) \times H^{2d-i}_{\mathrm{et}}(\mathcal{X}_K,\mathbb{Z}/m(d-n)) \to H^{2d}_{\mathrm{et}}(\mathcal{X}_K,\mathbb{Z}/m(d)) \to \mathbb{Z}/m.$$

However, this does not lift to a duality of integral groups [12]. For example, even for a curve $\mathcal{X}_K$, the dual of $H^1_{\mathrm{et}}(\mathcal{X}_K,\mathbb{Q}/\mathbb{Z}) \cong H^2_{\mathrm{et}}(\mathcal{X}_K,\mathbb{Z})$ has both contributions from $H^3_{\mathrm{et}}(\mathcal{X}_K,\mathbb{Z}(2))$ as well as from $H^4_{\mathrm{et}}(\mathcal{X}_K,\mathbb{Z}(2))$. The examples $H^1_{\mathrm{et}}(\mathcal{X}_K,\mathbb{Z}(1)) \cong K^\times$, or $H^2_{\mathrm{et}}(\mathcal{X}_K,\mathbb{Z}(1)) \cong \mathrm{Pic}(\mathcal{X}_K)$, which are both extensions of a finitely generated group by a finitely generated $\mathbb{Z}_p$-module, also suggest that the cohomology groups are topological groups. Thus, our goal is to construct topological cohomology groups which agree with étale cohomology groups with finite coefficients but satisfy a Pontryagin duality. More generally, we conjecture the existence of a cohomology theory on the category of separated schemes of finite type over $\mathrm{Spec}(\mathcal{O}_K)$, whose main expected properties are outlined in the last section of this paper.

Its existence was suggested by the 'Weil-Arakelov cohomology' of arithmetic schemes, which is conditionally defined in [2] for proper regular schemes over $\mathrm{Spec}(\mathbb{Z})$. The aim of this paper is to give a possible construction of such groups.

Let LCA be the quasi-abelian category of locally compact abelian groups and FLCA $\subseteq$ LCA the full subcategory consisting of locally compact abelian groups of finite ranks in the sense of [14]. We consider the bounded derived category $\mathbf{D}^b(\mathrm{LCA})$ and $\mathbf{D}^b(\mathrm{FLCA})$, respectively, [14]. The category $\mathbf{D}^b(\mathrm{FLCA})$ is a closed symmetric monoidal category with internal homomorphisms $R\underline{\mathrm{Hom}}(-,-)$ and tensor product $\underline{\otimes}^{\mathbb{L}}$. Assuming certain expected properties of Bloch's cycle complex $\mathbb{Z}(n)$, we construct for any $n \in \mathbb{Z}$ complexes in $\mathbf{D}^b(\mathrm{LCA})$ fitting in an exact triangle

$$R\Gamma_{ar}(\mathcal{X}_s, Ri^!\mathbb{Z}(n)) \to R\Gamma_{ar}(\mathcal{X},\mathbb{Z}(n)) \to R\Gamma_{ar}(\mathcal{X}_K,\mathbb{Z}(n)), \tag{1}$$

and we define

$$R\Gamma_{ar}(-,\mathbb{R}/\mathbb{Z}(n)) := R\Gamma_{ar}(-,\mathbb{Z}(n))\underline{\otimes}^{\mathbb{L}}\mathbb{R}/\mathbb{Z}.$$

We expect that this theory satisfies duality and many other properties (see Section 6).

To obtain unconditional results, we give an alternative construction, which conjecturally agrees with the above construction of the triangle (1) for $n = 0, d$ and show that this triangle belongs to $\mathbf{D}^b(\mathrm{FLCA})$. Then we prove Theorem 1.2 below under the following hypothesis.

Hypothesis 1.1. *The reduced scheme* $(\mathcal{X}_s)^{\mathrm{red}}$ *is a simple normal crossing scheme, and the complex* $R\Gamma_W(\mathcal{X}_s,\mathbb{Z}^c(0))$ *is a perfect complex of abelian groups, where* $\mathbb{Z}^c(0)$ *denotes the Bloch cycle complex in its homological notation [9] and* $R\Gamma_W(\mathcal{X}_s,-)$ *denotes Weil-étale cohomology.*

The homology groups of the complex $R\Gamma_W(\mathcal{X}_s,\mathbb{Z}^c(0))[1]$ are called arithmetic homology with compact support and denoted by $H_i^c(\mathcal{X}_{s,\mathrm{ar}},\mathbb{Z})$ in [10]. For $d \le 2$, the complex $R\Gamma_W(\mathcal{X}_s,\mathbb{Z}^c(0))$ is perfect, and perfectness, in general, follows from a special case of Parshin's conjecture [10, Proposition 4.2].

Theorem 1.2. *Suppose that either* $d \le 2$ *or that* $\mathcal{X}_s$ *satisfies Hypothesis 1.1. Then there is a trace map* $H^{2d}_{ar}(\mathcal{X}_K,\mathbb{R}/\mathbb{Z}(d)) \to \mathbb{R}/\mathbb{Z}$ *and an equivalence*

$$R\Gamma_{ar}(\mathcal{X}_K,\mathbb{R}/\mathbb{Z}(n)) \xrightarrow{\sim} R\underline{\mathrm{Hom}}(R\Gamma_{ar}(\mathcal{X}_K,\mathbb{Z}(d-n)),\mathbb{R}/\mathbb{Z}[-2d])$$

in $\mathbf{D}^b(\mathrm{FLCA})$*, for* $n = 0, d$.

Combining Theorem 1.2 with a consequence of Sato's work [22], we obtain

Corollary 1.3. *Suppose that* $\mathcal{X}/\mathcal{O}_K$ *has good or strictly semistable reduction, and suppose that* $R\Gamma_W(\mathcal{X}_s,\mathbb{Z}^c(0))$ *is a perfect complex of abelian groups. Then there is a perfect pairing of locally compact abelian groups*

$$H^i_{ar}(\mathcal{X}_K,\mathbb{R}/\mathbb{Z}(n)) \times H^{2d-i}_{ar}(\mathcal{X}_K,\mathbb{Z}(d-n)) \to H^{2d}_{ar}(\mathcal{X}_K,\mathbb{R}/\mathbb{Z}(d)) \to \mathbb{R}/\mathbb{Z}$$

for $n = 0, d$ *and any* $i \in \mathbb{Z}$.

In a forthcoming paper, we will give applications of this result to class field theory of schemes over local fields.

2. Locally compact abelian groups

In this section, we define and study the derived ∞-categories $\mathbf{D}^b(\mathrm{LCA})$ and $\mathbf{D}^b(\mathrm{FLCA})$. We also introduce a certain profinite completion functor.

2.1. Derived ∞-categories

Let A be an additive category. Let $\mathrm{C}^b(A)$ be the differential graded category of bounded complexes of objects in A, and let $\mathcal{N} \subset \mathrm{C}^b(A)$ be a full subcategory which is closed under the formation of shifts and under the formation of mapping cones. If $\mathrm{N}_{\mathrm{dg}}(-)$ denotes the differential graded nerve [18, Construction 1.3.1.6], then $\mathrm{N}_{\mathrm{dg}}(\mathrm{C}^b(A))$ is a stable ∞-category and $\mathrm{N}_{\mathrm{dg}}(\mathcal{N})$ is a stable ∞-subcategory of $\mathrm{N}_{\mathrm{dg}}(\mathrm{C}^b(A))$ [18, Proposition 1.3.2.10].

The Verdier quotient is defined [20, Theorem I.3.3] as the Dyer-Kan localisation

$$\mathrm{N}_{\mathrm{dg}}(\mathrm{C}^b(A))/\mathrm{N}_{\mathrm{dg}}(\mathcal{N}) := \mathrm{N}_{\mathrm{dg}}(\mathrm{C}^b(A))[W^{-1}],$$

where W is the set of arrows in $\mathrm{N}_{\mathrm{dg}}(\mathrm{C}^b(A))$ whose cone lies in $\mathrm{N}_{\mathrm{dg}}(\mathcal{N})$. The ∞-category $\mathrm{N}_{\mathrm{dg}}(\mathrm{C}^b(A))[W^{-1}]$ is stable. Moreover, the functor $\mathrm{N}_{\mathrm{dg}}(\mathrm{C}^b(A)) \to \mathrm{N}_{\mathrm{dg}}(\mathrm{C}^b(A))/\mathrm{N}_{\mathrm{dg}}(\mathcal{N})$ is exact and induces an equivalence from the category of exact functors $\mathrm{N}_{\mathrm{dg}}(\mathrm{C}^b(A))/\mathrm{N}_{\mathrm{dg}}(\mathcal{N}) \to \mathcal{E}$ to the category of exact functors $\mathrm{N}_{\mathrm{dg}}(\mathrm{C}^b(A)) \to \mathcal{E}$, which sends all objects of $\mathrm{N}_{\mathrm{dg}}(\mathcal{N})$ to zero objects in $\mathcal{E}$, for any (small) stable ∞-category $\mathcal{E}$. Finally, we have an equivalence of categories

$$h(\mathrm{N}_{\mathrm{dg}}(\mathrm{C}^b(A))/\mathrm{N}_{\mathrm{dg}}(\mathcal{N})) \simeq h(\mathrm{N}_{\mathrm{dg}}(\mathrm{C}^b(A)))/h(\mathrm{N}_{\mathrm{dg}}(\mathcal{N}))),$$

where $h(-)$ denotes the homotopy category and the right-hand side is the classical Verdier quotient. Note that the homotopy category of a stable ∞-category is triangulated [18, Theorem 1.1.2.14].

If A is a quasi-abelian category in the sense of [23], we define its bounded derived ∞-category

$$\mathbf{D}^b(A) := \mathrm{N}_{\mathrm{dg}}(\mathrm{C}^b(A))/\mathrm{N}_{\mathrm{dg}}(\mathcal{N}) \simeq \mathrm{N}_{\mathrm{dg}}(\mathrm{C}^b(A))[S^{-1}],$$

where $\mathcal{N} \subset \mathrm{C}^b(A)$ is the full subcategory of strictly acyclic complexes and S is the set of strict quasi-isomorphisms. The homotopy category

$$\mathrm{D}^b(A) := h(\mathbf{D}^b(A))$$

is equivalent to the bounded derived category of the quasi-abelian category A in the sense of [23].

2.2. The category $\mathbf{D}^b(\mathrm{LCA})$.

We denote by LCA the quasi-abelian category of locally compact abelian groups. A morphism of locally compact abelian groups $f : A \to B$ has a kernel $\mathrm{Ker}(f) = f^{-1}(0)$ and a cokernel $\mathrm{Coker}(f) = B/\overline{f(A)}$, where $\overline{f(A)}$ is the closure of $f(A)$ in B. The morphism f is said to be strict if the map $\mathrm{Coker}(\mathrm{Ker}(f)) \to \mathrm{Ker}(\mathrm{Coker}(f))$ is an isomorphism in

LCA. Then f is strict if and only if the induced monomorphism $\overline{f}: A/\mathrm{Ker}(f) \to B$ is a closed embedding. Let $\mathrm{FLCA} \subset \mathrm{LCA}$ be the quasi-abelian category [14, Corollary 2.11] of locally compact abelian groups of finite ranks in the sense of [14, Definition 2.6]. Recall that $A \in \mathrm{LCA}$ has finite ranks if the $\mathbb{R}$-vector spaces of continuous morphisms $\underline{\mathrm{Hom}}(\mathbb{R},A)$ and $\underline{\mathrm{Hom}}(A,\mathbb{R})$ are finite dimensional and $p: A \to A$ is strict with finite kernel and cokernel for any prime number p.

Let $\mathbf{D}^b(\mathrm{LCA})$ and $\mathbf{D}^b(\mathrm{FLCA})$ be the bounded derived ∞-category of LCA and FLCA, respectively. Then $\mathbf{D}^b(\mathrm{LCA})$ and $\mathbf{D}^b(\mathrm{FLCA})$ are stable ∞-categories in the sense of [18], whose homotopy categories are the bounded derived categories $\mathrm{D}^b(\mathrm{LCA})$ and $\mathrm{D}^b(\mathrm{FLCA})$ as defined in [14], respectively. It is more convenient to work with the derived ∞-category $\mathbf{D}^b(\mathrm{LCA})$ rather than with its homotopy category. For example, let $\mathrm{Fun}(\Delta^1,\mathbf{D}^b(\mathrm{LCA}))$ be the ∞-category of arrows in $\mathbf{D}^b(\mathrm{LCA})$. Taking the mapping fibre (or cofibre) of a morphism defines a functor (see [18, Remark 1.1.1.7])

$$\begin{array}{rccc} \mathrm{Fib}: & \mathrm{Fun}(\Delta^1,\mathbf{D}^b(\mathrm{LCA})) & \longrightarrow & \mathbf{D}^b(\mathrm{LCA}) \\ & C \to C' & \longmapsto & C \times_{C'} 0 \end{array}.$$

Let TA be the quasi-abelian category of topological abelian groups, and define $\mathbf{D}^b(\mathrm{TA})$ and $\mathrm{D}^b(\mathrm{TA})$ as above. The inclusions $\mathrm{FLCA} \subset \mathrm{LCA} \subset \mathrm{TA}$ send strict quasi-isomorphisms to strict quasi-isomorphisms, hence, induce functors

$$\mathbf{D}^b(\mathrm{FLCA}) \to \mathbf{D}^b(\mathrm{LCA}) \to \mathbf{D}^b(\mathrm{TA}).$$

The functor $\mathrm{disc}: \mathrm{TA} \to \mathrm{Ab}$, sending a topological abelian group to its underlying discrete abelian group, sends strict quasi-isomorphisms to usual quasi-isomorphisms. This yields a functor

$$\mathrm{disc}: \mathbf{D}^b(\mathrm{TA}) \to \mathbf{D}^b(\mathrm{Ab}).$$

Recall that the Pontryagin dual $X^D := \underline{\mathrm{Hom}}(X,\mathbb{R}/\mathbb{Z})$ of the locally compact abelian group X is the group of continuous homomorphisms $X \to \mathbb{R}/\mathbb{Z}$ endowed with the compact-open topology and that Pontryagin duality gives an isomorphism of locally compact groups

$$X \xrightarrow{\sim} X^{DD}.$$

The functor $(-)^D$ sends strict quasi-isomorphisms to strict quasi-isomorphisms and locally compact compact abelian groups of finite ranks to locally compact groups of finite ranks. We obtain equivalences of ∞-categories

$$\begin{array}{ccc} \mathbf{D}^b(\mathrm{LCA})^{\mathrm{op}} & \longrightarrow & \mathbf{D}^b(\mathrm{LCA}) \\ X & \longmapsto & X^D \end{array}$$

and

$$\begin{array}{ccc} \mathbf{D}^b(\mathrm{FLCA})^{\mathrm{op}} & \longrightarrow & \mathbf{D}^b(\mathrm{FLCA}) \\ X & \longmapsto & X^D \end{array}.$$

In [14], the authors define functors

$$R\mathrm{Hom}_{\mathrm{LCA}}(-,-): \mathrm{D}^b(\mathrm{LCA})^{\mathrm{op}} \times \mathrm{D}^b(\mathrm{LCA}) \to \mathrm{D}^b(\mathrm{TA})$$

and

$$R\mathrm{Hom}_{\mathrm{FLCA}}(-,-) : \mathrm{D}^b(\mathrm{FLCA})^{\mathrm{op}} \times \mathrm{D}^b(\mathrm{FLCA}) \to \mathrm{D}^b(\mathrm{FLCA}).$$

The construction of the functor $R\mathrm{Hom}_{\mathrm{FLCA}}(-,-)$ actually gives a functor of stable ∞-categories

$$R\underline{\mathrm{Hom}}(-,-) : \mathbf{D}^b(\mathrm{FLCA})^{\mathrm{op}} \times \mathbf{D}^b(\mathrm{FLCA}) \to \mathbf{D}^b(\mathrm{FLCA}).$$

Indeed, let I (respectively, P) be the additive category of divisible (respectively, codivisible) locally compact abelian groups I (respectively, P) of finite ranks, such that $I_{\mathbb{Z}} = 0$ (such that $P_{\mathbb{S}^1} = 0$) (see [14, Definition 3.2]). Define

$$\mathbf{D}^b(\mathrm{I}) := \mathrm{N}_{\mathrm{dg}}(\mathrm{C}^b(\mathrm{I}))/\mathrm{N}_{\mathrm{dg}}(\mathcal{N}_{\mathrm{I}}),$$

where $\mathcal{N}_{\mathrm{I}} \subset \mathrm{C}^b(\mathrm{I})$ is the dg-subcategory of strictly acyclic bounded complexes. We define similarly

$$\mathbf{D}^b(\mathrm{P}) := \mathrm{N}_{\mathrm{dg}}(\mathrm{C}^b(\mathrm{P}))/\mathrm{N}_{\mathrm{dg}}(\mathcal{N}_{\mathrm{P}}).$$

The exact functor

$$\mathrm{N}_{\mathrm{dg}}(\mathrm{C}^b(\mathrm{I})) \to \mathrm{N}_{\mathrm{dg}}(\mathrm{C}^b(\mathrm{FLCA})) \to \mathbf{D}^b(\mathrm{FLCA})$$

induces an exact functor

$$\mathbf{D}^b(\mathrm{I}) \to \mathbf{D}^b(\mathrm{FLCA}) \tag{2}$$

of stable ∞-categories which induces an equivalences between the corresponding homotopy categories by [14, Corollary 3.10]. It follows that (2) is an equivalence of stable ∞-categories. Similarly, $\mathbf{D}^b(\mathrm{P}) \to \mathbf{D}^b(\mathrm{FLCA})$ is an equivalence. We may, therefore, define

$$R\underline{\mathrm{Hom}}(-,-) : \mathbf{D}^b(\mathrm{FLCA})^{\mathrm{op}} \times \mathbf{D}^b(\mathrm{FLCA}) \xleftarrow{\sim} \mathbf{D}^b(\mathrm{P})^{\mathrm{op}} \times \mathbf{D}^b(\mathrm{I}) \to \mathbf{D}^b(\mathrm{FLCA})$$

since the functor

$$\begin{array}{ccc} \mathrm{C}^b(\mathrm{P})^{\mathrm{op}} \times \mathrm{C}^b(\mathrm{I}) & \longrightarrow & \mathrm{C}^b(\mathrm{FLCA}) \\ (P,I) & \longmapsto & \underline{\mathrm{Hom}}^{\bullet}(P,I) := \mathrm{Tot}(\underline{\mathrm{Hom}}(P,I)) \end{array}$$

sends a pair of strict quasi-isomorphisms to a strict quasi-isomorphism [14, Corollary 3.7]. Here, $\underline{\mathrm{Hom}}(P,I)$ is the double complex of continuous maps endowed with the compact-open topology and Tot denotes the total complex. Note that the Pontryagin dual X^D is given by the functor

$$R\underline{\mathrm{Hom}}(-,\mathbb{R}/\mathbb{Z}) : \begin{array}{ccc} \mathbf{D}^b(\mathrm{FLCA})^{\mathrm{op}} & \longrightarrow & \mathbf{D}^b(\mathrm{FLCA}) \\ X & \longmapsto & X^D \end{array}.$$

Following [14], we define the derived topological tensor product

$$\begin{array}{ccc} \mathbf{D}^b(\mathrm{FLCA}) \times \mathbf{D}^b(\mathrm{FLCA}) & \longrightarrow & \mathbf{D}^b(\mathrm{FLCA}) \\ (X,Y) & \longmapsto & X \underline{\otimes}^{\mathbb{L}} Y := R\underline{\mathrm{Hom}}(X,Y^D)^D \end{array}. \tag{3}$$

Lemma 2.1. *The functor* $\mathbf{D}^b(\mathrm{FLCA}) \to \mathbf{D}^b(\mathrm{LCA})$ *is an exact and fully faithful functor of stable* ∞*-categories.*

Proof. The functor

$$\mathrm{N}_{\mathrm{dg}}(\mathrm{C}^b(\mathrm{FLCA})) \to \mathrm{N}_{\mathrm{dg}}(\mathrm{C}^b(\mathrm{LCA})) \to \mathbf{D}^b(\mathrm{LCA})$$

induces an exact functor $\mathbf{D}^b(\mathrm{FLCA}) \to \mathbf{D}^b(\mathrm{LCA})$ by [20, Theorem I.3.3(i)]. It remains to check that this functor is fully faithful. The functors $R\mathrm{Hom}_{\mathrm{LCA}}(-,-)$ and $R\mathrm{Hom}_{\mathrm{FLCA}}(-,-)$ induce the same functor

$$\mathrm{D}^b(\mathrm{FLCA})^{\mathrm{op}} \times \mathrm{D}^b(\mathrm{FLCA}) \to \mathrm{D}^b(\mathrm{TA})$$

by [14, Remark 4.9]. Moreover, for any $X, Y \in \mathrm{D}^b(\mathrm{FLCA})$, we have [14, Proposition 4.12(i)]

$$\mathrm{disc}(H^0(R\mathrm{Hom}_{\mathrm{LCA}}(X,Y))) \simeq \mathrm{Hom}_{\mathrm{D}^b(\mathrm{LCA})}(X,Y)$$

and similarly[1]

$$\mathrm{disc}(H^0(R\mathrm{Hom}_{\mathrm{FLCA}}(X,Y))) \simeq \mathrm{Hom}_{\mathrm{D}^b(\mathrm{FLCA})}(X,Y).$$

Therefore, the map

$$\mathrm{Hom}_{\mathrm{D}^b(\mathrm{FLCA})}(X,Y) \to \mathrm{Hom}_{\mathrm{D}^b(\mathrm{LCA})}(X,Y)$$

is an isomorphism of abelian groups, that is, $\mathrm{D}^b(\mathrm{FLCA}) \to \mathrm{D}^b(\mathrm{LCA})$ is fully faithful. Hence,

$$\mathbf{D}^b(\mathrm{FLCA}) \to \mathbf{D}^b(\mathrm{LCA}) \tag{4}$$

is an exact functor of stable ∞-categories which induces a fully faithful functor between the corresponding homotopy categories. It follows that (4) is fully faithful. ☐

Therefore, we may identify $\mathbf{D}^b(\mathrm{FLCA})$ with its essential image in $\mathbf{D}^b(\mathrm{LCA})$. The stable ∞-category $\mathbf{D}^b(\mathrm{LCA})$ is endowed with a t-structure by [23, Section 1.2.2], since a t-structure on a stable ∞-category is defined as a t-structure on its homotopy category [18, Definition 1.2.1.4]. We denote its heart by $\mathcal{LH}(\mathrm{LCA})$. It is an abelian category containing LCA as a full subcategory This also applies to $\mathbf{D}^b(\mathrm{FLCA})$, and we denote $\mathcal{LH}(\mathrm{FLCA})$ the heart of the corresponding t-structure.

Remark 2.2. By [23, Corollary 1.2.21], an object in $\mathcal{LH}(\mathrm{LCA})$ can be represented by a monomorphism $E_1 \to E_0$ in LCA, where E_0 is in degree zero. A common example appearing below is a monomorphism $\mathbb{Z}^a \to \mathbb{Z}_p$. Its cokernel in LCA is trivial, but for the underlying discrete abelian groups the cokernel is

$$(\oplus_{l \neq p} \mathbb{Q}_l/\mathbb{Z}_l) \oplus (\mathbb{Q}/\mathbb{Z})^{a-1} \oplus D$$

with D uniquely divisible.

Remark 2.3. It follows from [14, Corollary 2.11] and [23, Proposition 1.2.19] that the fully faithful functor $\mathbf{D}^b(\mathrm{FLCA}) \to \mathbf{D}^b(\mathrm{LCA})$ is t-exact. Therefore, the induced functor $\mathcal{LH}(\mathrm{FLCA}) \hookrightarrow \mathcal{LH}(\mathrm{LCA})$ is exact and fully faithful.

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[1] One may adapt the proof of [14, Proposition 4.12(i)] to this case using [14, Corollary 3.10(iii)].

Notation 2.4. For any $X \in \mathbf{D}^b(\mathrm{LCA})$ and any $i \in \mathbb{Z}$, we consider

$$H^i(X) := \tau^{\geq 0}\tau^{\leq 0}(X[i]) \in \mathcal{LH}(\mathrm{LCA}).$$

In view of Remark 2.3, we identify $\mathcal{LH}(\mathrm{FLCA})$ with a full subcategory of $\mathcal{LH}(\mathrm{LCA})$.

Lemma 2.5. *Let* $X \in \mathbf{D}^b(\mathrm{LCA})$*. Then* $X \in \mathbf{D}^b(\mathrm{FLCA})$ *if and only if* $H^i(X) \in \mathcal{LH}(\mathrm{FLCA})$ *for any* $i \in \mathbb{Z}$*.*

Proof. If $X \to Y \to Z$ is a fibre sequence in $\mathbf{D}^b(\mathrm{LCA})$, such that $X, Z \in \mathbf{D}^b(\mathrm{FLCA})$, then $Y \in \mathbf{D}^b(\mathrm{FLCA})$. Indeed, a stable subcategory is closed under extensions. Let $X \in \mathbf{D}^b(\mathrm{LCA})$, such that $H^i(X) \in \mathcal{LH}(\mathrm{FLCA})$ for any i. Note that $H^i(X) = 0$ for all but finitely many $i \in \mathbb{Z}$. Therefore, X has a finite exhaustive filtration with i-graded piece $H^i(X)[-i] \in \mathbf{D}^b(\mathrm{FLCA})$, so that X belongs to $\mathbf{D}^b(\mathrm{FLCA})$ by induction.

The converse follows from the fact that the inclusion functor $\mathbf{D}^b(\mathrm{FLCA}) \to \mathbf{D}^b(\mathrm{LCA})$ is t-exact by Remark 2.3. □

Recall that Pontryagin duality gives an equivalence

$$\begin{array}{ccc} \mathbf{D}^b(\mathrm{LCA})^{\mathrm{op}} & \longrightarrow & \mathbf{D}^b(\mathrm{LCA}) \\ X & \longmapsto & X^D \end{array}.$$

Lemma 2.6. *Let* $X \in \mathbf{D}^b(\mathrm{LCA})$*, such that* $H^i(X) \in \mathcal{LH}(\mathrm{LCA})$ *belongs to* LCA *for any* $i \in \mathbb{Z}$*. Then for any* $i \in \mathbb{Z}$*, we have a canonical isomorphism in* LCA

$$H^i(X^D) \simeq (H^{-i}(X))^D.$$

Proof. Let

$$X = [\cdots \to X^{i-1} \xrightarrow{d_X^{i-1}} X^i \xrightarrow{d_X^i} X^{i+1} \to \cdots]$$

be an object of $\mathbf{D}^b(\mathrm{LCA})$, such that $H^i(X) \in \mathcal{LH}(\mathrm{LCA})$ belongs to LCA for any $i \in \mathbb{Z}$. We first observe that the differentials d_X^i all are strict morphisms. By [23, Proposition 1.2.19], the object $H^i(X)$ of $\mathcal{LH}(\mathrm{LCA})$ is given by the complex $[0 \to \mathrm{Coim}(d_X^{i-1}) \xrightarrow{\delta} \mathrm{Ker}(d_X^i) \to 0]$, where $\mathrm{Ker}(d_X^i)$ sits in degree 0 and δ is a monomorphism. Since $H^i(X) \in \mathrm{LCA}$, the map δ is strict by [23, Proposition 1.2.29], that is, δ is a closed embedding. Since $\mathrm{Ker}(d_X^i) \hookrightarrow X^i$ is a closed embedding as well, so is the map $\mathrm{Coim}(d_X^{i-1}) = X^{i-1}/\mathrm{Ker}(d_X^{i-1}) \to X^i$. Hence, d_X^{i-1} is strict.

We set $Y := X^D$ so that $Y^{-i} = (X^i)^D$ and $d_Y^{-i} : Y^{-i} \to Y^{-i+1}$ is the map $d_Y^{-i} = (d_X^{i-1})^D$. The differentials d_X^* are all strict morphisms, hence, so are their duals d_Y^*. We have the following isomorphisms of locally compact abelian groups:

$$\begin{aligned} H^i(X)^D &\simeq \left(\mathrm{Coker}(\mathrm{Coim}(d_X^{i-1}) \xrightarrow{\delta} \mathrm{Ker}(d_X^i))\right)^D && (5)\\ &\simeq \mathrm{Ker}\left(\mathrm{Coker}((d_X^i)^D) \to \mathrm{Im}((d_X^{i-1})^D)\right) && (6)\\ &\simeq \mathrm{Ker}\left(\mathrm{Coker}(d_Y^{-i-1}) \to \mathrm{Im}(d_Y^{-i})\right) && (7)\\ &\simeq \mathrm{Ker}\left(\mathrm{Coker}(d_Y^{-i-1}) \to Y^{-i+1}\right) && (8) \end{aligned}$$

$$\simeq \operatorname{Ker}\left(Y^{-i}/d_Y^{-i-1}(Y^{-i-1}) \to Y^{-i+1}\right) \tag{9}$$

$$\simeq \operatorname{Ker}(d_Y^{-i})/d_Y^{-i-1}(Y^{-i-1}) \tag{10}$$

$$\simeq \operatorname{Ker}(d_Y^{-i})/\operatorname{Coim}(d_Y^{-i-1}) \tag{11}$$

$$\simeq H^{-i}(Y), \tag{12}$$

where the kernels, cokernels, images and coimages are all computed in LCA. The isomorphism (5) is valid by [23, Proposition 1.2.29] since the map δ is strict, and (6) holds since Pontryagin duality $(-)^D : \mathrm{LCA}^{\mathrm{op}} \to \mathrm{LCA}$ is an equivalence of categories with kernels and cokernels. The identification (7) is given by definition of the maps d_Y^*, and (8) holds since $\operatorname{Im}(d_Y^{-i}) \to Y^{-i+1}$ is a monomorphism. We have (9) in view of $\operatorname{Coker}(d_Y^{-i-1}) = Y^{-i}/d_Y^{-i-1}(Y^{-i-1})$, which is valid since $d_Y^{-i-1}(Y^{-i-1})$ is closed in Y^{-i}, as d_Y^{-i-1} is strict. The isomorphism of locally compact abelian groups (10) is clear; (11) holds since $\operatorname{Coim}(d_Y^{-i-1}) \to d_Y^{-i-1}(Y^{-i-1}) = \operatorname{Im}(d_Y^{-i-1})$ is an isomorphism in LCA since d_Y^{-i-1} is strict. Finally, (12) holds by [23, Propositions 1.2.19 and 1.2.29] since $\operatorname{Coim}(d_Y^{-i-1}) \to \operatorname{Ker}(d_Y^{-i})$ is strict; indeed, $Y^{-i-1}/\operatorname{Ker}(d_Y^{-i-1}) \to Y^{-i}$ is a closed embedding, hence, so is $Y^{-i-1}/\operatorname{Ker}(d_Y^{-i-1}) \to \operatorname{Ker}(d_Y^{-i})$. □

The inclusion $\mathrm{Ab} \subset \mathrm{LCA}$ as discrete objects induces an exact functor

$$i : \mathbf{D}^b(\mathrm{Ab}) \to \mathbf{D}^b(\mathrm{LCA}).$$

Proposition 2.7. *The exact functor $i : \mathbf{D}^b(\mathrm{Ab}) \to \mathbf{D}^b(\mathrm{LCA})$ is fully faithful and left adjoint to*

$$\mathrm{disc} : \mathbf{D}^b(\mathrm{LCA}) \to \mathbf{D}^b(\mathrm{Ab}).$$

Proof. The functor

$$C^b(\mathrm{Ab}) \xrightarrow{i} C^b(\mathrm{LCA}) \xrightarrow{\mathrm{disc}} C^b(\mathrm{Ab})$$

is isomorphic to the identity functor of $C^b(\mathrm{Ab})$. We obtain a natural transformation

$$\mathrm{Id}_{\mathbf{D}^b(\mathrm{Ab})} \xrightarrow{\sim} \mathrm{disc} \circ i. \tag{13}$$

Similarly, there is a natural transformation

$$i \circ \mathrm{disc} \to \mathrm{Id}_{\mathbf{D}^b(\mathrm{LCA})}.$$

Let $X \in \mathbf{D}^b(\mathrm{Ab})$, and let $Y \in \mathbf{D}^b(\mathrm{LCA})$. Let $F \xrightarrow{\sim} X$ be a bounded flat resolution, and let $Y \xrightarrow{\sim} D$ be a strict quasi-isomorphism, where D is a bounded complex of divisible locally compact abelian groups. Then F is a bounded complex of codivisible[2] discrete groups F^i (in particular, such that $F^i_{\mathbb{S}^1} = 0$). Therefore, we have

$$R\mathrm{Hom}_{\mathrm{LCA}}(i(X), Y) \simeq \underline{\mathrm{Hom}}^{\bullet}(F, D) := \mathrm{Tot}(\underline{\mathrm{Hom}}(F, D))$$

---

[2] $A \in \mathrm{LCA}$ is said to be codivisible if A^D is divisible.

by [14, Corollary 4.7], where $\underline{\mathrm{Hom}}(F,D)$ is the double complex of continuous maps endowed with the compact-open topology and Tot denotes the total complex. We obtain

$$\begin{aligned}\mathrm{disc}\,(R\mathrm{Hom}_{\mathrm{LCA}}(i(X),Y)) &\simeq \mathrm{disc}\,(\underline{\mathrm{Hom}}^{\bullet}(F,D))\\ &\simeq \mathrm{Hom}^{\bullet}(F,\mathrm{disc}(D))\\ &\simeq R\mathrm{Hom}(X,\mathrm{disc}(Y)).\end{aligned}$$

In view of [14, Proposition 4.12], we have

$$\begin{aligned}H^0(\mathrm{disc}\,(R\mathrm{Hom}_{\mathrm{LCA}}(i(X),Y[-n]))) &\simeq \mathrm{disc}(H^0\,(R\mathrm{Hom}_{\mathrm{LCA}}(i(X),Y[-n])))\\ &\simeq \mathrm{Hom}_{\mathrm{D}^b(\mathrm{LCA})}(i(X),Y[-n])\\ &\simeq \pi_0(\mathrm{Map}_{\mathbf{D}^b(\mathrm{LCA})}(i(X),\Omega^n Y))\\ &\simeq \pi_n(\mathrm{Map}_{\mathbf{D}^b(\mathrm{LCA})}(i(X),Y)),\end{aligned}$$

where $\Omega(-) := 0\times_{(-)} 0$ is the loop space functor. Similarly, we have

$$H^0(R\mathrm{Hom}(X,\mathrm{disc}(Y[-n]))) \simeq \pi_n(\mathrm{Map}_{\mathbf{D}^b(\mathrm{Ab})}(X,\mathrm{disc}(Y))).$$

Hence, the map

$$\mathrm{Map}_{\mathbf{D}^b(\mathrm{LCA})}(i(X),Y) \to \mathrm{Map}_{\mathbf{D}^b(\mathrm{Ab})}(X,\mathrm{disc}(Y))$$

is an equivalence of ∞-groupoids. The result then follows from [17, Proposition 5.2.2.8] and from the fact that the unit transformation (13) is an equivalence. □

Definition 2.8. An object $X \in \mathbf{D}^b(\mathrm{LCA})$ lies in the essential image of the functor $i : \mathbf{D}^b(\mathrm{Ab}) \to \mathbf{D}^b(\mathrm{LCA})$ if and only if the counit map $i\circ \mathrm{disc}(X) \to X$ is an equivalence. Such an object $X \in \mathbf{D}^b(\mathrm{LCA})$ is called *discrete*.

Lemma 2.9. *Let $X,Y \in \mathbf{D}^b(\mathrm{Ab})$. If iX and iY belong to $\mathbf{D}^b(\mathrm{FLCA})$, then there is a canonical map*

$$i(R\mathrm{Hom}(X,Y)) \to R\underline{\mathrm{Hom}}(iX,iY).$$

Moreover, if X,Y are perfect complexes of abelian groups, then this map is an equivalence.

Proof. Let $P \xrightarrow{\sim} iX$ and $iY \xrightarrow{\sim} I$ be strict quasi-isomorphisms, where $P \in \mathrm{C}^b(\mathrm{P})$ (respectively, $I \in \mathrm{C}^b(\mathrm{I})$). We denote by $P^{\delta} := \mathrm{disc}(P)$ and $I^{\delta} := \mathrm{disc}(P)$ the underlying complexes of discrete abelian groups. Then the maps $P^{\delta} \xrightarrow{\sim} X$ and $Y \xrightarrow{\sim} I^{\delta}$ are quasi-isomorphisms in the usual sense. Hence, we have $\mathrm{Hom}^{\bullet}(X,I^{\delta}) \simeq R\mathrm{Hom}(X,Y)$, where $\mathrm{Hom}^{\bullet}$ denotes the total complex of the double complex of morphisms of abelian groups. We denote by $\underline{\mathrm{Hom}}^{\bullet}(P,I)$ the total complex of the double complex of continuous morphisms endowed with the compact-open topology. Then we have morphisms

$$R\mathrm{Hom}(X,Y) \simeq \mathrm{Hom}^{\bullet}(X,I^{\delta}) \to \underline{\mathrm{Hom}}^{\bullet}(iX,I) \to \underline{\mathrm{Hom}}^{\bullet}(P,I) \simeq R\underline{\mathrm{Hom}}(iX,iY).$$

Suppose now that X and Y are perfect complexes of abelian groups. We may suppose that X^n is a finitely generated free abelian group for all $n \in \mathbb{Z}$, zero for almost all n and similarly for Y. We have a strict quasi-isomorphism

$$iY \xrightarrow{\sim} I := \mathrm{Tot}[Y\otimes\mathbb{R} \to Y\otimes\mathbb{R}/\mathbb{Z}],$$

where $[Y\otimes\mathbb{R}\to Y\otimes\mathbb{R}/\mathbb{Z}]$ is seen as a double complex of locally compact abelian groups and Tot is the total complex. Then $iX\in \mathrm{C}^b(\mathrm{P})$ and $I\in\mathrm{C}^b(\mathrm{I})$, and we have a strict quasi-isomorphism

$$i\mathrm{Hom}^\bullet(X,Y)\overset{\sim}{\to}\underline{\mathrm{Hom}}^\bullet(iX,I).$$

We obtain

$$iR\mathrm{Hom}(X,Y)\simeq i\mathrm{Hom}^\bullet(X,Y)\overset{\sim}{\to}\underline{\mathrm{Hom}}^\bullet(iX,I)\simeq R\underline{\mathrm{Hom}}(iX,iY).$$

□

2.3. Profinite completion

Definition 2.10. We define a functor

$$\begin{array}{rccc}(-)\widehat{\underline{\otimes}}\widehat{\mathbb{Z}}: & \mathbf{D}^b(\mathrm{Ab}) & \longrightarrow & \mathbf{D}^b(\mathrm{LCA}) \\ & X & \longmapsto & (i(\mathrm{colim}\,R\mathrm{Hom}(X,\mathbb{Z}/m)))^D,\end{array}$$

where we compute $R\mathrm{Hom}(X,\mathbb{Z}/m)$ and the colimit $\mathrm{colim}\,R\mathrm{Hom}(X,\mathbb{Z}/m)$ over m in the ∞-category $\mathbf{D}^b(\mathrm{Ab})$. We define similarly

$$\begin{array}{rccc}(-)\widehat{\underline{\otimes}}\mathbb{Z}_p: & \mathbf{D}^b(\mathrm{Ab}) & \longrightarrow & \mathbf{D}^b(\mathrm{LCA}) \\ & X & \longmapsto & (i(\mathrm{colim}\,R\mathrm{Hom}(X,\mathbb{Z}/p^\bullet)))^D.\end{array}$$

For any $X\in\mathbf{D}^b(\mathrm{LCA})$, we define[3]

$$R\underline{\mathrm{Hom}}(X,\mathbb{Z}/m):=\mathrm{Fib}(X^D\xrightarrow{m}X^D)$$

and

$$X\underline{\otimes}^{\mathbb{L}}\mathbb{Z}/m:=\mathrm{Cofib}(X\xrightarrow{m}X).$$

Proposition 2.11. *Let $X\in\mathbf{D}^b(\mathrm{Ab})$. Suppose that $R\underline{\mathrm{Hom}}(i(X),\mathbb{Z}/m)\in\mathbf{D}^b(\mathrm{LCA})$ is discrete for any m. Then we have an equivalence*

$$X\widehat{\underline{\otimes}}\widehat{\mathbb{Z}}\simeq\varprojlim\left(i(X)\underline{\otimes}^{\mathbb{L}}\mathbb{Z}/m\right),$$

where the limit is computed in the ∞-category $\mathbf{D}^b(\mathrm{LCA})$ and an equivalence

$$\mathrm{disc}(X\widehat{\underline{\otimes}}\widehat{\mathbb{Z}})\simeq X\widehat{\otimes}\widehat{\mathbb{Z}}:=\varprojlim(X\otimes^{\mathbb{L}}\mathbb{Z}/m)\in\mathbf{D}^b(\mathrm{Ab}).$$

Proof. The co-unit map

$$i\circ\mathrm{disc}\,R\underline{\mathrm{Hom}}(i(X),\mathbb{Z}/m)\to R\underline{\mathrm{Hom}}(i(X),\mathbb{Z}/m)$$

is an equivalence by assumption. Moreover, we have

$$R\mathrm{Hom}(X,\mathbb{Z}/m)\simeq\mathrm{disc}\,R\underline{\mathrm{Hom}}(i(X),\mathbb{Z}/m),$$

---

[3] This is compatible with the definition given in Section 2.2, which is only valid if $X\in\mathbf{D}^b(\mathrm{FLCA})$.

hence,

$$i\,R\mathrm{Hom}(X,\mathbb{Z}/m)\xrightarrow{\sim}R\underline{\mathrm{Hom}}(i(X),\mathbb{Z}/m).$$

We obtain

$$\begin{aligned}X\widehat{\underline{\otimes}}\widehat{\mathbb{Z}}&:=(i(\mathrm{colim}\,R\mathrm{Hom}(X,\mathbb{Z}/m)))^D\\&\simeq(\mathrm{colim}\,(i\,R\mathrm{Hom}(X,\mathbb{Z}/m)))^D\\&\simeq\lim\left((i\,R\mathrm{Hom}(X,\mathbb{Z}/m))^D\right)\\&\simeq\lim\left(R\underline{\mathrm{Hom}}(i(X),\mathbb{Z}/m)^D\right)\\&\simeq\lim\left(i(X)\underline{\otimes}^{\mathbb{L}}\mathbb{Z}/m\right)\end{aligned}$$

since the left adjoint functor i commutes with arbitrary colimits and since $(-)^D$ transforms colimits into limits. Hence, we have

$$\begin{aligned}\mathrm{disc}(X\widehat{\underline{\otimes}}\widehat{\mathbb{Z}})&\simeq\mathrm{disc}\left(\lim(i(X)\underline{\otimes}^{\mathbb{L}}\mathbb{Z}/m)\right)\\&\simeq\lim\left(\mathrm{disc}(i(X)\underline{\otimes}^{\mathbb{L}}\mathbb{Z}/m)\right)\\&\simeq\lim\Big(\mathrm{Cofib}(\mathrm{disc}\circ i(X)\xrightarrow{m}\mathrm{disc}\circ i(X))\Big)\\&\simeq\lim\left(X\otimes^{\mathbb{L}}\mathbb{Z}/m\right)\end{aligned}$$

since the right adjoint functor disc commutes with arbitrary limits. ☐

Remark 2.12. Suppose that $X\in\mathbf{D}^b(\mathrm{Ab})$ is, such that, the cohomology groups of $X\otimes^{\mathbb{L}}\mathbb{Z}/m$ are all finite. Then $R\underline{\mathrm{Hom}}(i(X),\mathbb{Z}/m)$ is discrete.

Remark 2.13. We have

$$X\widehat{\underline{\otimes}}\widehat{\mathbb{Z}}\simeq R\underline{\mathrm{Hom}}\left(i\,\mathrm{colim}\,R\mathrm{Hom}(X\otimes^{\mathbb{L}}\mathbb{Z}/m,\mathbb{Q}/\mathbb{Z}),\mathbb{R}/\mathbb{Z}\right).$$

Lemma 2.14. *We have a canonical map $iX\to X\widehat{\underline{\otimes}}\widehat{\mathbb{Z}}$ in $\mathbf{D}^b(\mathrm{LCA})$.*

Proof. The composite map

$$i(R\mathrm{Hom}(X,\mathbb{Z}/m))\xrightarrow{\sim}i\circ\mathrm{disc}(R\underline{\mathrm{Hom}}(iX,\mathbb{Z}/m))\to R\underline{\mathrm{Hom}}(iX,\mathbb{Z}/m)$$

$$\to R\underline{\mathrm{Hom}}(iX,\mathbb{R}/\mathbb{Z})\simeq(iX)^D$$

induces

$$i(\mathrm{colim}\,R\mathrm{Hom}(X,\mathbb{Z}/m))\simeq\mathrm{colim}\,i(R\mathrm{Hom}(X,\mathbb{Z}/m))\to(iX)^D.$$

We obtain

$$iX\xrightarrow{\sim}(iX)^{DD}\to(i(\mathrm{colim}\,R\mathrm{Hom}(X,\mathbb{Z}/m)))^D=:X\widehat{\underline{\otimes}}\widehat{\mathbb{Z}}.$$

☐

Remark 2.15. Let X be an object of $\mathbf{D}^b(\mathrm{Ab})$ whose image $iX \in \mathbf{D}^b(\mathrm{LCA})$ belongs to $\mathbf{D}^b(\mathrm{FLCA})$. Then one may consider $iX\underline{\otimes}^{\mathbb{L}}\widehat{\mathbb{Z}}$ and $iX\underline{\otimes}^{\mathbb{L}}\widehat{\mathbb{Z}}_p$, where $\underline{\otimes}^{\mathbb{L}}$ is the tensor product (3) in $\mathbf{D}^b(\mathrm{FLCA})$. There are canonical maps $iX\underline{\otimes}^{\mathbb{L}}\widehat{\mathbb{Z}} \to X\widehat{\underline{\otimes}}\widehat{\mathbb{Z}}$ and $iX\underline{\otimes}^{\mathbb{L}}\mathbb{Z}_p \to X\widehat{\underline{\otimes}}\mathbb{Z}_p$ but those maps are not equivalences, in general. For example, we have

$$\mathbb{Q}_p/\mathbb{Z}_p\underline{\otimes}^{\mathbb{L}}\mathbb{Z}_p \simeq \mathbb{Q}_p/\mathbb{Z}_p \quad\text{and}\quad \mathbb{Q}\underline{\otimes}^{\mathbb{L}}\mathbb{Z}_p \simeq \mathbb{Q}_p,$$

while

$$\mathbb{Q}_p/\mathbb{Z}_p\widehat{\underline{\otimes}}\mathbb{Z}_p \simeq \mathbb{Z}_p[1] \quad\text{and}\quad \mathbb{Q}\widehat{\underline{\otimes}}\mathbb{Z}_p \simeq 0.$$

Notation 2.16. In the next sections, given $X \in \mathbf{D}^b(\mathrm{Ab})$, we often simply denote by X its image iX in $\mathbf{D}^b(\mathrm{LCA})$. In particular, for $X,Y \in \mathbf{D}^b(\mathrm{Ab})$, we denote by $R\mathrm{Hom}(X,Y) \in \mathbf{D}^b(\mathrm{Ab}) \subseteq \mathbf{D}^b(\mathrm{LCA})$ the usual $R\mathrm{Hom}$ seen as an object of $\mathbf{D}^b(\mathrm{LCA})$.

3. Duality for schemes over finite fields

Let Y be a proper scheme over a finite field. If Y is smooth, then the Weil-étale cohomology $R\Gamma_W(Y,\mathbb{Z})$ of [8] is a perfect complex of abelian groups. In general, the Weil-h cohomology $R\Gamma_{Wh}(Y,\mathbb{Z})$ of [8] is a perfect complex of abelian groups provided resolution of singularities [8, Definition 2.4] holds (see Proposition 3.2 below). We show that if Y is a simple normal crossing scheme, then the Weil-étale cohomology $R\Gamma_W(Y,\mathbb{Z})$ is a perfect complex of abelian groups, and that the canonical map $R\Gamma_W(Y,\mathbb{Z}) \to R\Gamma_{Wh}(Y,\mathbb{Z})$ is an equivalence under resolution of singularities. In Section 3.3, we show that $R\Gamma_W(Y,\mathbb{Z})$ is dual to $R\Gamma_W(Y,\mathbb{Z}^c(0))$ under the assumption that $R\Gamma_W(Y,\mathbb{Z}^c(0))$ is perfect, where $\mathbb{Z}^c(0)$ is the cycle complex.

3.1. Finite generation of cohomology

Definition 3.1. Let k be a finite field with algebraic closure $\bar{k}$ and W_k be its Weil group. For a scheme Y over k, we let $\bar{Y} = Y \times_k \bar{k}$. For a scheme Y of finite type Y over k, we define the Wh-cohomology of the constant sheaf $\mathbb{Z}$ to be

$$R\Gamma_{Wh}(Y,\mathbb{Z}) := R\Gamma(W_k, R\Gamma_{eh}(\bar{Y},\mathbb{Z})).$$

Proposition 3.2. *Let Y be a proper scheme over a finite field k. Assume resolution of singularities for schemes over k of dimension $\leq \dim(Y)$ [8, Definition 2.4]. Then $R\Gamma_{Wh}(Y,\mathbb{Z})$ is a perfect complex of abelian groups.*

Proof. To prove perfectness of $R\Gamma_{Wh}(Y,\mathbb{Z})$, one first reduces to the smooth and projective case by [8, Proposition 3.2], in which case, one can conclude with loc. cit. Theorem 4.3 and [16]. ☐

Definition 3.3. Let k be a field, and let Y be a pure dimensional proper scheme over k with irreducible components Y_i, $i = 1,\ldots,c$. Then Y is said to be a *simple normal crossing scheme* if for all $I \subseteq \{1,\ldots,c\}$, $Y_I = \bigcap_{i\in I} Y_i$ is regular of codimension $|I|-1$ in Y.

In fact, for all the results in this paper, we only need that $(Y_I)^{\mathrm{red}}$ is regular.

Lemma 3.4. *Consider a Cartesian square*

$$\begin{tikzcd} Y' \arrow[r, "i'"] \arrow[d, "q"'] & T' \arrow[d, "\pi"] \\ Y \arrow[r, "i"] & T \end{tikzcd}$$

of schemes of finite type over a field k with i a closed embedding and π finite, such that $\pi|T'-Y'$ is an isomorphism to $T-Y$. Then there is a distinguished triangle

$$R\Gamma_{et}(T,\mathbb{Z}) \to R\Gamma_{et}(T',\mathbb{Z}) \oplus R\Gamma_{et}(Y,\mathbb{Z}) \to R\Gamma_{et}(Y',\mathbb{Z}).$$

In particular, if k is a finite field, we obtain a triangle

$$R\Gamma_W(T,\mathbb{Z}) \to R\Gamma_W(T',\mathbb{Z}) \oplus R\Gamma_W(Y,\mathbb{Z}) \to R\Gamma_W(Y',\mathbb{Z}).$$

Proof. To get the first triangle, noting that i_* and π_* are exact, it suffices to show that the exact sequence

$$0 \to \mathbb{Z} \to \pi_*\mathbb{Z} \oplus i_*\mathbb{Z} \to (\pi \circ i')_*\mathbb{Z} \to 0$$

of étale sheaves on T is exact. But this follows by considering stalks at points $t \in T$. If $t \notin Y$, then the sequence reduces to the isomorphism $\mathbb{Z} \cong \pi_*\mathbb{Z}$, and if $t \in Y$, then $\mathbb{Z} \cong i_*\mathbb{Z}$ and $\pi_*\mathbb{Z} \cong (\pi \circ i')_*\mathbb{Z}$.

The second triangle can be obtained by applying $R\Gamma(W,-)$ to the first triangle after base extension to the algebraic closure. □

Proposition 3.5. *If T^{red} is a strict normal crossing scheme, then $R\Gamma_W(T,\mathbb{Z})$ is a perfect complex of abelian groups. Under resolution of singularities, we have a quasi-isomorphic $R\Gamma_W(T,\mathbb{Z}) \simeq R\Gamma_{Wh}(T,\mathbb{Z})$.*

Proof. Since étale cohomology with coefficients in $\mathbb{Z}$ does not change if we replace T by T^{red}, we can assume that T is reduced. We proceed by induction on dimension of T and the number of irreducible components of T. If the number of components is one, then T^{red} is smooth and proper and the result follows from [8, Theorem 4.3]. In general, let $T = \cup_{i\in I} S_i$ and set $Y = S_1$ and $T' = \cup_{i\neq 1} S_i$. Then the hypotheses of Lemma 3.4 are satisfied, Y is smooth, T' is a normal crossing scheme with fewer irreducible components and Y' a normal crossing scheme of smaller dimension. Hence, we obtain the first statement on perfectness and the second statement by comparing with the corresponding triangle for Wh-cohomology. □

Note that we can have $H^2_W(T,\mathbb{Z}) \neq H^2_{Wh}(T,\mathbb{Z})$ for normal proper surfaces [8, Proposition 8.2].

3.2. Finite generation of homology

For later use, we record the following conditional results on finite generation of homology. Recall the following conjecture from [10].

Conjecture $P_n(X)$: For the smooth and proper scheme X over a finite field, the group $CH_n(X,i)$ is torsion for all $i > 0$.

Conjecture $P_n(X)$ is known for all n if X is a curve. In general, it is a particular case of Parshin's conjecture, which is equivalent to the statement $P_n(X)$ for all n. Parshin's conjecture, in turn, is implied by the Beilinson-Tate conjecture [6, Theorem 1.2]. By the projective bundle formula, conjecture $P_n(X)$ for all X of dimension, at most, d implies conjecture $P_{n-1}(X)$ for all X of dimension $d-1$. The following proposition is [10, Proposition 4.2].

Proposition 3.6. *If conjecture $P_0(X)$ holds for all smooth and proper schemes of dimension, at most, $\dim Y$, then the cohomology groups of $R\Gamma_W(Y,\mathbb{Z}^c(0))$ are finitely generated and vanish for almost all indices.*

If Y is a simple normal crossing scheme, then it suffices to assume that $P_0(Y_I)$ holds for all multiple intersections Y_I.

Proposition 3.7. *If resolution of singularities and conjecture $P_{-1}(X)$ holds for all schemes of dimension, at most, $d-1$, then the cohomology groups of $R\Gamma_{Wh}(\mathcal{X}_s,\mathbb{Z}(d))$ are finite and vanish for almost all indices.*

Proof. Using blow-up squares and induction on the dimension, it suffices to prove the statement for smooth and proper schemes T of dimension, at most, $d-1$. By [8, Corollary 5.5], the Weil-eh cohomology groups agree with Weil-étale cohomology groups. By conjecture $P_{-1}(T)$, they are torsion, hence, finite by comparison with étale cohomology groups. □

3.3. Duality

Theorem 3.8. *Let Y be a simple normal crossing scheme over a finite field k, such that $R\Gamma_W(Y,\mathbb{Z}^c(0))$ is a perfect complex of abelian groups. Then there is a perfect pairing*

$$R\Gamma_W(Y,\mathbb{Z}) \otimes^{\mathbb{L}} R\Gamma_W(Y,\mathbb{Z}^c(0)) \to \mathbb{Z}[-1]$$

of perfect complexes of abelian groups.

Proof. Let $f: Y \to \operatorname{Spec}(k) = s$ be the structure morphism. The pushforward map [9, Corollary 3.2]

$$Rf_*\mathbb{Z}^c(0)^Y \to \mathbb{Z}^c(0)^s \simeq \mathbb{Z}[0]$$

induces a trace map

$$R\Gamma_W(Y,\mathbb{Z}^c(0)) \to R\Gamma_W(s,\mathbb{Z}) \to \mathbb{Z}[-1].$$

We consider the map

$$R\Gamma_W(Y,\mathbb{Z}) \longrightarrow R\mathrm{Hom}(R\Gamma_W(Y,\mathbb{Z}^c(0)),\mathbb{Z}[-1]) \tag{14}$$

induced by the pairing

$$R\Gamma_W(Y,\mathbb{Z}) \otimes^{\mathbb{L}} R\Gamma_W(Y,\mathbb{Z}^c(0)) \to R\Gamma_W(Y,\mathbb{Z}^c(0)) \to R\Gamma_W(s,\mathbb{Z}^c(0)) \to \mathbb{Z}[-1],$$

which, in turn, is induced by the obvious pairing $\mathbb{Z} \otimes^{\mathbb{L}} \mathbb{Z}^c(0) \to \mathbb{Z}^c(0)$. In order to show that the morphism of perfect complexes (14) is an equivalence, it is enough to show that

(14) $\otimes^{\mathbb{L}} \mathbb{Z}/m\mathbb{Z}$ is an equivalence for any integer m. But (14) $\otimes^{\mathbb{L}} \mathbb{Z}/m\mathbb{Z}$ may be identified with the canonical map

$$R\Gamma_{et}(Y,\mathbb{Z}/m\mathbb{Z}) \longrightarrow R\mathrm{Hom}(R\Gamma_{et}(Y,\mathbb{Z}^c(0)/m),\mathbb{Q}/\mathbb{Z}[-1]) \tag{15}$$

since we have an equivalence of lax symmetric monoidal functors

$$R\Gamma_W(Y,-)\otimes^{\mathbb{L}}_{\mathbb{Z}}\mathbb{Z}/m\mathbb{Z} \simeq R\Gamma_{et}(Y,(-)\otimes^{\mathbb{L}}_{\mathbb{Z}}\mathbb{Z}/m\mathbb{Z}).$$

But (15) is an equivalence by [9, Theorem 5.1]. Hence, (14) is an equivalence as well. ☐

4. The complexes $R\Gamma_{ar}(\mathcal{X}_K,\mathbb{Z}(n))$ in $\mathbf{D}^b(\mathrm{LCA})$

Under the assumption that the pair $(\mathcal{X},n)$ satisfies Hypothesis 4.1 below, we give in Section 4.2 a construction of complexes in a fibre sequence in $\mathbf{D}^b(\mathrm{LCA})$

$$R\Gamma_{ar}(\mathcal{X}_s,Ri^!\mathbb{Z}(n)) \to R\Gamma_{ar}(\mathcal{X},\mathbb{Z}(n)) \to R\Gamma_{ar}(\mathcal{X}_K,\mathbb{Z}(n)), \tag{16}$$

where

$$R\Gamma_{ar}(\mathcal{X}_s,Ri^!\mathbb{Z}(n)) := R\Gamma_W(\mathcal{X}_s,Ri^!\mathbb{Z}(n))$$

is defined below. Hypothesis 4.1 is known for $n=0,1$ and arbitrary $\mathcal{X}$, hence, this construction is unconditional in those cases.

In Sections 4.3 and 4.4, we give an alternative definition of the triangle (16) for $n=0$ and $n=d:=\dim(\mathcal{X})$, respectively, which is expected to coincide with the conditional definition of Section 4.2.

In Section 4.5, we show that these complexes, in fact, belong to $\mathbf{D}^b(\mathrm{FLCA})$ under some conditions. In Section 4.6, we show that the cohomology of these complexes consists of locally compact abelian groups for $n=0,d$.

4.1. Notation

Let p be a prime number, let $K/\mathbb{Q}_p$ be a finite extension, let $\mathcal{O}_K$ be its ring of integers and let $\bar{K}/K$ be an algebraic closure. We denote by K^{un} the maximal unramified extension of K inside $\bar{K}$. Let $\mathcal{X}/\mathcal{O}_K$ be a regular, proper and flat scheme over $\mathrm{Spec}(\mathcal{O}_K)$. Suppose that $\mathcal{X}$ is connected of Krull dimension d. Let $\mathcal{X}_s$ be its special fibre, where $s \in \mathrm{Spec}(\mathcal{O}_K)$ is the closed point. We consider the following diagram.

$$\begin{tikzcd}
\mathcal{X}_{K^{un}} \arrow[r, "\bar{j}"] \arrow[d] & \mathcal{X}_{\mathcal{O}_{K^{un}}} \arrow[d] & \mathcal{X}_{\bar{s}} \arrow[l, "\bar{i}"'] \arrow[d] \\
\mathcal{X}_K \arrow[r, "j"] & \mathcal{X} & \mathcal{X}_s. \arrow[l, "i"']
\end{tikzcd}$$

For any $n \geq 0$, we denote by $\mathbb{Z}(n)$ Bloch's cycle complex in its cohomological notation considered as a complex of étale sheaves. For any $n<0$, we define

$$\mathbb{Z}(n) := \mathbb{Q}/\mathbb{Z}(n)[-1] := (\oplus_{l\neq p}\mathrm{colim}\,\mu_{l^\bullet}^{\otimes n} \oplus j_!\mathrm{colim}\mu_{p^\bullet}^{\otimes n})[-1].$$

For any $n \in \mathbb{Z}$, we set $\mathbb{Z}/m(n) := \mathbb{Z}(n) \otimes^{\mathbb{L}} \mathbb{Z}/m$. We denote by $G_{\kappa(s)} \simeq \widehat{\mathbb{Z}}$ and by $W_{\kappa(s)} \simeq \mathbb{Z}$ the Galois group and the Weil group of the finite field $\kappa(s)$, respectively. We define Weil-étale cohomology groups

$$\begin{aligned} R\Gamma_W(\mathcal{X}_K,\mathbb{Z}(n)) &:= R\Gamma(W_{\kappa(s)},R\Gamma_{et}(\mathcal{X}_{K^{un}},\mathbb{Z}(n))), \\ R\Gamma_W(\mathcal{X},\mathbb{Z}(n)) &:= R\Gamma(W_{\kappa(s)},R\Gamma_{et}(\mathcal{X}_{\mathcal{O}_{K^{un}}},\mathbb{Z}(n))), \\ R\Gamma_W(\mathcal{X}_s,Ri^!\mathbb{Z}(n)) &:= R\Gamma(W_{\kappa(s)},R\Gamma_{et}(\mathcal{X}_{\bar{s}},R\bar{i}^!\mathbb{Z}(n))). \end{aligned}$$

If one replaces $W_{\kappa(s)}$ by $G_{\kappa(s)}$, one obtains étale motivic cohomology $R\Gamma_{et}(\mathcal{X}_K,\mathbb{Z}(n))$, $R\Gamma_{et}(\mathcal{X},\mathbb{Z}(n))$ and $R\Gamma_{et}(\mathcal{X}_s,Ri^!\mathbb{Z}(n))$. Applying $R\Gamma(W_{\kappa(s)},-)$ to the fibre sequence

$$R\Gamma_{et}(\mathcal{X}_{\bar{s}},R\bar{i}^!\mathbb{Z}(n)) \to R\Gamma_{et}(\mathcal{X}_{\mathcal{O}_{K^{un}}},\mathbb{Z}(n)) \to R\Gamma_{et}(\mathcal{X}_{K^{un}},\mathbb{Z}(n)),$$

we obtain the fibre sequence

$$R\Gamma_W(\mathcal{X}_s,Ri^!\mathbb{Z}(n)) \to R\Gamma_W(\mathcal{X},\mathbb{Z}(n)) \to R\Gamma_W(\mathcal{X}_K,\mathbb{Z}(n)). \tag{17}$$

4.2. Uniform conditional definition

Recall from Definition 3.1 the eh-motivic cohomology $R\Gamma_{eh}(-,\mathbb{Z}(n))$ and Wh-motivic cohomology

$$R\Gamma_{Wh}(\mathcal{X}_s,\mathbb{Z}(n)) := R\Gamma(W_{\kappa(s)},R\Gamma_{eh}(\mathcal{X}_{\bar{s}},\mathbb{Z}(n))).$$

Hypothesis 4.1. *We have a reduction map*

$$\bar{i}^* : R\Gamma_{et}(\mathcal{X}_{\mathcal{O}_{K^{un}}},\mathbb{Z}(n)) \to R\Gamma_{eh}(\mathcal{X}_{\bar{s}},\mathbb{Z}(n)),$$

and the complexes $R\Gamma_{et}(\mathcal{X},\mathbb{Z}(n))$, $R\Gamma_{eh}(\mathcal{X}_s,\mathbb{Z}(n))$ and $R\Gamma_{et}(\mathcal{X}_s,Ri^!\mathbb{Z}(n))$ are cohomologically bounded.

Definition 4.2. Under hypothesis 4.1, we apply the functor $R\Gamma(W_{\kappa(s)},-)$ to the reduction map $\bar{i}^*$, and we obtain a map

$$R\Gamma_W(\mathcal{X},\mathbb{Z}(n)) \to R\Gamma_{Wh}(\mathcal{X}_s,\mathbb{Z}(n)). \tag{18}$$

We denote the cofibre of (18) by $C_W(\mathcal{X},n)$, so that we have a cofibre sequence

$$R\Gamma_W(\mathcal{X},\mathbb{Z}(n)) \to R\Gamma_{Wh}(\mathcal{X}_s,\mathbb{Z}(n)) \to C_W(\mathcal{X},n) \tag{19}$$

in $\mathbf{D}^b(\mathrm{Ab})$.

Proposition 4.3. *Assume Hypothesis 4.1. Then there exist $\mathbf{R\Gamma}_{ar}(\mathcal{X},\mathbb{Z}(n)) \in \mathbf{D}^b(\mathrm{LCA})$ and $\mathbf{R\Gamma}_{ar}(\mathcal{X}_K,\mathbb{Z}(n)) \in \mathbf{D}^b(\mathrm{LCA})$ endowed with fibre sequences*

$$\mathbf{R\Gamma}_{ar}(\mathcal{X},\mathbb{Z}(n)) \to R\Gamma_{Wh}(\mathcal{X}_s,\mathbb{Z}(n)) \to C_W(\mathcal{X},n)\underline{\widehat{\otimes}}\widehat{\mathbb{Z}} \tag{20}$$

and

$$R\Gamma_W(\mathcal{X}_s,Ri^!\mathbb{Z}(n)) \to \mathbf{R\Gamma}_{ar}(\mathcal{X},\mathbb{Z}(n)) \to \mathbf{R\Gamma}_{ar}(\mathcal{X}_K,\mathbb{Z}(n))$$

in $\mathbf{D}^b(\mathrm{LCA})$.

Proof. Composing the morphism in $\mathbf{D}^b(\mathrm{Ab})$

$$R\Gamma_{Wh}(\mathcal{X}_s,\mathbb{Z}(n)) \to C_W(\mathcal{X},n)$$

and the morphism in $\mathbf{D}^b(\mathrm{LCA})$

$$C_W(\mathcal{X},n) \to C_W(\mathcal{X},n)\underline{\widehat{\otimes}}\widehat{\mathbb{Z}}$$

given by Lemma 2.14, we obtain a morphism in $\mathbf{D}^b(\mathrm{LCA})$

$$R\Gamma_{Wh}(\mathcal{X}_s,\mathbb{Z}(n)) \to C_W(\mathcal{X},n)\underline{\widehat{\otimes}}\widehat{\mathbb{Z}}. \tag{21}$$

We define $\mathbf{R\Gamma}_{ar}(\mathcal{X},\mathbb{Z}(n))$ as the fibre of (21), and we obtain the fibre sequence (20) in $\mathbf{D}^b(\mathrm{LCA})$. Lemma 2.14 gives a map from (19) to (20), hence, a map

$$R\Gamma_W(\mathcal{X},\mathbb{Z}(n)) \to \mathbf{R\Gamma}_{ar}(\mathcal{X},\mathbb{Z}(n)).$$

Then we define $\mathbf{R\Gamma}_{ar}(\mathcal{X}_K,\mathbb{Z}(n)) \in \mathbf{D}^b(\mathrm{LCA})$ as the cofibre of the composite map

$$R\Gamma_W(\mathcal{X}_s,Ri^!\mathbb{Z}(n)) \to R\Gamma_W(\mathcal{X},\mathbb{Z}(n)) \to \mathbf{R\Gamma}_{ar}(\mathcal{X},\mathbb{Z}(n)).$$

□

Remark 4.4. Since $\mathbb{Z}(0)\cong\mathbb{Z}$ and $\mathbb{Z}(1)\cong\mathbb{G}_m[-1]$, Hypothesis 4.1 holds for $n=0$ and $n=1$, so that $\mathbf{R\Gamma}_{ar}(\mathcal{X},\mathbb{Z}(n))$ and $\mathbf{R\Gamma}_{ar}(\mathcal{X}_K,\mathbb{Z}(n))$ are unconditionally defined in these cases.

4.3. Working definition for the Tate twist $n=0$.

We assume that $\mathcal{X}_s^{\mathrm{red}}$ is a simple normal crossing scheme. To obtain unconditional definitions for $n=0$, we replace $R\Gamma_{Wh}(\mathcal{X}_s,\mathbb{Z})$ by $R\Gamma_W(\mathcal{X}_s,\mathbb{Z})$ in the construction of Section 4.2. In view of Corollary 3.5, this will agree with the definition of Section 4.2 provided that resolution of singularities for schemes of dimension, at most, $\dim(\mathcal{X}_s)$ exist.

There is a canonical map

$$R\Gamma_W(\mathcal{X},\mathbb{Z}) \to R\Gamma_W(\mathcal{X}_s,\mathbb{Z}),$$

whose cofibre we again denote by $C_W(\mathcal{X},0)$. Following the construction of Section 4.2, we define $R\Gamma_{ar}(\mathcal{X},\mathbb{Z})\in\mathbf{D}^b(\mathrm{LCA})$ and $R\Gamma_{ar}(\mathcal{X}_K,\mathbb{Z})\in\mathbf{D}^b(\mathrm{LCA})$ endowed with fibre sequences

$$R\Gamma_{ar}(\mathcal{X},\mathbb{Z}) \to R\Gamma_W(\mathcal{X}_s,\mathbb{Z}) \to C_W(\mathcal{X},0)\underline{\widehat{\otimes}}\widehat{\mathbb{Z}}$$

and

$$R\Gamma_W(\mathcal{X}_s,Ri^!\mathbb{Z}) \to R\Gamma_{ar}(\mathcal{X},\mathbb{Z}) \to R\Gamma_{ar}(\mathcal{X}_K,\mathbb{Z})$$

in $\mathbf{D}^b(\mathrm{LCA})$. We used bold letters for the complexes defined in Section 4.2 in order to distinguish them from the complexes defined in this section.

Proposition 4.5. *If $\mathcal{X}_s^{\mathrm{red}}$ is a simple normal crossing scheme, then the map*

$$R\Gamma_{ar}(\mathcal{X},\mathbb{Z}) \to R\Gamma_W(\mathcal{X}_s,\mathbb{Z})$$

is an equivalence. For arbitrary $\mathcal{X}$, the map

$$\mathbf{R\Gamma}_{ar}(\mathcal{X},\mathbb{Z}) \to R\Gamma_{Wh}(\mathcal{X}_s,\mathbb{Z})$$

is an equivalence.

Proof. By proper base change, the map

$$R\Gamma_{et}(\mathcal{X},\mathbb{Z}/m) \to R\Gamma_{et}(\mathcal{X}_s,\mathbb{Z}/m)$$

is an equivalence, hence, $C_W(\mathcal{X},0)\otimes^{\mathbb{L}}\mathbb{Z}/m \simeq 0$. We obtain $C_W(\mathcal{X},0)\widehat{\underline{\otimes}}\widehat{\mathbb{Z}} \simeq 0$. The first equivalence of the proposition follows. The second equivalence is obtained the same way, in view of the fact that

$$R\Gamma_{et}(\mathcal{X},\mathbb{Z}/m) \to R\Gamma_{et}(\mathcal{X}_s,\mathbb{Z}/m) \to R\Gamma_{eh}(\mathcal{X}_s,\mathbb{Z}/m)$$

is an equivalence, again, by proper base change. ☐

Proposition 4.6. *If $\mathcal{X}_s^{\mathrm{red}}$ is a simple normal crossing scheme, then there is a canonical map of fibre sequences*

$$\begin{tikzcd}
R\Gamma_W(\mathcal{X}_s,Ri^!\mathbb{Z}) \arrow[r] \arrow[d, equal] & R\Gamma_{ar}(\mathcal{X},\mathbb{Z}) \arrow[r] \arrow[d] & R\Gamma_{ar}(\mathcal{X}_K,\mathbb{Z}) \arrow[d] \\
R\Gamma_W(\mathcal{X}_s,Ri^!\mathbb{Z}) \arrow[r] & \mathbf{R\Gamma}_{ar}(\mathcal{X},\mathbb{Z}) \arrow[r] & \mathbf{R\Gamma}_{ar}(\mathcal{X}_K,\mathbb{Z}).
\end{tikzcd}$$

If resolution of singularities for schemes over $\kappa(s)$ of dimension, at most, $d-1$ [8, Definition 2.4] exists, then this morphism of fibre sequences is an equivalence.

Proof. This follows from Corollary 3.5 and Proposition 4.5. ☐

In particular, we obtain that

$$R\Gamma_{ar}(\mathcal{X},\mathbb{Z}) \xrightarrow{\sim} \mathbf{R\Gamma}_{ar}(\mathcal{X},\mathbb{Z})$$

is an equivalence if $d \leq 3$.

Notation 4.7. If $\mathcal{X}_s^{\mathrm{red}}$ is a simple normal crossing scheme, we denote by $R\Gamma_{ar}(\mathcal{X},\mathbb{Z})$ and $R\Gamma_{ar}(\mathcal{X}_K,\mathbb{Z})$ the complexes defined above. In view of Proposition 4.6, we set $R\Gamma_{ar}(\mathcal{X},\mathbb{Z}) := \mathbf{R\Gamma}_{ar}(\mathcal{X},\mathbb{Z})$ and $R\Gamma_{ar}(\mathcal{X}_K,\mathbb{Z}) := \mathbf{R\Gamma}_{ar}(\mathcal{X}_K,\mathbb{Z})$ for arbitrary regular $\mathcal{X}$ of dimension, at most, 3 or when we are assuming resolution of singularities.

4.4. Working definition for the Tate twist $n = d$.

The complex $R\Gamma_W(\mathcal{X},\mathbb{Z}(d))$ is not known to be bounded below. However, the complex

$$R\Gamma_W(\mathcal{X},\mathbb{Q}/\mathbb{Z}(d)) \simeq R\Gamma_{et}(\mathcal{X},\mathbb{Q}/\mathbb{Z}(d))$$

is bounded, as can be seen by duality, hence, the cohomology groups $H^i_W(\mathcal{X},\mathbb{Z}(d))$ are $\mathbb{Q}$-vector spaces for $i \ll 0$. In particular, for $a < b \ll 0$, the map

$$\tau^{>a}R\Gamma_W(\mathcal{X},\mathbb{Z}(d)) \to \tau^{>b}R\Gamma_W(\mathcal{X},\mathbb{Z}(d))$$

induces an equivalence

$$(\tau^{>a}R\Gamma_W(\mathcal{X},\mathbb{Z}(d)))\underline{\widehat{\otimes}}\widehat{\mathbb{Z}} \xrightarrow{\sim} (\tau^{>b}R\Gamma_W(\mathcal{X},\mathbb{Z}(d)))\underline{\widehat{\otimes}}\widehat{\mathbb{Z}}.$$

Definition 4.8. Let $a \ll 0$. We define

$$R\Gamma_{ar}(\mathcal{X},\mathbb{Z}(d)) := (\tau^{>a}R\Gamma_W(\mathcal{X},\mathbb{Z}(d)))\underline{\widehat{\otimes}}\widehat{\mathbb{Z}}.$$

If $R\Gamma_W(\mathcal{X}_s,\mathbb{Z}^c(0))$ is cohomologically bounded, we define $R\Gamma_{ar}(\mathcal{X}_K,\mathbb{Z}(d))$ as the cofibre of the composite map

$$R\Gamma_W(\mathcal{X}_s,Ri^!\mathbb{Z}(d)) \to \tau^{>a}R\Gamma_W(\mathcal{X},\mathbb{Z}(d)) \to R\Gamma_{ar}(\mathcal{X},\mathbb{Z}(d))$$

in $\mathbf{D}^b(\mathrm{LCA})$.

Remark 4.9. On the connected, d-dimensional and regular scheme $\mathcal{X}$, we have $\mathbb{Z}(d)^{\mathcal{X}} = \mathbb{Z}^c(0)^{\mathcal{X}}[-2d]$ by definition. By [9, Corollary 7.2], we have $Ri^!\mathbb{Z}^c(0)^{\mathcal{X}} = \mathbb{Z}^c(0)^{\mathcal{X}_s}$, hence, $Ri^!\mathbb{Z}(d)^{\mathcal{X}} = \mathbb{Z}^c(0)^{\mathcal{X}_s}[-2d]$.

Proposition 4.10. *Suppose that $\mathcal{X}$ satisfies Hypothesis 4.1 for $n = d$, and suppose that $R\Gamma_W(\mathcal{X}_K,\mathbb{Z}(d))$ and $R\Gamma_W(\mathcal{X}_s,\mathbb{Z}^c(0))$ are cohomologically bounded. Then there is a canonical map of fibre sequences*

$$\begin{tikzcd}
R\Gamma_W(\mathcal{X}_s,Ri^!\mathbb{Z}(d)) \arrow[r] \arrow[d, equal] & \mathbf{R\Gamma}_{ar}(\mathcal{X},\mathbb{Z}(d)) \arrow[r] \arrow[d] & \mathbf{R\Gamma}_{ar}(\mathcal{X}_K,\mathbb{Z}(d)) \arrow[d] \\
R\Gamma_W(\mathcal{X}_s,Ri^!\mathbb{Z}(d)) \arrow[r] & R\Gamma_{ar}(\mathcal{X},\mathbb{Z}(d)) \arrow[r] & R\Gamma_{ar}(\mathcal{X}_K,\mathbb{Z}(d)).
\end{tikzcd}$$

If $R\Gamma_{Wh}(\mathcal{X}_s,\mathbb{Z}(d))$ has finite cohomology groups, then this morphism of fibre sequences is an equivalence.

Cohomological boundedness of $R\Gamma_W(\mathcal{X}_K,\mathbb{Z}(d))$ in negative degrees is a special case of the Beilinson-Soulé conjecture stating that there is no negative motivic cohomology, and in positive degrees, it follows for finite cohomological dimension reasons (see Propositions 3.6 and 3.7 for the other boundedness conditions).

Proof. If $R\Gamma_W(\mathcal{X}_K,\mathbb{Z}(d))$ and $R\Gamma_W(\mathcal{X}_s,Ri^!\mathbb{Z}(d)) \simeq R\Gamma_W(\mathcal{X}_s,\mathbb{Z}^c(0))[-2d]$ are cohomologically bounded, then the same holds for $R\Gamma_W(\mathcal{X},\mathbb{Z}(d))$ by the localisation triangle

$$\cdots \to R\Gamma_W(\mathcal{X}_s,Ri^!\mathbb{Z}(d)) \to R\Gamma_W(\mathcal{X},\mathbb{Z}(d)) \to R\Gamma_W(\mathcal{X}_K,\mathbb{Z}(d)) \to \cdots.$$

In this case, $R\Gamma_W(\mathcal{X},\mathbb{Z}(d)) \xrightarrow{\sim} \tau^{>a}R\Gamma_W(\mathcal{X},\mathbb{Z}(d))$ is an equivalence for $a << 0$, hence,

$$R\Gamma_{ar}(\mathcal{X},\mathbb{Z}(d)) \simeq R\Gamma_W(\mathcal{X},\mathbb{Z}(d))\underline{\widehat{\otimes}}\widehat{\mathbb{Z}}.$$

In view of (19) and (20), we obtain a map of fibre sequences

$$\begin{tikzcd}
\mathbf{R\Gamma}_{ar}(\mathcal{X},\mathbb{Z}(d)) \arrow[r] \arrow[d] & R\Gamma_{Wh}(\mathcal{X}_s,\mathbb{Z}(d)) \arrow[r] \arrow[d] & C_W(\mathcal{X},d)\underline{\widehat{\otimes}}\widehat{\mathbb{Z}} \arrow[d, equal] \\
R\Gamma_{ar}(\mathcal{X},\mathbb{Z}(d)) \arrow[r] & R\Gamma_{Wh}(\mathcal{X}_s,\mathbb{Z}(d))\underline{\widehat{\otimes}}\widehat{\mathbb{Z}} \arrow[r] & C_W(\mathcal{X},d)\underline{\widehat{\otimes}}\widehat{\mathbb{Z}}
\end{tikzcd}$$

since $(-)\underline{\widehat{\otimes}}\widehat{\mathbb{Z}}$ is an exact functor. If the cohomology of $R\Gamma_{Wh}(\mathcal{X}_s,\mathbb{Z}(d))$ consists of finite groups, then the middle vertical map

$$R\Gamma_{Wh}(\mathcal{X}_s,\mathbb{Z}(d)) \to R\Gamma_{Wh}(\mathcal{X}_s,\mathbb{Z}(d))\underline{\widehat{\otimes}}\widehat{\mathbb{Z}}$$

in the above diagram is an equivalence by Proposition 2.11 and the two fibre sequences are equivalent. □

Remark 4.11. It follows from Proposition 2.11 and Remark 2.12 that we have

$$\begin{aligned}\operatorname{disc}(R\Gamma_{ar}(\mathcal{X},\mathbb{Z}(d))) &\simeq R\lim\left(\tau^{>a}R\Gamma_W(\mathcal{X},\mathbb{Z}(d))\otimes^{\mathbb{L}}\mathbb{Z}/m\right)\\ &\simeq R\lim R\Gamma_W(\mathcal{X},\mathbb{Z}/m(d))\\ &\simeq R\lim R\Gamma_{et}(\mathcal{X},\mathbb{Z}/m(d))\\ &=: R\Gamma_{et}(\mathcal{X},\widehat{\mathbb{Z}}(d)),\end{aligned}$$

where we denote $\mathbb{Z}/m(d) := \mathbb{Z}(d)\otimes^{\mathbb{L}}\mathbb{Z}/m$.

4.5. Finite ranks

Lemma 4.12. *If $\mathcal{X}$ is normal, then we have an isomorphism*

$$H^j_{et}(\mathcal{X}_s,Ri^!\mathbb{Q}/\mathbb{Z}) \cong H^{j+1}_{et}(\mathcal{X}_s,Ri^!\mathbb{Z})$$

of abelian groups of finite ranks for all $j\in\mathbb{Z}$.

Proof. The isomorphism follows because normality of $\mathcal{X}$ implies that $\mathbb{Q}\cong Rj_*j^*\mathbb{Q}$, hence, $Ri^!\mathbb{Q}\cong 0$. Since $H^j_{et}(\mathcal{X}_s,Ri^!\mathbb{Q}/\mathbb{Z})$ is torsion and discrete, it is both of finite $\mathbb{Z}$-rank and of finite $\mathbb{S}^1$-rank. It remains to see that it is of finite p-rank for any prime number p. But $H^j_{et}(\mathcal{X}_s,Ri^!\mathbb{Z}/p\mathbb{Z})$ is a finite group for any $j\in\mathbb{Z}$, because of the fibre sequence

$$R\Gamma_{et}(\mathcal{X}_s,Ri^!\mathbb{Z}/p\mathbb{Z}) \to R\Gamma_{et}(\mathcal{X},\mathbb{Z}/p\mathbb{Z}) \to R\Gamma_{et}(\mathcal{X}_K,\mathbb{Z}/p\mathbb{Z}),$$

and classical finiteness results in étale and Galois cohomology. □

Proposition 4.13. a) *Assume that $\mathcal{X}_s^{\mathrm{red}}$ is a simple normal crossing scheme, or assume resolution of singularities for schemes over $\kappa(s)$ of dimension, at most, $d-1$ [8, Definition 2.4]. Then $R\Gamma_{ar}(\mathcal{X},\mathbb{Z})$ and $R\Gamma_{ar}(\mathcal{X}_K,\mathbb{Z})$ belong to $\mathbf{D}^b(\mathrm{FLCA})$.*

b) *Assume that $R\Gamma_W(\mathcal{X}_s,\mathbb{Z}^c(0))$ is a perfect complex of abelian groups. Then $R\Gamma_{ar}(\mathcal{X},\mathbb{Z}(d))$ and $R\Gamma_{ar}(\mathcal{X}_K,\mathbb{Z}(d))$ belong to $\mathbf{D}^b(\mathrm{FLCA})$.*

Proof. a) Under the hypothesis, the complexes $R\Gamma_W(\mathcal{X}_s,\mathbb{Z})$ and $R\Gamma_{Wh}(\mathcal{X}_s,\mathbb{Z})$ are perfect complexes of abelian groups by Propositions 3.2 and 3.5, respectively, hence, they belong to $\mathbf{D}^b(\mathrm{FLCA})$ by Lemma 2.5. The result for $R\Gamma_{ar}(\mathcal{X},\mathbb{Z})$ then follows from Proposition 4.5 (using Notation 4.7), and the result for $R\Gamma_{ar}(\mathcal{X}_K,\mathbb{Z})$ follows from Proposition 4.6 and Lemma 4.12.

b) By the proof of Proposition 5.4, $R\Gamma_{ar}(\mathcal{X},\mathbb{Z}(d))$ is (up to a shift) dual to $R\Gamma_{et}(\mathcal{X}_s,Ri^!\mathbb{Q}/\mathbb{Z})$. Hence, the result follows from Lemmas 2.5 and 4.12. The statement for $R\Gamma_{ar}(\mathcal{X}_K,\mathbb{Z}(d))$ follows from the statement for $R\Gamma_{ar}(\mathcal{X},\mathbb{Z}(d))$ together with the perfectness of $R\Gamma_W(\mathcal{X}_s,Ri^!\mathbb{Z}(d))\simeq R\Gamma_W(\mathcal{X}_s,\mathbb{Z}^c(0))[-2d]$ by hypothesis. □

4.6. The topology on cohomology groups

Recall from Section 4.1 that $\mathcal{X}$ denotes a regular connected scheme which is proper and flat over $\mathcal{O}_K$. We refer to [1, Section 2.16] for the following definition.

Definition 4.14. Let $\mathcal{X}_{s,i}, i \in I$ be the irreducible components of $\mathcal{X}_s$. We set $\mathcal{X}_{s,J} = \cap_{i\in J}\mathcal{X}_{s,i}$ for any nonempty subset $J \subseteq I$. We say that $\mathcal{X}/\mathcal{O}_K$ has *strictly semistable reduction* if $\mathcal{X}_s$ is reduced, $\mathcal{X}_{s,i}$ is a divisor on $\mathcal{X}$ and for each nonempty $J \subseteq I$, the scheme $\mathcal{X}_{s,J}$ is smooth over $\kappa(s)$ and has codimension $|J|$ in $\mathcal{X}$.

If $\mathcal{X}/\mathcal{O}_K$ has strictly semistable reduction, then $\mathcal{X}_s$ is a simple normal crossing scheme over $\kappa(s)$, in the sense of Definition 3.3.

Theorem 4.15. *Suppose that $\mathcal{X}/\mathcal{O}_K$ has strictly semistable reduction. Then for any $i \in \mathbb{Z}$, the map*

$$H^i_{et}(\mathcal{X},\mathbb{Q}_p(d)) \to H^i_{et}(\mathcal{X}_K,\mathbb{Q}_p(d)) \tag{22}$$

is injective.

Proof. Since $\mathcal{X}/\mathcal{O}_K$ has strictly semistable reduction, the morphism $\mathcal{X} \to \mathrm{Spec}(\mathcal{O}_K)$ is log smooth with respect to the log structures associated with $\mathcal{X}_s$ and s, respectively, where s is the closed point of $\mathrm{Spec}(\mathcal{O}_K)$, and $\mathcal{X}_s$ is a normal crossing divisor on $\mathcal{X}$. Therefore, the results of [22] apply. We have isomorphisms

$$H^i_{et}(\mathcal{X},\mathbb{Q}_p(d)) \simeq H^i_{et}(\mathcal{X},\mathbb{Q}_p^S(d))$$

compatible with the map (22), where

$$R\Gamma_{et}(\mathcal{X},\mathbb{Q}_p^S(d)) := R\lim R\Gamma_{et}(\mathcal{X},\mathfrak{T}_r(d)) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$$

is the complex studied in [21] and [22]. Indeed, this follows from the equivalences

$$\begin{aligned} R\Gamma(\mathcal{X}_{et},\mathfrak{T}_r(d)) &\simeq R\mathrm{Hom}(R\Gamma_{\mathcal{X}_s}(\mathcal{X}_{et},\mathbb{Z}/p^r),\mathbb{Z}/p^r)[-2d-1] \\ &\simeq R\Gamma(\mathcal{X}_{et},\mathbb{Z}/p^r(d)), \end{aligned}$$

given by [21, Theorem 10.1.1] and [9, Proof of Theorem 7.5] and from the fact that (22) is induced by the dual of the map

$$R\Gamma(\mathcal{X}_{K,et},\mathbb{Z}/p^r)[-1] \to R\Gamma_{\mathcal{X}_s}(\mathcal{X}_{et},\mathbb{Z}/p^r).$$

Hence, we are reduced to show that the map

$$H^i_{et}(\mathcal{X},\mathbb{Q}_p^S(d)) \to H^i_{et}(\mathcal{X}_K,\mathbb{Q}_p(d))$$

is injective. By [22, Proposition 3.4(1)], [22, Section 4.1] and [22, Theorem 5.3], there is a morphism of spectral sequences from

$$H^i_f(G_K,H^j_{et}(\mathcal{X}_{\bar{K}},\mathbb{Q}_p(d))) \Rightarrow H^{i+j}_{et}(\mathcal{X},\mathbb{Q}_p^S(d))$$

to

$$H^i(G_K,H^j_{et}(\mathcal{X}_{\bar{K}},\mathbb{Q}_p(d))) \Rightarrow H^{i+j}_{et}(\mathcal{X}_K,\mathbb{Q}_p(d)),$$

where the first spectral sequence degenerates into isomorphisms

$$H^j_{et}(\mathcal{X},\mathbb{Q}^S_p(d)) \xrightarrow{\sim} H^1_f(G_K,H^{j-1}_{et}(\mathcal{X}_{\bar{K}},\mathbb{Q}_p(d))).$$

Since we have [22, Proposition 5.10(1)]

$$H^0_f(G_K,H^j_{et}(\mathcal{X}_{\bar{K}},\mathbb{Q}_p(d))) = H^0(G_K,H^j_{et}(\mathcal{X}_{\bar{K}},\mathbb{Q}_p(d))) = 0$$

for any $j \in \mathbb{Z}$, we obtain a commutative square

$$\begin{tikzcd}
H^j_{et}(\mathcal{X},\mathbb{Q}^S_p(d)) \arrow[r] \arrow[d, "\simeq"] & H^j_{et}(\mathcal{X}_K,\mathbb{Q}_p(d)) \arrow[d] \\
H^1_f(G_K,H^{j-1}_{et}(\mathcal{X}_{\bar{K}},\mathbb{Q}(d))) \arrow[r] & H^1(G_K,H^{j-1}_{et}(\mathcal{X}_{\bar{K}},\mathbb{Q}_p(d))),
\end{tikzcd}$$

where the vertical maps are edge morphisms of the corresponding spectral sequences. Here, the left vertical map is an isomorphism and the lower horizontal map is injective. It follows that the upper horizontal map is injective as well. □

Lemma 4.16. *Suppose that $\mathcal{X}_s$ is a simple normal crossing scheme. Then the group*

$$H^i_{et}(\mathcal{X},\widehat{\mathbb{Z}}'(d)) := H^i(R\mathrm{lim}_{p\nmid m}\, R\Gamma_{et}(\mathcal{X},\mathbb{Z}(d)) \otimes^{\mathbb{L}} \mathbb{Z}/m)$$

is finite for any $i \in \mathbb{Z}$.

Proof. By definition, we have $\mathbb{Z}(d) \cong \mathbb{Z}^c(0)[-2d]$, and by [9, Proposition 7.10 a)] and Gabber's purity theorem [4, Section 8], we have

$$\mathbb{Z}^c/m(0)[-2d] \cong Rf^!\mathbb{Z}^c/m(0)[-2d] \cong Rf^!\mathbb{Z}/m[-2d] \cong \mu_m^{\otimes d}$$

on $\mathcal{X}$ for any m prime to p. Moreover, by the proper base change theorem

$$R\Gamma_{et}(\mathcal{X},\mu_m^{\otimes d}) \simeq R\Gamma_{et}(\mathcal{X}_s,\mu_m^{\otimes d}).$$

Thus, it suffices to show that the cohomology of the right-hand side of

$$R\Gamma_{et}(\mathcal{X},\widehat{\mathbb{Z}}'(d)) \simeq R\mathrm{lim}_{p\nmid m}\, R\Gamma_{et}(\mathcal{X}_s,\mu_m^{\otimes d})$$

if finite. By the analog of Proposition 3.4 and induction on the number of irreducible components of $\mathcal{X}_s$, it suffices to show that the cohomology $R\Gamma_{et}(Y,\mathbb{Z}_l(d))$ of each connected component Y of each $\mathcal{X}_s^{(i)}$ is finite for all $l \neq p$ and zero for almost all l. Since Y is smooth and proper, this is known for the extension $\bar{Y}$ to the algebraic closure by Gabber's theorem [5], [24], and this extends to Y by a weight argument because $d > \dim Y$, hence, the Frobenius does not have eigenvalue one on $R\Gamma_{et}(Y,\mathbb{Z}_l(d))$. □

Theorem 4.17. a) *Suppose that $(\mathcal{X}_s)^{\mathrm{red}}$ is a simple normal crossing scheme. Then for any $i \in \mathbb{Z}$, the object $H^i_{ar}(\mathcal{X}_K,\mathbb{Z}) \in \mathcal{LH}(\mathrm{LCA})$ is a discrete abelian group. More precisely, $H^j_{ar}(\mathcal{X}_K,\mathbb{Z}) \in \mathcal{LH}(\mathrm{LCA})$ is an extension of a torsion abelian group by a finitely generated abelian group.*

b) *Suppose that $\mathcal{X}/\mathcal{O}_K$ has strictly semistable reduction, and suppose that $R\Gamma_W(\mathcal{X}_s,\mathbb{Z}^c(0))$ is a perfect complex of abelian groups. Then for any $i \in \mathbb{Z}$, the object*

$H^i_{ar}(\mathcal{X}_K,\mathbb{Z}(d))$ is a locally compact abelian group. More precisely, $H^i_{ar}(\mathcal{X}_K,\mathbb{Z}(d))$ is an extension of a finitely generated abelian group by a finitely generated $\mathbb{Z}_p$-module endowed with the p-adic topology.

Proof. a) We have a long exact sequence in the abelian category $\mathcal{LH}(\mathrm{LCA})$

$$\cdots \to H^j_{ar}(\mathcal{X},\mathbb{Z}) \to H^j_{ar}(\mathcal{X}_K,\mathbb{Z}) \to H^{j+1}_W(\mathcal{X}_s,Ri^!\mathbb{Z}) \to \cdots,$$

where $H^{j+1}_W(\mathcal{X}_s,Ri^!\mathbb{Z}) \simeq H^j_W(\mathcal{X}_s,Ri^!\mathbb{Q}/\mathbb{Z})$ is a discrete torsion abelian group (see the proof of Proposition 5.4) and $H^j_{ar}(\mathcal{X},\mathbb{Z}) \simeq H^j_W(\mathcal{X}_s,\mathbb{Z})$ is a discrete finitely generated abelian group by Proposition 3.5. Hence, $H^j_{ar}(\mathcal{X}_K,\mathbb{Z}) \in \mathcal{LH}(\mathrm{LCA})$ is an extension of a torsion abelian group by a finitely generated abelian group. It follows that $H^j_{ar}(\mathcal{X}_K,\mathbb{Z}) \in \mathrm{LCA}$ since $\mathrm{LCA} \subset \mathcal{LH}(\mathrm{LCA})$ is stable under extensions [23, Proposition 1.2.29(c)].

b) We have a long exact sequence in $\mathcal{LH}(\mathrm{LCA})$

$$H^j_W(\mathcal{X}_s,Ri^!\mathbb{Z}(d)) \to H^j_{ar}(\mathcal{X},\mathbb{Z}(d)) \to H^j_{ar}(\mathcal{X}_K,\mathbb{Z}(d)) \to H^{j+1}_W(\mathcal{X}_s,Ri^!\mathbb{Z}(d)),$$

where $H^j_W(\mathcal{X}_s,Ri^!\mathbb{Z}(d))$ is a discrete finitely generated abelian group by assumption. Moreover, $H^j_{ar}(\mathcal{X},\mathbb{Z}(d)) \in \mathcal{LH}(\mathrm{LCA})$ is the group (see Remark 4.11)

$$H^j_{et}(\mathcal{X},\widehat{\mathbb{Z}}(d)) \simeq \prod_l H^j_{et}(\mathcal{X},\mathbb{Z}_l(d)) := \prod_l H^j(R\lim(R\Gamma_{et}(\mathcal{X},\mathbb{Z}(d)) \otimes^{\mathbb{L}} \mathbb{Z}/l^\bullet))$$

which by Lemma 4.16 is the product of a finite group and the finitely generated $\mathbb{Z}_p$-module $H^j_{et}(\mathcal{X},\mathbb{Z}_p(d))$ endowed with the p-adic topology. If we can show that the image of the map $H^j_W(\mathcal{X}_s,Ri^!\mathbb{Z}(d)) \to H^j_{et}(\mathcal{X},\widehat{\mathbb{Z}}(d))$ is finite, then it will follow that $H^j_{ar}(\mathcal{X}_K,\mathbb{Z}(d))$ is an extension of a finitely generated abelian group by a profinite abelian group. Since we have an isomorphism of finitely generated $\mathbb{Z}_p$-modules

$$H^j_W(\mathcal{X}_s,Ri^!\mathbb{Z}(d)) \otimes_{\mathbb{Z}} \mathbb{Z}_p \simeq H^j_{et}(\mathcal{X}_s,Ri^!\mathbb{Z}_p(d)),$$

it is enough to show that the image of the map

$$H^j_{et}(\mathcal{X}_s,Ri^!\mathbb{Z}_p(d)) \to H^j_{et}(\mathcal{X},\mathbb{Z}_p(d))$$

is finite, or equivalently, that the map

$$H^j_{et}(\mathcal{X}_s,Ri^!\mathbb{Q}_p(d)) \to H^j_{et}(\mathcal{X},\mathbb{Q}_p(d))$$

is the zero map. This follows from Theorem 4.15 by the localisation sequence. □

5. Duality theorems

The goal of this section is to prove various duality theorems. In particular, we prove Theorem 1.2 and Corollary 1.3 of the Introduction. Throughout this section, we use the notation and definitions introduced in Sections 4.3 and 4.4 and we assume the following,

Hypothesis 5.1. *At least one of the following conditions holds:*

- *we have $d \leq 2$;*
- *the scheme $(\mathcal{X}_s)^{\mathrm{red}}$ is a simple normal crossing scheme and $R\Gamma_W(\mathcal{X}_s,\mathbb{Z}^c(0))$ is a perfect complex of abelian groups.*

In view of Proposition 3.6, $d \leq 2$ implies that $R\Gamma_W(\mathcal{X}_s,\mathbb{Z}^c(0))$ is a perfect complex of abelian groups.

5.1. Duality with $\mathbb{Z}$-coefficients

Theorem 5.2. *Assume Hypothesis 5.1. Then there is a perfect pairing*

$$R\Gamma_{ar}(\mathcal{X}_K,\mathbb{Z}(d))\underline{\otimes}^{\mathbb{L}}R\Gamma_{ar}(\mathcal{X}_K,\mathbb{Z}) \longrightarrow \mathbb{Z}[-2d]$$

in $\mathbf{D}^b(\mathrm{FLCA})$.

The rest of Section 5.1 is devoted to the proof of Theorem 5.2. We assume Hypothesis 5.1 throughout.

Proof. Recall from Proposition 4.13 that $R\Gamma_{ar}(\mathcal{X}_K,\mathbb{Z}(n))$ belongs to $\mathbf{D}^b(\mathrm{FLCA})$ for $n = 0,d$, so that the tensor product

$$R\Gamma_{ar}(\mathcal{X}_K,\mathbb{Z}(d))\underline{\otimes}^{\mathbb{L}}R\Gamma_{ar}(\mathcal{X}_K,\mathbb{Z}),$$

defined in Section 2, makes sense. Moreover, the equivalence $Ri^!\mathbb{Z}^c(0)^{\mathcal{X}} \simeq \mathbb{Z}^c(0)^{\mathcal{X}_s}$ of Remark 4.9 and the pushforward map $Rf_*\mathbb{Z}^c(0)^{\mathcal{X}_s} \to \mathbb{Z}^c(0)^s \simeq \mathbb{Z}[0]$ of [9, Corollary 3.2] induce trace maps

$$R\Gamma_W(\mathcal{X}_s,Ri^!\mathbb{Z}(d)) \simeq R\Gamma_W(\mathcal{X}_s,\mathbb{Z}^c(0)[-2d]) \to R\Gamma_W(s,\mathbb{Z}[-2d]) \to \mathbb{Z}[-2d-1]$$

and

$$R\Gamma_{et}(\mathcal{X}_s,Ri^!\mathbb{Z}/m(d)) \to R\Gamma_{et}(s,\mathbb{Z}/m[-2d]) \to \mathbb{Z}/m[-2d-1].$$

We start with the following,

Proposition 5.3. *The canonical product map* $\mathbb{Z}\otimes^{\mathbb{L}}\mathbb{Z}(d) \to \mathbb{Z}(d)$ *in the derived* ∞-*category of étale sheaves over* $\mathcal{X}_{\mathcal{O}_{K^{un}}}$ *and* $\mathcal{X}_{K^{un}}$ *induce perfect pairings*

$$R\Gamma_{et}(\mathcal{X}_s,Ri^!\mathbb{Z}/m)\otimes^{\mathbb{L}}R\Gamma_{et}(\mathcal{X},\mathbb{Z}/m(d)) \to R\Gamma_{et}(\mathcal{X}_s,Ri^!\mathbb{Z}/m(d)) \to \mathbb{Z}/m[-2d-1]$$

and

$$R\Gamma_{et}(\mathcal{X}_s,Ri^!\mathbb{Z}/m(d))\otimes^{\mathbb{L}}R\Gamma_{et}(\mathcal{X},\mathbb{Z}/m) \to R\Gamma_{et}(\mathcal{X}_s,Ri^!\mathbb{Z}/m(d)) \to \mathbb{Z}/m[-2d-1]$$

for any m.

Proof. Consider the commutative square

$$\begin{tikzcd}
R\Gamma_{et}(\mathcal{X},\mathbb{Z})\otimes^{\mathbb{L}}R\Gamma_{et}(\mathcal{X},\mathbb{Z}(d)) \arrow[r] \arrow[d] & R\Gamma_{et}(\mathcal{X},\mathbb{Z}(d)) \arrow[d] \\
R\Gamma_{et}(\mathcal{X}_K,\mathbb{Z})\otimes^{\mathbb{L}}R\Gamma_{et}(\mathcal{X},\mathbb{Z}(d)) \arrow[r] & R\Gamma_{et}(\mathcal{X}_K,\mathbb{Z}(d)).
\end{tikzcd}$$

Taking the fibres of the vertical arrows induces the product map

$$R\Gamma_{et}(\mathcal{X}_s,Ri^!\mathbb{Z})\otimes^{\mathbb{L}}R\Gamma_{et}(\mathcal{X},\mathbb{Z}(d)) \to R\Gamma_{et}(\mathcal{X}_s,Ri^!\mathbb{Z}(d)), \tag{23}$$

and the product map

$$R\Gamma_{et}(\mathcal{X}_s,Ri^!\mathbb{Z}(d))\otimes^{\mathbb{L}}R\Gamma_{et}(\mathcal{X},\mathbb{Z}) \to R\Gamma_{et}(\mathcal{X}_s,Ri^!\mathbb{Z}(d)) \tag{24}$$

is obtained similarly. By [9, Theorem 7.5] applied to $\mathcal{F}=\mathbb{Z}/m$, the pairing

$$R\Gamma_{et}(\mathcal{X}_s,Ri^!\mathbb{Z}/m)\otimes^{\mathbb{L}}R\Gamma_{et}(\mathcal{X},\mathbb{Z}/m(d))\to R\Gamma_{et}(\mathcal{X}_s,Ri^!\mathbb{Z}/m(d))\to\mathbb{Z}/m[-2d-1],$$

induced by (23), is perfect. The pairing induced by (24)

$$R\Gamma_{et}(\mathcal{X}_s,Ri^!\mathbb{Z}/m(d))\otimes^{\mathbb{L}}R\Gamma_{et}(\mathcal{X},\mathbb{Z}/m)\to R\Gamma_{et}(\mathcal{X}_s,Ri^!\mathbb{Z}/m(d))\to\mathbb{Z}/m[-2d-1]$$

is perfect as well, since it reduces, by purity and proper base change, to

$$R\Gamma_{et}(\mathcal{X}_s,\mathbb{Z}^c/m[-2d])\otimes^{\mathbb{L}}R\Gamma_{et}(\mathcal{X}_s,\mathbb{Z}/m)\to R\Gamma_{et}(\mathcal{X}_s,\mathbb{Z}^c/m[-2d])\to\mathbb{Z}/m[-2d-1]$$

which is perfect by [9, Theorem 5.1] applied to $\mathcal{F}=\mathbb{Z}/m$. □

For $n=0$ or $n=d$, consider the product map

$$R\Gamma_{et}(\mathcal{X}_{\bar{s}},R\bar{i}^!\mathbb{Z}(n))\otimes^{\mathbb{L}}R\Gamma_{et}(\mathcal{X}_{\mathcal{O}_{K^{un}}},\mathbb{Z}(d-n))\to R\Gamma_{et}(\mathcal{X}_{\bar{s}},R\bar{i}^!\mathbb{Z}(d)).$$

This product map is induced by the obvious product maps $\mathbb{Z}\otimes^{\mathbb{L}}\mathbb{Z}(d)\to\mathbb{Z}(d)$ in the derived ∞-category of étale sheaves over $\mathcal{X}_{\mathcal{O}_{K^{un}}}$ and $\mathcal{X}_{K^{un}}$, as in the proof of Proposition 5.3. Applying $R\Gamma(W_{\kappa(s)},-)$ and composing with the trace map, we obtain

$$R\Gamma_W(\mathcal{X}_s,Ri^!\mathbb{Z}(n))\otimes^{\mathbb{L}}R\Gamma_W(\mathcal{X},\mathbb{Z}(d-n))\to R\Gamma_W(\mathcal{X}_s,Ri^!\mathbb{Z}(d))\to\mathbb{Z}[-2d-1]. \tag{25}$$

This yields the morphisms

$$R\Gamma_W(\mathcal{X},\mathbb{Z}(d))\to R\mathrm{Hom}(R\Gamma_W(\mathcal{X}_s,Ri^!\mathbb{Z}),\mathbb{Z}[-2d-1]),$$

which, in turn, induce

$$\tau^{>a}R\Gamma_W(\mathcal{X},\mathbb{Z}(d))\to R\mathrm{Hom}(R\Gamma_W(\mathcal{X}_s,Ri^!\mathbb{Z}),\mathbb{Z}[-2d-1]) \tag{26}$$

for $a\ll 0$, since the right-hand side is bounded. Composing (26) with the canonical map (see Lemma 2.9)

$$R\mathrm{Hom}(R\Gamma_W(\mathcal{X}_s,Ri^!\mathbb{Z}),\mathbb{Z}[-2d-1])\to R\underline{\mathrm{Hom}}(R\Gamma_W(\mathcal{X}_s,Ri^!\mathbb{Z}),\mathbb{Z}[-2d-1]),$$

we obtain

$$\tau^{>a}R\Gamma_W(\mathcal{X},\mathbb{Z}(d))\to R\underline{\mathrm{Hom}}(R\Gamma_W(\mathcal{X}_s,Ri^!\mathbb{Z}),\mathbb{Z}[-2d-1]). \tag{27}$$

Proposition 5.4. *The map (27) factors through an equivalence*

$$R\Gamma_{ar}(\mathcal{X},\mathbb{Z}(d))\xrightarrow{\sim}R\underline{\mathrm{Hom}}(R\Gamma_W(\mathcal{X}_s,Ri^!\mathbb{Z}),\mathbb{Z}[-2d-1]) \tag{28}$$

in $\mathbf{D}^b(\mathrm{FLCA})$.

Proof. One has

$$\begin{aligned}R\Gamma_{ar}(\mathcal{X},\mathbb{Z}(d))&:=\tau^{>a}R\Gamma_W(\mathcal{X},\mathbb{Z}(d))\underline{\widehat{\otimes}}\widehat{\mathbb{Z}}\\&\simeq(\mathrm{hocolim}\,R\mathrm{Hom}(R\Gamma_W(\mathcal{X},\mathbb{Z}/m(d)),\mathbb{Q}/\mathbb{Z}))^D\\&\simeq(\mathrm{hocolim}\,R\mathrm{Hom}(R\Gamma_{et}(\mathcal{X},\mathbb{Z}/m(d)),\mathbb{Q}/\mathbb{Z}))^D\\&\simeq R\underline{\mathrm{Hom}}(\mathrm{hocolim}\,R\mathrm{Hom}(R\Gamma_{et}(\mathcal{X},\mathbb{Z}/m(d)),\mathbb{Q}/\mathbb{Z}[-2d-1]),\mathbb{R}/\mathbb{Z}[-2d-1])\\&\xrightarrow{\sim}R\underline{\mathrm{Hom}}(\mathrm{hocolim}\,R\Gamma_{et}(\mathcal{X}_s,Ri^!\mathbb{Z}/m),\mathbb{R}/\mathbb{Z}[-2d-1])\end{aligned}$$

$$\begin{aligned}&\simeq R\underline{\mathrm{Hom}}(R\Gamma_W(\mathcal{X}_s,Ri^!\mathbb{Q}/\mathbb{Z}),\mathbb{R}/\mathbb{Z}[-2d-1])\\ &\simeq R\underline{\mathrm{Hom}}(R\Gamma_W(\mathcal{X}_s,Ri^!\mathbb{Z}[1]),\mathbb{R}/\mathbb{Z}[-2d-1])\\ &\simeq R\underline{\mathrm{Hom}}(R\Gamma_W(\mathcal{X}_s,Ri^!\mathbb{Z}),\mathbb{Z}[-2d-1]),\end{aligned}$$

where we use Proposition 5.3, the vanishing

$$R\underline{\mathrm{Hom}}(R\Gamma_W(\mathcal{X}_s,Ri^!\mathbb{Z}[1]),\mathbb{R})\simeq 0$$

proven in Lemma 5.5 below and $Ri^!\mathbb{Q}\simeq 0$. □

Lemma 5.5. *We have*

$$R\underline{\mathrm{Hom}}(R\Gamma_W(\mathcal{X}_s,Ri^!\mathbb{Z}[1]),\mathbb{R})\simeq R\underline{\mathrm{Hom}}(\mathbb{R},R\Gamma_W(\mathcal{X}_s,Ri^!\mathbb{Z}[1]))\simeq 0.$$

Proof. As observed above, we have $R\Gamma_W(\mathcal{X}_s,Ri^!\mathbb{Z}[1])\simeq R\Gamma_W(\mathcal{X}_s,Ri^!\mathbb{Q}/\mathbb{Z})$. Since $R\underline{\mathrm{Hom}}(\mathbb{R},-)$ and $R\underline{\mathrm{Hom}}(-,\mathbb{R})$ are exact functors, and using the t-structure on $\mathbf{D}^b(\mathrm{FLCA})$, we may suppose that $R\Gamma_W(\mathcal{X}_s,Ri^!\mathbb{Q}/\mathbb{Z})$ is cohomologically concentrated in one degree. Hence, one is reduced to show that

$$R\underline{\mathrm{Hom}}(A,\mathbb{R})\simeq R\underline{\mathrm{Hom}}(\mathbb{R},A)\simeq 0$$

for any torsion discrete abelian group of finite ranks A. This follows from [14, Proposition 4.15 (i) and (vii)]. □

Corollary 5.6. *We have*

$$R\underline{\mathrm{Hom}}(R\Gamma_{ar}(\mathcal{X},\mathbb{Z}(d)),\mathbb{R})\simeq R\underline{\mathrm{Hom}}(\mathbb{R},R\Gamma_{ar}(\mathcal{X},\mathbb{Z}(d)))\simeq 0.$$

Proof. In the proof of Proposition 5.4, we have shown that $R\Gamma_{ar}(\mathcal{X},\mathbb{Z}(d))$ is, up to a shift, Pontryagin dual to $R\Gamma_W(\mathcal{X}_s,Ri^!\mathbb{Z}[1])$. Hence, the corollary follows from Lemma 5.5, since $R\underline{\mathrm{Hom}}(X,Y)\simeq R\underline{\mathrm{Hom}}(Y^D,X^D)$ for any $X,Y\in\mathbf{D}^b(\mathrm{FLCA})$. □

Similarly, we have the

Proposition 5.7. *The map*

$$R\Gamma_W(\mathcal{X}_s,Ri^!\mathbb{Z}(d))\to R\mathrm{Hom}(R\Gamma_W(\mathcal{X},\mathbb{Z}),\mathbb{Z}[-2d-1]),\tag{29}$$

induced by (25), factors through an equivalence

$$R\Gamma_W(\mathcal{X}_s,Ri^!\mathbb{Z}(d))\xrightarrow{\sim}R\underline{\mathrm{Hom}}(R\Gamma_{ar}(\mathcal{X},\mathbb{Z}),\mathbb{Z}[-2d-1]).\tag{30}$$

Proof. Recall from Remark 4.9 that we have

$$Ri^!\mathbb{Z}(d)=Ri^!\mathbb{Z}^c(0)[-2d]\simeq\mathbb{Z}^c(0)[-2d].$$

If $\mathcal{X}_s^{\mathrm{red}}$ is a simple normal crossing scheme, we may, therefore, identify the map

$$R\Gamma_W(\mathcal{X}_s,Ri^!\mathbb{Z}(d))[2d]\xrightarrow{\sim}R\underline{\mathrm{Hom}}(R\Gamma_{ar}(\mathcal{X},\mathbb{Z}),\mathbb{Z}[-2d-1])[2d]\tag{31}$$

with the composite morphism

$$R\Gamma_W(\mathcal{X}_s,\mathbb{Z}^c(0))\xrightarrow{\sim}R\mathrm{Hom}(R\Gamma_W(\mathcal{X}_s,\mathbb{Z}),\mathbb{Z}[-1])\xrightarrow{\sim}R\underline{\mathrm{Hom}}(R\Gamma_W(\mathcal{X}_s,\mathbb{Z}),\mathbb{Z}[-1]),$$

which is an equivalence of perfect complexes of abelian groups by Proposition 3.5, Theorem 3.8 and Lemma 2.9. If $d \leq 2$, we may identify (31) with the morphism

$$R\Gamma_W(\mathcal{X}_s,\mathbb{Z}^c(0)) \xrightarrow{\sim} R\mathrm{Hom}(R\Gamma_{Wh}(\mathcal{X}_s,\mathbb{Z}),\mathbb{Z}[-1]) \xrightarrow{\sim} R\underline{\mathrm{Hom}}(R\Gamma_{Wh}(\mathcal{X}_s,\mathbb{Z}),\mathbb{Z}[-1]),$$

which is an equivalence of perfect complexes of abelian groups by Proposition 3.2 and [11, Theorem 4.2] (using the fact that for a curve, étale and eh-cohomology agree).

Note that, if $d \leq 2$, then the following diagram in $\mathbf{D}^b(\mathrm{LCA})$

$$\begin{tikzcd}
R\Gamma_W(\mathcal{X}_s,Ri^!\mathbb{Z}(d)) \arrow[r, "(30)"] \arrow[d, "(29)"] \arrow[dr, "\simeq"] & R\underline{\mathrm{Hom}}(R\Gamma_{ar}(\mathcal{X},\mathbb{Z}),\mathbb{Z}[-2d-1]) \\
R\mathrm{Hom}(R\Gamma_W(\mathcal{X},\mathbb{Z}),\mathbb{Z}[-2d-1]) & R\mathrm{Hom}(R\Gamma_{Wh}(\mathcal{X}_s,\mathbb{Z}),\mathbb{Z}[-2d-1]) \arrow[l] \arrow[u, "\simeq"']
\end{tikzcd}$$

commutes. If $\mathcal{X}_s^{\mathrm{red}}$ is a simple normal crossing scheme, then the same diagram with $R\Gamma_{Wh}(\mathcal{X}_s,\mathbb{Z})$ replaced by $R\Gamma_W(\mathcal{X}_s,\mathbb{Z})$ commutes as well. □

We now combine Propositions 5.4 and 5.7 to prove our result for the generic fibre.

Proposition 5.8. *There is an equivalence*

$$R\Gamma_{ar}(\mathcal{X}_K,\mathbb{Z}(d)) \xrightarrow{\sim} R\underline{\mathrm{Hom}}(R\Gamma_{ar}(\mathcal{X}_K,\mathbb{Z}),\mathbb{Z}[-2d]),$$

such that, for any m, there is a commutative square

$$\begin{tikzcd}
R\Gamma_{ar}(\mathcal{X}_K,\mathbb{Z}(d)) \arrow[r, "\sim"] \arrow[d] & R\underline{\mathrm{Hom}}(R\Gamma_{ar}(\mathcal{X}_K,\mathbb{Z}),\mathbb{Z}[-2d]) \arrow[d] \\
R\Gamma_{et}(\mathcal{X}_K,\mathbb{Z}/m(d)) \arrow[r, "\sim"] & R\underline{\mathrm{Hom}}(R\Gamma_{et}(\mathcal{X}_K,\mathbb{Z}/m),\mathbb{Q}/\mathbb{Z}[-2d]),
\end{tikzcd}$$

where the lower horizontal map is induced by duality for the usual étale cohomology of the variety $\mathcal{X}_K$.

Proof. We start with the commutative diagram:

$$\begin{tikzcd}
R\Gamma_{et}(\mathcal{X}_{\bar{s}},R\bar{i}^!\mathbb{Z})\otimes^{\mathbb{L}} R\Gamma_{et}(\mathcal{X}_{\mathcal{O}_{K^{un}}},\mathbb{Z}(d)) \arrow[dr] & \\
R\Gamma_{et}(\mathcal{X}_{\bar{s}},R\bar{i}^!\mathbb{Z})\otimes^{\mathbb{L}} R\Gamma_{et}(\mathcal{X}_{\bar{s}},R\bar{i}^!\mathbb{Z}(d)) \arrow[u] \arrow[d] \arrow[r] & R\Gamma_{et}(\mathcal{X}_{\bar{s}},R\bar{i}^!\mathbb{Z}(d)) \\
R\Gamma_{et}(\mathcal{X}_{\mathcal{O}_{K^{un}}},\mathbb{Z})\otimes^{\mathbb{L}} R\Gamma_{et}(\mathcal{X}_{\bar{s}},R\bar{i}^!\mathbb{Z}(d)) \arrow[ur] & ,
\end{tikzcd}$$

where the map

$$R\Gamma_{et}(\mathcal{X}_{\bar{s}}, R\bar{i}^!\mathbb{Z}) \otimes^{\mathbb{L}} R\Gamma_{et}(\mathcal{X}_{\bar{s}}, R\bar{i}^!\mathbb{Z}(d)) \to R\Gamma_{et}(\mathcal{X}_{\bar{s}}, R\bar{i}^!\mathbb{Z}(d))$$

is induced by the map $\mathbb{Z} \otimes^{\mathbb{L}} \mathbb{Z}(d) \to \mathbb{Z}(d)$ over $\mathcal{X}_{\mathcal{O}_{K^{un}}}$ as follows. Consider the morphism

$$\bar{i}_* R\bar{i}^!\mathbb{Z} \to \mathbb{Z} \to R\mathbf{Hom}_{\mathcal{X}_{\mathcal{O}_{K^{un}}}}(\mathbb{Z}(d), \mathbb{Z}(d))$$

$$\to R\mathbf{Hom}_{\mathcal{X}_{\mathcal{O}_{K^{un}}}}(\bar{i}_* R\bar{i}^!\mathbb{Z}(d), \mathbb{Z}(d)) \simeq \bar{i}_* R\mathbf{Hom}_{\mathcal{X}_{\bar{s}}}(R\bar{i}^!\mathbb{Z}(d), R\bar{i}^!\mathbb{Z}(d)),$$

and apply $\bar{i}^*$, where **Hom** denotes the internal Hom in the category of sheaves on the small étale site of the corresponding scheme. Applying $R\Gamma(W_{\kappa(s)}, -)$ to the diagram above, we obtain the following commutative diagram in $\mathbf{D}(\mathrm{Ab})$, where tr is the trace map:

$$\begin{tikzcd}
R\Gamma_W(\mathcal{X}_s, Ri^!\mathbb{Z}) \otimes^{\mathbb{L}} R\Gamma_W(\mathcal{X}, \mathbb{Z}(d)) \arrow[rd] & & \\
R\Gamma_W(\mathcal{X}_s, Ri^!\mathbb{Z}) \otimes^{\mathbb{L}} R\Gamma_W(\mathcal{X}_s, Ri^!\mathbb{Z}(d)) \arrow[u] \arrow[d] \arrow[r] & R\Gamma_W(\mathcal{X}_s, Ri^!\mathbb{Z}(d)) \arrow[r, "tr"] & \mathbb{Z}[-2d-1] \\
R\Gamma_W(\mathcal{X}, \mathbb{Z}) \otimes^{\mathbb{L}} R\Gamma_W(\mathcal{X}_s, Ri^!\mathbb{Z}(d)) \arrow[ru] & . &
\end{tikzcd}$$

It gives the following commutative diagram in $\mathbf{D}(\mathrm{Ab})$

$$\begin{tikzcd}
R\Gamma_W(\mathcal{X}_s, Ri^!\mathbb{Z}(d)) \arrow[r] \arrow[d] & R\mathrm{Hom}(R\Gamma_W(\mathcal{X}, \mathbb{Z}), \mathbb{Z}[-2d-1]) \arrow[d] \\
R\Gamma_W(\mathcal{X}, \mathbb{Z}(d)) \arrow[r] \arrow[d] & R\mathrm{Hom}(R\Gamma_W(\mathcal{X}_s, Ri^!\mathbb{Z}), \mathbb{Z}[-2d-1]) \\
\tau^{>a} R\Gamma_W(\mathcal{X}, \mathbb{Z}(d)) \arrow[ru] & .
\end{tikzcd}$$

We obtain the following commutative diagram

$$\begin{tikzcd}
R\Gamma_W(\mathcal{X}_s, Ri^!\mathbb{Z}(d)) \arrow[r, "(29)"] \arrow[d] & R\mathrm{Hom}(R\Gamma_W(\mathcal{X}, \mathbb{Z}), \mathbb{Z}[-2d-1]) \arrow[d] \\
\tau^{>a} R\Gamma_W(\mathcal{X}, \mathbb{Z}(d)) \arrow[r, "(27)"] & R\mathrm{Hom}(R\Gamma_W(\mathcal{X}_s, Ri^!\mathbb{Z}), \mathbb{Z}[-2d-1]) \arrow[d] \\
 & R\underline{\mathrm{Hom}}(R\Gamma_W(\mathcal{X}_s, Ri^!\mathbb{Z}), \mathbb{Z}[-2d-1])
\end{tikzcd}$$

in the derived ∞-category $\mathbf{D}^b(\mathrm{LCA})$, where the lower-right map is given by Lemma 2.9. By construction of the maps (28) and (30), we obtain the following commutative diagram

$$\begin{tikzcd}
 & R\underline{\mathrm{Hom}}(R\Gamma_{ar}(\mathcal{X},\mathbb{Z}),\mathbb{Z}[-2d-1]) \arrow[d] \\
R\Gamma_W(\mathcal{X}_s,Ri^!\mathbb{Z}(d)) \arrow[r] \arrow[d] \arrow[ur, "(30)"] & R\mathrm{Hom}(R\Gamma_W(\mathcal{X},\mathbb{Z}),\mathbb{Z}[-2d-1]) \arrow[d] \\
\tau^{>a}R\Gamma_W(\mathcal{X},\mathbb{Z}(d)) \arrow[r] \arrow[d] & R\underline{\mathrm{Hom}}(R\Gamma_W(\mathcal{X}_s,Ri^!\mathbb{Z}),\mathbb{Z}[-2d-1]) \\
R\Gamma_{ar}(\mathcal{X},\mathbb{Z}(d)) \arrow[ur, "(28)"'] & ,
\end{tikzcd}$$

hence, the upper square in the commutative square

$$\begin{tikzcd}
R\Gamma_W(\mathcal{X}_s,Ri^!\mathbb{Z}(d)) \arrow[r, "(30)", "\sim"'] \arrow[d] & R\underline{\mathrm{Hom}}(R\Gamma_{ar}(\mathcal{X},\mathbb{Z}),\mathbb{Z}[-2d-1]) \arrow[d] \\
R\Gamma_{ar}(\mathcal{X},\mathbb{Z}(d)) \arrow[r, "(28)", "\sim"'] \arrow[d] & R\underline{\mathrm{Hom}}(R\Gamma_W(\mathcal{X}_s,Ri^!\mathbb{Z}),\mathbb{Z}[-2d-1]) \arrow[d] \\
R\Gamma_{ar}(\mathcal{X}_K,\mathbb{Z}(d)) \arrow[r] & R\underline{\mathrm{Hom}}(R\Gamma_{ar}(\mathcal{X}_K,\mathbb{Z}),\mathbb{Z}[-2d]).
\end{tikzcd}$$

It follows that the diagram is an equivalence of cofibre sequences in $\mathbf{D}^b(\mathrm{LCA})$. Tensoring the upper commutative square with $\mathbb{Z}/m$ gives a square equivalent to the commutative square

$$\begin{tikzcd}
R\Gamma_{et}(\mathcal{X}_s,Ri^!\mathbb{Z}/m(d)) \arrow[r] \arrow[d] & R\underline{\mathrm{Hom}}(R\Gamma_{et}(\mathcal{X},\mathbb{Z}/m),\mathbb{Q}/\mathbb{Z}[-2d-1]) \arrow[d] \\
R\Gamma_{et}(\mathcal{X},\mathbb{Z}/m(d)) \arrow[r] & R\underline{\mathrm{Hom}}(R\Gamma_{et}(\mathcal{X}_s,Ri^!\mathbb{Z}/m),\mathbb{Q}/\mathbb{Z}[-2d-1]),
\end{tikzcd}$$

where the horizontal maps are induced by the perfect pairings of Proposition 5.3. This yields the commutative square of Proposition 5.8. □

It remains to prove that

$$R\Gamma_{ar}(\mathcal{X}_K,\mathbb{Z}) \to R\underline{\mathrm{Hom}}(R\Gamma_{ar}(\mathcal{X}_K,\mathbb{Z}(d)),\mathbb{Z}[-2d])$$

is an equivalence.

Lemma 5.9. *The map*

$$R\Gamma_{ar}(\mathcal{X}_K,\mathbb{Z}) \to R\underline{\mathrm{Hom}}(R\underline{\mathrm{Hom}}(R\Gamma_{ar}(\mathcal{X}_K,\mathbb{Z}),\mathbb{Z}),\mathbb{Z})$$

is an equivalence.

Proof. We have

$$R\Gamma_{ar}(\mathcal{X},\mathbb{Z}) \simeq R\underline{\mathrm{Hom}}(R\underline{\mathrm{Hom}}(R\Gamma_{ar}(\mathcal{X},\mathbb{Z}),\mathbb{Z}),\mathbb{Z})$$

by Lemma 2.9, since $R\Gamma_{ar}(\mathcal{X},\mathbb{Z})$ is a perfect complex of abelian groups by Propositions 3.2, 3.5 and 4.5. In view of the cofibre sequence

$$R\Gamma_W(\mathcal{X}_s,Ri^!\mathbb{Z}) \to R\Gamma_{ar}(\mathcal{X},\mathbb{Z}) \to R\Gamma_{ar}(\mathcal{X}_K,\mathbb{Z}),$$

one is reduced to check that the map

$$R\Gamma_W(\mathcal{X}_s,Ri^!\mathbb{Z}) \to R\underline{\mathrm{Hom}}(R\underline{\mathrm{Hom}}(R\Gamma_W(\mathcal{X}_s,Ri^!\mathbb{Z}),\mathbb{Z}),\mathbb{Z})$$

is an equivalence. Recall from the proof of Proposition 5.4 that we have

$$\begin{aligned}
&R\underline{\mathrm{Hom}}(R\underline{\mathrm{Hom}}(R\Gamma_W(\mathcal{X}_s,Ri^!\mathbb{Z}),\mathbb{Z}),\mathbb{Z})\\
&\quad\simeq R\underline{\mathrm{Hom}}(R\Gamma_{ar}(\mathcal{X},\mathbb{Z}(d))[2d+1],\mathbb{Z})\\
&\quad\simeq R\underline{\mathrm{Hom}}(R\Gamma_{ar}(\mathcal{X},\mathbb{Z}(d))[2d+1],\mathbb{R}/\mathbb{Z}[-1])\\
&\quad\simeq R\Gamma_{ar}(\mathcal{X},\mathbb{Z}(d))^D[-2d-2]\\
&\quad\simeq (\mathrm{hocolim}\, R\mathrm{Hom}(R\Gamma_W(\mathcal{X},\mathbb{Z}/m(d)),\mathbb{Q}/\mathbb{Z}))^{DD}[-2d-2]\\
&\quad\simeq \mathrm{hocolim}\, R\mathrm{Hom}(R\Gamma_W(\mathcal{X},\mathbb{Z}/m(d)),\mathbb{Q}/\mathbb{Z}[-2d-1])[-1]\\
&\quad\simeq R\Gamma_W(\mathcal{X}_s,Ri^!\mathbb{Q}/\mathbb{Z})[-1]\\
&\quad\simeq R\Gamma_W(\mathcal{X}_s,Ri^!\mathbb{Z}),
\end{aligned}$$

where the second equivalence follows from Corollary 5.6. □

Consider the pairing

$$R\Gamma_{ar}(\mathcal{X}_K,\mathbb{Z}(d))\underline{\otimes}^{\mathbb{L}}R\Gamma_{ar}(\mathcal{X}_K,\mathbb{Z}) \longrightarrow \mathbb{Z}[-2d] \tag{32}$$

induced by the equivalence of Proposition 5.8. The induced map

$$R\Gamma_{ar}(\mathcal{X}_K,\mathbb{Z}(d)) \xrightarrow{\sim} R\underline{\mathrm{Hom}}(R\Gamma_{ar}(\mathcal{X}_K,\mathbb{Z}),\mathbb{Z}[-2d]) \tag{33}$$

is (tautologically) the equivalence of Proposition 5.8. Moreover, the map

$$R\Gamma_{ar}(\mathcal{X}_K,\mathbb{Z}) \xrightarrow{\sim} R\underline{\mathrm{Hom}}(R\Gamma_{ar}(\mathcal{X}_K,\mathbb{Z}(d)),\mathbb{Z}[-2d]) \tag{34}$$

induced by (32) is an equivalence as well. Indeed, applying $R\underline{\mathrm{Hom}}(-,\mathbb{Z}[-2d])$ to (33) and using Lemma 5.9, we obtain the composite equivalence

$$\begin{aligned}
R\Gamma_{ar}(\mathcal{X}_K,\mathbb{Z}) &\xrightarrow{\sim} R\underline{\mathrm{Hom}}(R\underline{\mathrm{Hom}}(R\Gamma_{ar}(\mathcal{X}_K,\mathbb{Z}),\mathbb{Z}[-2d]),\mathbb{Z}[-2d])\\
&\xrightarrow{\sim} R\underline{\mathrm{Hom}}(R\Gamma_{ar}(\mathcal{X}_K,\mathbb{Z}(d)),\mathbb{Z}[-2d]),
\end{aligned}$$

which is, up to equivalence, the map (34). □

5.2. Pontryagin duality

Recall that we denote by FLCA the category of locally compact abelian group of finite ranks in the sense of [14]. It follows from (3) and Proposition 4.13 that the following definition makes sense.

Definition 5.10. Assume Hypothesis 5.1. For $n = 0, d$, we define

$$R\Gamma_{ar}(\mathcal{X}_K, \mathbb{R}/\mathbb{Z}(n)) := R\Gamma_{ar}(\mathcal{X}_K, \mathbb{Z}(n)) \underline{\otimes}^{\mathbb{L}} \mathbb{R}/\mathbb{Z};$$

$$R\Gamma_{ar}(\mathcal{X}, \mathbb{R}/\mathbb{Z}(n)) := R\Gamma_{ar}(\mathcal{X}, \mathbb{Z}(n)) \underline{\otimes}^{\mathbb{L}} \mathbb{R}/\mathbb{Z}.$$

Corollary 5.11. *Assume Hypothesis 5.1. Then one has equivalences*

$$R\Gamma_{ar}(\mathcal{X}_K, \mathbb{R}/\mathbb{Z}) \xrightarrow{\sim} R\underline{\mathrm{Hom}}(R\Gamma_{ar}(\mathcal{X}_K, \mathbb{Z}(d)), \mathbb{R}/\mathbb{Z}[-2d])$$

and

$$R\Gamma_{ar}(\mathcal{X}_K, \mathbb{Z}) \xrightarrow{\sim} R\underline{\mathrm{Hom}}(R\Gamma_{ar}(\mathcal{X}_K, \mathbb{R}/\mathbb{Z}(d)), \mathbb{R}/\mathbb{Z}[-2d])$$

in $\mathbf{D}^b(\mathrm{FLCA})$.

Proof. By Theorem 5.2 and [14, Remark 4.3(ii)], we have

$$\begin{aligned} R\Gamma_{ar}(\mathcal{X}_K, \mathbb{Z}(d)) &\xrightarrow{\sim} R\underline{\mathrm{Hom}}(R\Gamma_{ar}(\mathcal{X}_K, \mathbb{Z}), \mathbb{Z}[-2d]) \\ &\xrightarrow{\sim} R\underline{\mathrm{Hom}}(R\Gamma_{ar}(\mathcal{X}_K, \mathbb{Z}), R\underline{\mathrm{Hom}}(\mathbb{R}/\mathbb{Z}, \mathbb{R}/\mathbb{Z}[-2d]) \\ &\simeq R\underline{\mathrm{Hom}}(R\Gamma_{ar}(\mathcal{X}_K, \mathbb{Z}) \underline{\otimes}^{\mathbb{L}} \mathbb{R}/\mathbb{Z}, \mathbb{R}/\mathbb{Z}[-2d]) \\ &:= R\underline{\mathrm{Hom}}(R\Gamma_{ar}(\mathcal{X}_K, \mathbb{R}/\mathbb{Z}), \mathbb{R}/\mathbb{Z}[-2d]). \end{aligned}$$

Applying the functor $R\underline{\mathrm{Hom}}(-, \mathbb{R}/\mathbb{Z}[-2d])$ and using Pontryagin duality, we obtain the first equivalence of the corollary.

Similarly, we have

$$\begin{aligned} R\Gamma_{ar}(\mathcal{X}_K, \mathbb{Z}) &\xrightarrow{\sim} R\underline{\mathrm{Hom}}(R\Gamma_{ar}(\mathcal{X}_K, \mathbb{Z}(d)), \mathbb{Z}[-2d]) \\ &\simeq R\underline{\mathrm{Hom}}(R\Gamma_{ar}(\mathcal{X}_K, \mathbb{Z}(d)) \underline{\otimes}^{\mathbb{L}} \mathbb{R}/\mathbb{Z}, \mathbb{R}/\mathbb{Z}[-2d]) \\ &:= R\underline{\mathrm{Hom}}(R\Gamma_{ar}(\mathcal{X}_K, \mathbb{R}/\mathbb{Z}(d)), \mathbb{R}/\mathbb{Z}[-2d]). \end{aligned}$$

□

Corollary 5.12. *Suppose that* $\mathcal{X}/\mathcal{O}_K$ *has strictly semistable reduction, and suppose that* $R\Gamma_W(\mathcal{X}_s, \mathbb{Z}^c(0))$ *is a perfect complex of abelian groups. Then for any* $i \in \mathbb{Z}$, *we have an isomorphism of locally compact groups*

$$H^i_{ar}(\mathcal{X}_K, \mathbb{R}/\mathbb{Z}) \xrightarrow{\sim} H^{2d-i}_{ar}(\mathcal{X}_K, \mathbb{Z}(d))^D$$

and an isomorphism of discrete groups

$$H^i_{ar}(\mathcal{X}_K, \mathbb{Z}) \xrightarrow{\sim} H^{2d-i}_{ar}(\mathcal{X}_K, \mathbb{R}/\mathbb{Z}(d))^D.$$

Proof. In view of Theorem 4.17 and Lemma 2.6, the equivalence in $\mathbf{D}^b(\mathrm{FLCA})$

$$R\Gamma_{ar}(\mathcal{X}_K, \mathbb{R}/\mathbb{Z}) \xrightarrow{\sim} R\Gamma(\mathcal{X}_K, \mathbb{Z}(d))^D[-2d]$$

induces isomorphisms

$$H^i_{ar}(\mathcal{X}_K, \mathbb{R}/\mathbb{Z}) \xrightarrow{\sim} H^i(R\Gamma(\mathcal{X}_K, \mathbb{Z}(d))^D[-2d]) \simeq H^{2d-i}_{ar}(\mathcal{X}_K, \mathbb{Z}(d))^D$$

of locally compact abelian groups. Similarly, the equivalence in $\mathbf{D}^b(\mathrm{FLCA})$

$$R\Gamma_{ar}(\mathcal{X}_K,\mathbb{R}/\mathbb{Z}(d)) \xrightarrow{\sim} R\Gamma(\mathcal{X}_K,\mathbb{Z})^D[-2d]$$

induces isomorphisms

$$H^i_{ar}(\mathcal{X}_K,\mathbb{R}/\mathbb{Z}(d)) \xrightarrow{\sim} H^i(R\Gamma(\mathcal{X}_K,\mathbb{Z})^D[-2d]) \simeq H^{2d-i}_{ar}(\mathcal{X}_K,\mathbb{Z})^D$$

of compact abelian groups. □

Remark 5.13. Corollary 5.11 as well as Corollary 5.12 can be extended to Tate twists $n>d$, or equivalently, $n<0$. Let $\mathcal{X}$ be a regular, proper and flat scheme over $\mathcal{O}_K$. Assume that $\mathcal{X}$ is connected of Krull dimension d, and let $n>d$. We, moreover, assume that $R\Gamma_{et}(\mathcal{X}_K,\mathbb{Z}(n))$ is bounded[4] .

Then we define

$$R\Gamma_{ar}(\mathcal{X}_K,\mathbb{Z}(n)) := R\Gamma_{et}(\mathcal{X}_K,\mathbb{Z}(n))\widehat{\underline{\otimes}}\widehat{\mathbb{Z}} \simeq \varprojlim \left(R\Gamma_{et}(\mathcal{X}_K,\mathbb{Z}(n))\underline{\otimes}^{\mathbb{L}}\mathbb{Z}/m\right),$$

where the limit is computed in the ∞-category $\mathbf{D}^b(\mathrm{LCA})$ (see Proposition 2.11 and Remark 2.12).

Dually, we define

$$\begin{aligned} R\Gamma_{ar}(\mathcal{X}_K,\mathbb{Z}(d-n)) &:= R\Gamma_{et}(\mathcal{X}_K,\mathbb{Q}/\mathbb{Z}(d-n))[-1] \\ &\simeq \varinjlim R\Gamma_{et}(\mathcal{X}_K,\mathbb{Z}/m\mathbb{Z}(d-n))[-1] \\ &\simeq \varinjlim R\Gamma_{et}(\mathcal{X}_K,\mu_m^{\otimes d-n})[-1], \end{aligned}$$

where the colimit is computed in the ∞-category $\mathbf{D}^b(\mathrm{LCA})$ (see Proposition 2.7). Here, we follow the abuse of Notation 2.16. We have

$$R\Gamma_{ar}(\mathcal{X}_K,\mathbb{R}(d-n)) := R\Gamma_{ar}(\mathcal{X}_K,\mathbb{Z}(d-n))\underline{\otimes}^{\mathbb{L}}\mathbb{R} \simeq 0,$$

hence,

$$R\Gamma_{ar}(\mathcal{X}_K,\mathbb{R}/\mathbb{Z}(d-n)) \simeq R\Gamma_{et}(\mathcal{X}_K,\mathbb{Q}/\mathbb{Z}(d-n)) \simeq \varinjlim R\Gamma_{et}(\mathcal{X}_K,\mu_m^{\otimes d-n}).$$

Poincaré duality for étale cohomology of $\mathcal{X}_{\overline{K}}$ together with Tate duality for Galois cohomology of the local field K gives an equivalence

$$R\Gamma_{et}(\mathcal{X}_K,\mu_m^{\otimes n}) \xrightarrow{\sim} R\underline{\mathrm{Hom}}(R\Gamma_{et}(\mathcal{X}_K,\mu_m^{\otimes d-n}),\mathbb{R}/\mathbb{Z}[-2d])$$

of discrete complexes in $\mathbf{D}^b(\mathrm{FLCA})$. We obtain an equivalence

$$R\Gamma_{et}(\mathcal{X}_K,\mathbb{Z}(n)) \xrightarrow{\sim} R\underline{\mathrm{Hom}}(R\Gamma_{et}(\mathcal{X}_K,\mathbb{R}/\mathbb{Z}(d-n)),\mathbb{R}/\mathbb{Z}[-2d])$$

in $\mathbf{D}^b(\mathrm{FLCA})$ and an isomorphism of compact groups of finite ranks

$$H^i_{ar}(\mathcal{X}_K,\mathbb{Z}(n)) \xrightarrow{\sim} H^{2d-i}_{ar}(\mathcal{X}_K,\mathbb{R}/\mathbb{Z}(d-n))^D$$

for any $i \in \mathbb{Z}$.

---

[4] This rather strong condition can be avoided using the trick of Definition 4.8.

Remark 5.14. It might also be possible, although probably not so trivial, to prove the analogue of Corollaries 5.11 and 5.12 in the case $n = 1, d = 2$. We refer to the work of Karpuk [15, Theorem 4.2.2] for a first step in this direction. It would be interesting to translate Karpuk's result in the LCA-language used in this paper, in order to obtain a perfect Pontryagin duality between locally compact abelian groups of finite ranks.

6. The conjectural picture

Let $K/\mathbb{Q}_p$ be a finite extension. We conjecture the existence of a cohomology theory on the category of separated schemes of finite type over $\mathrm{Spec}(\mathcal{O}_K)$, with values in $\mathbf{D}^b(\mathrm{FLCA})$, which we denote by

$$R\Gamma_{ar}(-,A(n))$$

for any $A \in \mathrm{FLCA}$ and any $n \in \mathbb{Z}$. Furthermore, we conjecture that the conclusion of Theorem 1.2 holds in full generality: For any smooth proper scheme $\mathcal{X}_K$ over K of pure dimension $d-1$, any Tate twist $n \in \mathbb{Z}$ and any $A \in \mathrm{FLCA}$, there is an equivalence

$$R\Gamma_{ar}(\mathcal{X}_K, A^D(n)) \xrightarrow{\sim} R\underline{\mathrm{Hom}}(R\Gamma_{ar}(\mathcal{X}_K, A(d-n)), \mathbb{R}/\mathbb{Z}[-2d])$$

in $\mathbf{D}^b(\mathrm{FLCA})$, which is induced by a trace map $H^{2d}_{ar}(\mathcal{X}_K, \mathbb{R}/\mathbb{Z}(d)) \to \mathbb{R}/\mathbb{Z}$, where A^D denotes the Pontryagin dual of A. Similarly, for any regular proper flat scheme $\mathcal{X}/\mathcal{O}_K$ of pure Krull dimension d, any $n \in \mathbb{Z}$ and any $A \in \mathrm{FLCA}$, there is an equivalence

$$R\Gamma_{ar}(\mathcal{X}, A^D(n)) \xrightarrow{\sim} R\underline{\mathrm{Hom}}(R\Gamma_{ar}(\mathcal{X}_s, Ri^! A(d-n)), \mathbb{R}/\mathbb{Z}[-2d-1]) \tag{35}$$

in $\mathbf{D}^b(\mathrm{FLCA})$, which is induced by a trace map $H^{2d+1}_{ar}(\mathcal{X}_s, Ri^!\mathbb{R}/\mathbb{Z}(d)) \to \mathbb{R}/\mathbb{Z}$.

However, we do not expect the analog of Corollary 1.3 to be true in general, since the groups $H^i_{ar}(\mathcal{X}_K, \mathbb{Z}(n))$ cannot be expected to be locally compact for arbitrary Tate twist n, as one can see from [13] for $n = 1$. Instead, they could be seen as condensed abelian groups in the sense of Clausen-Scholze, or, more precisely, as objects of the heart $\mathcal{LH}(\mathrm{FLCA})$ of the left t-structure on $\mathbf{D}^b(\mathrm{FLCA})$ in the sense of [14] and [23]. In contrast, we do expect isomorphisms of *compact* abelian groups

$$H^i_{ar}(\mathcal{X}, A(n)) \simeq H^{2d+1-i}_{ar}(\mathcal{X}_s, Ri^! A^D(d-n))^D$$

for any $i,n \in \mathbb{Z}$ and any *compact* $A \in \mathrm{FLCA}$. Concerning the relationship between $R\Gamma_{ar}(-,A(n))$ and known cohomology theories, we expect the following, for $\mathcal{X}$, a regular proper flat scheme over $\mathrm{Spec}(\mathcal{O}_K)$ of pure Krull dimension d.

- For any $n \in \mathbb{Z}$ and any positive integer m, we have

  $$R\Gamma_{ar}(\mathcal{X}_K, \mathbb{Z}/m(n)) \simeq R\Gamma_{et}(\mathcal{X}_K, \mathbb{Z}/m(n)),$$

  where the right-hand side denotes étale cohomology with coefficients in $\mathbb{Z}/m(n) \simeq \mu_m^{\otimes n}$. In particular, for any prime l, one has equivalences

  $$R\Gamma_{ar}(\mathcal{X}_K, \mathbb{Z}_l(n)) \simeq R\Gamma_{ar}(\mathcal{X}_K, \mathbb{Z}(n)) \widehat{\underline{\otimes}} \mathbb{Z}_l \simeq R\Gamma_{et}(\mathcal{X}_K, \mathbb{Z}_l(n)),$$

  where $(-)\widehat{\underline{\otimes}}\mathbb{Z}_l := R\lim(-\underline{\otimes}^{\mathbb{L}}\mathbb{Z}/l^\bullet)$ is the l-adic completion functor.

- The canonical map
$$R\Gamma_{ar}(-,\mathbb{Z}(n))\underline{\otimes}^{\mathbb{L}}A \xrightarrow{\sim} R\Gamma_{ar}(-,A(n))$$
is an equivalence for $(-)=\mathcal{X},\mathcal{X}_s$ and any ring object A, and for $(-)=\mathcal{X}_K$, if A has no topological p-torsion[5]. For example, the map
$$R\Gamma_{ar}(\mathcal{X}_K,\mathbb{Z}(n))\underline{\otimes}^{\mathbb{L}}A \xrightarrow{\sim} R\Gamma_{ar}(\mathcal{X}_K,A(n))$$
is an equivalence for $A=\mathbb{R}$ and $A=\mathbb{Q}_l$ if $l\neq p$.
- For any $n\in\mathbb{Z}$, we have
$$R\Gamma_{ar}(\mathcal{X}_s,\mathbb{Z}(n))\simeq R\Gamma_{Wh}(\mathcal{X}_s,\mathbb{Z}(n)),$$
where $R\Gamma_{Wh}(\mathcal{X}_s,\mathbb{Z}(n))$ is motivic Wh-cohomology in the sense of [8] (see Section 4.2). Moreover, the cofibre
$$C_{ar}(\mathcal{X},n):=\mathrm{Cofib}\,(R\Gamma_{ar}(\mathcal{X},\mathbb{Z}(n))\to R\Gamma_{ar}(\mathcal{X}_s,\mathbb{Z}(n)))$$
is a perfect complex of $\mathbb{Z}_p$-modules, such that
$$C_{ar}(\mathcal{X},n)\underline{\otimes}^{\mathbb{L}}\mathbb{Q}\simeq R\Gamma(\mathcal{X}_K,\Omega^{<n}_{\mathcal{X}_K/K}), \tag{36}$$
where the right-hand side denotes de Rham cohomology modulo, the n-step of the Hodge filtration. Finally, $C_{ar}(\mathcal{X},n)\simeq 0$ for any $n\leq 0$.
- For any $n\in\mathbb{Z}$, we have equivalences
$$\begin{aligned} R\Gamma_{ar}(\mathcal{X},\mathbb{Z}(n))\underline{\otimes}^{\mathbb{L}}\mathbb{Z}_p &\simeq R\Gamma_{ar}(\mathcal{X},\mathbb{Z}_p(n)) \\ &\simeq R\Gamma_{et}(\mathcal{X},\mathbb{Z}(n))\widehat{\underline{\otimes}}\mathbb{Z}_p \\ &=: R\Gamma_{et}(\mathcal{X},\mathbb{Z}_p(n)), \end{aligned} \tag{37}$$
where $R\Gamma_{et}(\mathcal{X},\mathbb{Z}(n))$ denotes étale motivic cohomology, as defined in Section 4.1. Note that $R\Gamma_{et}(\mathcal{X},\mathbb{Z}(n))\widehat{\underline{\otimes}}\mathbb{Q}_p$ is equivalent to the syntomic cohomology of Fontaine-Messing [3], at least if $\mathcal{X}/\mathcal{O}_K$ is smooth and $0\leq n<p-1$ (see [7, Theorem 1.3] and [2, Proposition 7.21, Remark 7.23]). For general regular proper flat $\mathcal{X}$ and arbitrary $n\geq 0$, a conjectural syntomic description of $R\Gamma_{et}(\mathcal{X},\mathbb{Z}(n))\widehat{\underline{\otimes}}\mathbb{Q}_p$ is given by [2, Corollary 7.17].
- For any $n\in\mathbb{Z}$, one has
$$R\Gamma_{ar}(\mathcal{X}_s,Ri^!\mathbb{Z}(n))\simeq R\Gamma_{W}(\mathcal{X}_s,Ri^!\mathbb{Z}(n)),$$
where the right-hand side is defined as in Section 4.1.
  - Let $n\geq 1$. On the one hand, Bloch's cycle complex $\mathbb{Z}(n)$, seen as a complex of étale sheaves over $\mathcal{X}$, is expected[6] to satisfy
$$\tau^{\leq n+1}Ri^!\mathbb{Z}(n)\simeq \mathbb{Z}^c(d-n)[-2d].$$

---

[5] The locally compact group A has a unique filtration by closed subgroups with graded pieces $A_{\mathbb{S}^1}$, $A_{\mathbb{A}}$ and $A_{\mathbb{Z}}$ of type $\mathbb{S}^1$, $\mathbb{A}$ and $\mathbb{Z}$, respectively. Then $A_{\mathbb{A}}$ is the direct sum of a finite dimensional $\mathbb{R}$-vector space and topological torsion group A_{toptor}, which, in turn, has a topological p-torsion component A_p (see [14, Section 2]). We say that A has no topological p-torsion if $A_p=0$.

[6] This is known, at least, if $\mathcal{X}/\mathcal{O}_K$ is smooth by [7, Theorem 1.2.1].

Here, $\mathbb{Z}^c(d-n)$ denotes Bloch's cycle complex in its homological notation as in [9]. On the other hand, we expect

$$\begin{aligned} H^i_W(\mathcal{X}_s,\tau^{>n+1}Ri^!\mathbb{Z}(n)) &\simeq H^i_{et}(\mathcal{X}_s,\tau^{>n+1}Ri^!\mathbb{Z}(n)) \\ &\xrightarrow{\sim} H^{2d+1-i}(C_{ar}(\mathcal{X},d-n))^D \end{aligned} \tag{38}$$

for any $i\in\mathbb{Z}$, where $H^{2d+1-i}(C_{ar}(\mathcal{X},d-n))$ is a finitely generated $\mathbb{Z}_p$-module. In particular, we have

$$R\Gamma_W(\mathcal{X}_s,\tau^{>n+1}Ri^!\mathbb{Z}(n))\underline{\otimes}^{\mathbb{L}}\mathbb{Q}_p\simeq 0.$$

Thus, we expect

$$\begin{aligned} R\Gamma_{ar}(\mathcal{X}_s,Ri^!\mathbb{Z}(n))\underline{\otimes}^{\mathbb{L}}\mathbb{Q}_p &\simeq R\Gamma_W(\mathcal{X}_s,\mathbb{Z}^c(d-n))\underline{\otimes}^{\mathbb{L}}\mathbb{Q}_p[-2d] \\ &\simeq R\Gamma_W(\mathcal{X}_s,\mathbb{Z}^c(d-n))\widehat{\underline{\otimes}}\mathbb{Q}_p[-2d] \qquad (39) \\ &\simeq R\Gamma_{et}(\mathcal{X}_s,\mathbb{Z}^c(d-n))\widehat{\underline{\otimes}}\mathbb{Q}_p[-2d] \\ &=: R\Gamma_{et}(\mathcal{X}_s,\mathbb{Q}_p^c(d-n))[-2d] \qquad (40) \end{aligned}$$

for any $n\geq 1$. The equivalence (39) is justified by the fact that $R\Gamma_W(\mathcal{X}_s,\mathbb{Z}^c(d-n))$ is expected to be a perfect complex of abelian groups. We also expect[7] an equivalence

$$R\Gamma_W(\mathcal{X}_s,\mathbb{Z}^c(d-n))\xrightarrow{\sim} R\mathrm{Hom}(R\Gamma_{Wh}(\mathcal{X}_s,\mathbb{Z}(d-n)),\mathbb{Z}[-1])$$

of perfect complexes of abelian groups.

– Suppose now that $n\leq 0$. Then we have

$$\begin{aligned} H^i_{ar}(\mathcal{X}_s,Ri^!\mathbb{Z}(n)) &\simeq H^i_W(\mathcal{X}_s,Ri^!\mathbb{Z}(n)) \\ &\simeq H^i_{et}(\mathcal{X}_s,Ri^!\mathbb{Z}(n)) \\ &\xrightarrow{\sim} H^{2d+1-i}_{ar}(\mathcal{X},\mathbb{R}/\mathbb{Z}(d-n))^D \\ &\simeq H^{2d+2-i}_{ar}(\mathcal{X},\mathbb{Z}(d-n))^D, \end{aligned}$$

where $H^{2d+1-i}_{ar}(\mathcal{X},\mathbb{R}/\mathbb{Z}(d-n))$ is isomorphic to $H^{2d+1-i}(C_{ar}(\mathcal{X},d-n))$ up to finite groups. Hence, we have

$$R\Gamma_{ar}(\mathcal{X}_s,Ri^!\mathbb{Z}(n))\underline{\otimes}^{\mathbb{L}}\mathbb{Q}_p\simeq 0 \qquad \text{for any } n\leq 0. \tag{41}$$

- For any $n\in\mathbb{Z}$, one has

$$\begin{aligned} R\Gamma_{ar}(\mathcal{X}_s,Ri^!\mathbb{Z}(n))\widehat{\underline{\otimes}}\mathbb{Q}_p &\simeq R\Gamma_{ar}(\mathcal{X}_s,Ri^!\mathbb{Q}_p(n)) \\ &\simeq R\Gamma_W(\mathcal{X}_s,Ri^!\mathbb{Z}(n))\widehat{\underline{\otimes}}\mathbb{Q}_p \\ &\simeq R\Gamma_{et}(\mathcal{X}_s,Ri^!\mathbb{Z}(n))\widehat{\underline{\otimes}}\mathbb{Q}_p \\ &=: R\Gamma_{et}(\mathcal{X}_s,Ri^!\mathbb{Q}_p(n)). \end{aligned}$$

Note that for any $n\leq 0$, the complex $R\Gamma_{ar}(\mathcal{X}_s,Ri^!\mathbb{Q}_p(n))$ is nontrivial by (35), (36) and the fact that $R\Gamma_{ar}(\mathcal{X}_s,\mathbb{Q}_p(d-n))\simeq 0$ as $d-n>\dim(\mathcal{X}_s)$. Therefore, the

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[7] See Theorem 3.8 for a special case.

map

$$R\Gamma_{ar}(\mathcal{X}_s, Ri^!\mathbb{Z}(n))\underline{\otimes}^{\mathbb{L}}\mathbb{Q}_p \to R\Gamma_{ar}(\mathcal{X}_s, Ri^!\mathbb{Z}(n))\widehat{\underline{\otimes}}\mathbb{Q}_p \tag{42}$$

is expected to be an equivalence if and only if one has

$$n \geq 1 \text{ and } R\Gamma_{et}(\mathcal{X}_s, \tau^{>n+1}Ri^!\mathbb{Z}(n))\widehat{\underline{\otimes}}\mathbb{Q}_p \simeq 0. \tag{43}$$

Now we observe that (43) holds if and only if $n \geq d$. This condition is indeed sufficient by [9, Corollary 7.2 (a)]. By (38) and by the de Rham description (36) of $C_{ar}(\mathcal{X}, d-n)$, the condition $n \geq d$ is also necessary for (43) to hold. Hence, the map (42) is an equivalence if and only if $n \geq d$. Moreover, it follows from (36) and (38) that

$$\begin{aligned}
&\dim_{\mathbb{Q}_p} H^{n+1}(R\Gamma_{ar}(\mathcal{X}_s, \tau^{>n+1}Ri^!\mathbb{Z}(n))\widehat{\underline{\otimes}}\mathbb{Q}_p) \\
&= \operatorname{corank}_p \left(H^{n+2}_{ar}(\mathcal{X}_s, \tau^{>n+1}Ri^!\mathbb{Z}(n))\right) \\
&= \operatorname{rank}_{\mathbb{Z}_p} H^{2d+1-n-2}(C_{ar}(\mathcal{X}, d-n)) \\
&= \dim_{\mathbb{Q}_p} H^{2d-1-n}(\mathcal{X}_K, \Omega^{<d-n}_{\mathcal{X}_K/K}) \\
&= 0
\end{aligned}$$

for any $n \geq 1$, since

$$2d - 1 - n > \dim(\mathcal{X}_K) + (d-n-1) = 2d - 2 - n.$$

Hence, the map (42) is an equivalence in cohomological degrees $\leq n+1$, for any $n \geq 1$. For $n \leq 0$, the left- (respectively, right-) hand side of (42) vanishes (respectively, is concentrated in cohomological degrees $> n+1$). Hence, (42) is an equivalence in cohomological degrees $\leq n+1$, for any Tate twist $n \in \mathbb{Z}$.

- For any $n \geq 0$, one has an equivalence

$$R\Gamma_{ar}(\mathcal{X}_K, \mathbb{Z}(n))\underline{\otimes}^{\mathbb{L}}\mathbb{Q}_p \simeq R\Gamma_{syn}(\mathcal{X}_K, n), \tag{44}$$

where the right-hand side is the Nekovar-Niziol syntomic cohomology [19]. Indeed, by (37), (40) and (41), one has a cofibre sequence

$$R\Gamma_{et}(\mathcal{X}_s, \mathbb{Q}_p^c(d-n))[-2d] \to R\Gamma_{et}(\mathcal{X}, \mathbb{Q}_p(n)) \to R\Gamma_{ar}(\mathcal{X}_K, \mathbb{Z}(n))\underline{\otimes}^{\mathbb{L}}\mathbb{Q}_p,$$

where the left map is induced by the adjunction maps $\tau^{\leq n+1}Ri^!\mathbb{Z}(n) \to Ri^!\mathbb{Z}(n)$ and $Ri_*Ri^! \to \mathrm{Id}$. But $R\Gamma_{syn}(\mathcal{X}_K, n)$ lies in the same cofibre sequence by [2, Corollaries 7.13 and 7.17], hence, (44) follows for any $n \geq 0$. For $n < 0$, the left-hand side of (44) vanishes. The induced map

$$R\Gamma_{syn}(\mathcal{X}_K, n) \simeq R\Gamma_{ar}(\mathcal{X}_K, \mathbb{Z}(n))\underline{\otimes}^{\mathbb{L}}\mathbb{Q}_p \to R\Gamma_{ar}(\mathcal{X}_K, \mathbb{Z}(n))\widehat{\underline{\otimes}}\mathbb{Q}_p \simeq R\Gamma_{et}(\mathcal{X}_K, \mathbb{Q}_p(n)) \tag{45}$$

is an equivalence if and only if $n \geq d$, as (42) is an equivalence if and only if $n \geq d$. For any Tate twist $n \geq 0$, the map (45) is an equivalence in cohomological degrees $\leq n$, as (42) is an equivalence in cohomological degrees $\leq n+1$.

- We have

$$\dim_{\mathbb{Q}_l} H^i_{ar}(\mathcal{X}_K, \mathbb{Q}_l(n)) = \dim_{\mathbb{R}} H^i_{ar}(\mathcal{X}_K, \mathbb{R}(n))$$

  for any $i,n \in \mathbb{Z}$ and any prime $l \neq p$. In particular, the left-hand side is independent on $l \neq p$.

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