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Disjoint Hypercyclicity and Weighted Translations on Discrete Groups

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Abstract. Let $1 \le p < \infty$, and let G be a discrete group. We give a sufficient and necessary condition for weighted translation operators on the Lebesgue space $\ell^p(G)$ to be densely disjoint hypercyclic. The characterization for the dual of a weighted translation to be densely disjoint hypercyclic is also obtained.

1 Introduction

An operator T on a separable Banach space X is called *hypercyclic* if there is a vector $x \in X$ such that the orbit $\{x, Tx, \dots, T^n x, \dots\}$ is dense in X, where T^n denotes the n-th iterate of T and x is called a *hypercyclic vector* for T. In the investigation on hypercyclicity, the weighted shifts on $\ell^p(\mathbb{N}_0)$ or $\ell^p(\mathbb{Z})$ are concrete examples for researchers to construct and demonstrate the theory of hypercyclicity. For unilateral shifts T on $\ell^p(\mathbb{N}_0)$, Rolewicz [19] showed that αT is hypercyclic whenever $|\alpha| > 1$, which is the first example of a hypercyclic operator on a separable Banach space. In [21], Salas characterized hypercyclic bilateral weighted shifts on $\ell^p(\mathbb{Z})$. Also, Costakis and Sambarino [13] gave a sufficient and necessary condition for bilateral weighted shifts on $\ell^p(\mathbb{Z})$ to be mixing. In [14], Grosse-Erdmann characterized chaotic bilateral weighted shifts on $\ell^p(\mathbb{Z})$. The characterization for bilateral weighted shifts on $\ell^p(\mathbb{Z})$ to be Cesàro hypercyclic was given in [17].

Based on these works about weighted shifts, we have extended some results in [13, 14, 17, 21] to the setting of weighted translations on the Lebesgue space of a locally compact group in [8–12]. For more results and recent works on hypercyclicity, we refer the readers to these two books [1, 15] on the subject.

Over a decade ago, Bernal-González, Bès, and Peris introduced the new notion of hypercyclicity, namely, *disjoint* (or *diagonal*) *hypercyclicity* in [2] and [7], respectively. Since then, disjoint hypercyclicity has been studied intensively in [3–6, 18, 22–25]. For example, the existence of disjoint hypercyclic operators on separable, infinite-dimensional topological vector spaces was investigated in [24, 25]. In [3, 4], Bès, Martin, and Peris studied disjoint hypercyclic composition operators on spaces of holomorphic functions. Also, Salas considered dual disjoint hypercyclic operators in [22]. The characterizations for weighted shifts and powers of weighted shifts on $\ell^p(\mathbb{Z})$ to be disjoint hypercyclic and supercyclic were given in [6, 7, 18], respectively. Inspired

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by these works in [6, 7, 18, 22] together with our previous work, in this note we will characterize disjoint hypercyclic weighted translation operators on locally compact groups.

First we recall the definition of disjoint hypercyclicity introduced by Bernal-González, Bès, and Peris independently.

Definition 1.1 Given $N \geq 2$, the operators $T_1, T_2, \dots, T_N$ acting on a separable Banach space X are *disjoint hypercyclic*, or *diagonally hypercyclic* (*d-hypercyclic*) if there is some vector $(x, x, \dots, x)$ in the diagonal of $X^N = X \times X \times \cdots \times X$ such that

$$\{(x, x, \dots, x), (T_1 x, T_2 x, \dots, T_N x), (T_1^2 x, T_2^2 x, \dots, T_N^2 x), \dots\}$$

is dense in X^N where $x \in X$ is called a *d-hypercyclic vector* associated with the operators $T_1, T_2, \dots, T_N$. Moreover, $T_1, T_2, \dots, T_N$ are called *densely d-hypercyclic* if they have a dense set of d-hypercyclic vectors.

For a single operator T, it is known that hypercyclicity is equivalent to topological transitivity. An operator T is *topologically transitive* if given two nonempty open subsets $U, V \subset X$, there is some $n \in \mathbb{N}$ such that $T^n(U) \cap V \neq \varnothing$. If $T^n(U) \cap V \neq \varnothing$ from some n onward, then T is called *topologically mixing*. In [7], Bès and Peris extended the definitions of topological transitivity and mixing to the setting of disjointness.

Definition 1.2 Given $N \geq 2$, the operators $T_1, T_2, \dots, T_N$ on a separable Banach space X are *d-topologically transitive* if given nonempty open sets $V_0, V_1, \dots, V_N \subset X$, there is some $n \in \mathbb{N}$ such that

$$\varnothing \neq V_0 \cap T_1^{-n}(V_1) \cap T_2^{-n}(V_2) \cap \cdots \cap T_N^{-n}(V_N).$$

If the above condition is satisfied from some n onward, then $T_1, T_2, \dots, T_N$ are called *d-mixing*.

We note that in [7, Proposition 2.3], the operators $T_1, T_2, \dots, T_N$ are d-topologically transitive if and only if $T_1, T_2, \dots, T_N$ have a dense set of d-hypercyclic vectors. Also, the disjoint blow up/collapse property implies d-topological transitivity in [7, Proposition 2.4]. An operator T on a separable Banach space X is said to have the *blow up/collapse* property if for any three nonempty open subsets U, V, W of X with $0 \in W$, there exists some $n \in \mathbb{N}$ so that $W \cap T^{-n}(U) \neq \varnothing$ and $V \cap T^{-n}(W) \neq \varnothing$. Accordingly, the definition of *disjoint blow up/collapse* property is given in [7].

Definition 1.3 Given $N \geq 2$, the operators $T_1, T_2, \dots, T_N$ on a separable Banach space X satisfy the *disjoint blow up/collapse* property if given any nonempty open sets $W, V_0, V_1, \dots, V_N \subset X$ with $0 \in W$, there is some $n \in \mathbb{N}$ such that

$$\varnothing \neq W \cap T_1^{-n}(V_1) \cap T_2^{-n}(V_2) \cap \cdots \cap T_N^{-n}(V_N),$$
$$\varnothing \neq V_0 \cap T_1^{-n}(W) \cap T_2^{-n}(W) \cap \cdots \cap T_N^{-n}(W).$$

We note that it has been shown in [6, Theorem 2.1] that the bilateral weighted shifts on $\ell^p(\mathbb{Z})$ are d-hypercyclic if and only if they satisfy the disjoint blow up/collapse property. In fact, the weighted shifts on $\ell^p(\mathbb{Z})$ are a special case of the weighted translations on the Lebesgue space of a locally compact group. Hence, the aim of this note is to characterize densely disjoint hypercyclic weighted translations and show that dense d-hypercyclicity and the disjoint blow up/collapse property can only occur simultaneously on weighted translations.

2 Disjoint Hypercyclicity

As in [8–12], the result on the ℓ^p-space of a discrete group in this note can be extended without difficulty to the L^p-space of a locally compact group. For simplicity and exposing the essential idea, we will confine our discussion to discrete groups.

Throughout, let G be a discrete group with identity e and an invariant Haar measure λ that is the counting measure here. We denote by $\ell^p(G)$ $(1 \le p < \infty)$ the real Lebesgue space, with respect to λ. A bounded function $w: G \to (-\infty, 0) \cup (0, \infty)$ is called a *weight* on G. Let $a \in G$ and let δ_a be the unit point mass at a. A *weighted translation* on G is a weighted convolution operator $T_{a,w}: \ell^p(G) \to \ell^p(G)$ defined by

$$T_{a,w}(f) = wT_a(f) \qquad \big(f \in \ell^p(G)\big),$$

where w is a weight on G and $T_a(f) = f * \delta_a \in \ell^p(G)$ is the convolution:

$$(f * \delta_a)(x) = \sum_{y \in G} f(xy^{-1})\delta_a(y) = f(xa^{-1}) \qquad (x \in G).$$

We also define a self-map $S_{a,w}$ on the subspace $\ell^p_c(G)$ of functions in $\ell^p(G)$ with finite support by

$$S_{a,w}(h) = \frac{h}{w} * \delta_{a^{-1}} \qquad (h \in \ell^p_c(G))$$

so that

$$T_{a,w}S_{a,w}(h) = h \qquad (h \in \ell^p_c(G)).$$

Since the weighted translation $T_{a,w}$ is generated by the group element a and the weight function w, some elements $a \in G$ and weights w should be excluded from our consideration. For example, if $\|w\|_\infty \le 1$, then $\|T_{a,w}\| \le 1$ and $T_{a,w}$ is never hypercyclic. Also, it has been shown in [12] that $T_{a,w}$ is not hypercyclic if a is a torsion element of G. Hence the weighted translation operators cannot be disjoint hypercyclic if they are generated by torsion elements. An element a in a group G is called a *torsion element* if it is of finite order. It was characterized in [11, Lemma 2.7] that an element a of a discrete group G is not torsion (non-torsion) if and only if for any finite set $K \subset G$, there exists some $N \in \mathbb{N}$ such that $K \cap Ka^{\pm n} = \varnothing$ for all $n > N$.

Now we are ready to prove the main result by applying the property of non-torsion elements and the following useful result from [11, Lemma 2.6].

Lemma 2.1 ([11, Lemma 2.6]) *Let G be a discrete group. For each $n \in \mathbb{N}$, let $\varphi_n: G \to (-\infty, 0) \cup (0, \infty)$ be a function defined on G. Then the following conditions are equivalent.*

(i) *Given $\varepsilon > 0$, a finite set $K \subset G$ and $N \in \mathbb{N}$, there exists $m > N$ satisfying $|\varphi_m(x)| < \varepsilon$ for all $x \in K$.*
(ii) *The sequence $(|\varphi_n|)$ admits a subsequence $(|\varphi_{n_k}|)$ that converges uniformly to* 0 *on each finite subset D of G.*

Here we follow the ideas in the proof of [6, Theorem 2.1] to obtain the result. It is interesting to learn that Han and Liang recently also characterized disjoint hypercyclic weighted translations on groups in [16] by using a different approach.

Theorem 2.2 *Let G be a discrete group and let a be a non-torsion element in G. Let $1 \le p < \infty$. Given some $N \ge 2$, let $w_l: G \to (-\infty, 0) \cup (0, \infty)$ be a weight on G, and let $T_l = T_{a,w_l}$ be a weighted translation on the real space $\ell^p(G)$, generated by a and w_l for $1 \le l \le N$. Define*

$$\varphi_{l,n} = \prod_{j=1}^{n} w_l * \delta_{a^{-1}}^j, \quad \widetilde{\varphi}_{l,n} = \Big(\prod_{j=0}^{n-1} w_l * \delta_a^j\Big)^{-1} \quad \text{and} \quad \phi_{l,n} = \frac{\widetilde{\varphi}_{1,n}}{\widetilde{\varphi}_{l,n}} = \frac{\prod_{j=0}^{n-1} w_l * \delta_a^j}{\prod_{j=0}^{n-1} w_1 * \delta_a^j}.$$

Then the following conditions are equivalent.

(i) *$T_1, T_2, \ldots, T_N$ have a dense set of d-hypercyclic vectors.*
(ii) *$T_1, T_2, \ldots, T_N$ satisfy the disjoint blow up/collapse property.*
(iii) *There exists a sequence $(n_k)_{k\in\mathbb{N}} \subset \mathbb{N}$ such that $(|\varphi_{l,n_k}|)$ and $(|\widetilde{\varphi}_{1,n_k}|)$ converge to 0 pointwise in G for $1 \le l \le N$, and the set*

$$\big\{ (\phi_{2,n_k}(x), \phi_{3,n_k}(x), \ldots, \phi_{N,n_k}(x)) : k \in \mathbb{N} \big\}$$

is dense in $\mathbb{R}^{N-1}$ with respect to the product topology for all $x \in G$.

Proof (iii) $\Rightarrow$ (ii). Let $W, V_0, V_1, \ldots, V_N$ be non-empty open subsets of $\ell^p(G)$ with $0 \in W$. Since the space $C_c(G)$ of continuous functions on G with finite support is dense in $\ell^p(G)$, we can pick $f_0, f_1, \ldots, f_N \in C_c(G)$ with $f_l \in V_l$ for each l. Let K be the union of the finite supports of all f_l. We can further assume $f_1(x) \ne 0$ for all $x \in K$. Let the sequences $(\varphi_{l,n_k}), (\widetilde{\varphi}_{1,n_k})$, and (ϕ_{l,n_k}) satisfy condition (iii). Choose $\varepsilon > 0$ such that $B(0,\varepsilon) := \{g \in \ell^p(G) : \|g - 0\|_p < \varepsilon\} \subseteq W$, and $B(f_l, \varepsilon) \subseteq V_l$ for $0 \le l \le N$. By condition (iii), there exists $N_1 \in \mathbb{N}$ such that for $k \ge N_1$, we have $|\varphi_{l,n_k}| < \frac{\varepsilon}{\|f_0\|_p}$ on K and $\|\phi_{l,n_k} f_1 - f_l\|_p < \varepsilon$ for $2 \le l \le N$.

First, for $1 \le l \le N$ and $k \ge N_1$, we have

$$\begin{aligned}\|T_l^{n_k} f_0\|_p^p &= \sum_G \Big|\prod_{j=0}^{n_k-1} w_l(xa^{-j}) f_0(xa^{-n_k})\Big|^p = \sum_G \Big|\prod_{j=1}^{n_k} w_l(xa^j) f_0(x)\Big|^p \\ &= \sum_K |\varphi_{l,n_k}(x)|^p |f_0(x)|^p < \varepsilon^p,\end{aligned}$$

which says $T_l^{n_k} f_0 \in W$; that is,

$$\varnothing \ne V_0 \cap T_1^{-n_k}(W) \cap \cdots \cap T_N^{-n_k}(W).$$

Applying a similar argument to the iterates $S_1^{n_k}$, and making use of the sequence $(\widetilde{\varphi}_{1,n_k})$, one has

$$\begin{aligned}\|S_1^{n_k} f_1\|_p^p &= \sum_G \frac{1}{|\prod_{j=1}^{n_k} w_1(xa^j)|^p} |f_1(xa^{n_k})|^p = \sum_G \frac{1}{|\prod_{j=0}^{n_k-1} w_l(xa^{-j})|^p} |f_1(x)|^p \\ &= \sum_K |\widetilde{\varphi}_{1,n_k}(x)|^p |f_1(x)|^p \longrightarrow 0\end{aligned}$$

as $k \to \infty$, which implies $S_1^{n_k} f_1 \in W$. Also, we have

$$\|T_l^{n_k} S_1^{n_k} f_1 - f_l\|_p = \|\phi_{l,n_k} f_1 - f_l\|_p < \varepsilon$$

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saying $S_1^{n_k} f_1 \in T_l^{-n_k} V_l$ for $2 \le l \le N$. Together with $T_1^{n_k} S_1^{n_k} f_1 = f_1 \in V_1$, we have

$$\varnothing \ne W \cap T_1^{-n_k}(V_1) \cap \cdots \cap T_N^{-n_k}(V_N).$$

Therefore, $T_1, T_2, \dots, T_N$ satisfy the disjoint blow up/collapse property.

Since condition (ii) implies (i) by [7, Propositions 2.3 and 2.4], we only need to show condition (i) implies (iii).

(i) $\Rightarrow$ (iii). For each $n \in \mathbb{N}$ and $2 \le l \le N$, let $h_{l,n}$ be a bounded function defined on G. Assume that $\{(h_{2,n_k}(x), h_{3,n_k}(x), \dots, h_{N,n_k}(x)) : k \in \mathbb{N}\}$ is a dense set in $\mathbb{R}^{N-1}$ for each $x \in G$. Let $K \subset G$ be a finite set, and let $\chi_K \in \ell^p(G)$ be the characteristic function of K. Let $\varepsilon \in (0,1)$. Then there exist a d-hypercyclic vector $f \in \ell^p(G)$ and some $r \in \mathbb{N}$ such that

$$\|f - \chi_K\|_p < \varepsilon \qquad \text{and} \qquad \|T_l^r f - \chi_K\|_p < \varepsilon$$

for $l = 1, 2, \dots, N$, which says

$$|f(x)| > 1 - \varepsilon \qquad (x \in K), \qquad |f(x)| < \varepsilon \quad \text{for} \quad x \in G \setminus K \tag{2.1}$$

and

$$|\widetilde{\varphi}_{l,r}(x)|^{-1} |f(xa^{-r})| > 1 - \varepsilon \qquad (x \in K, 1 \le l \le N). \tag{2.2}$$

Since a is non-torsion, there is some N_2 such that $K \cap Ka^{\pm n} = \varnothing$ for all $n > N_2$. Hence there is some $m \in \mathbb{N}$ such that $m - r > N_2$,

$$\|T_1^{m+r} f - (\varphi_{1,r} * \delta_a^r)(\chi_K * \delta_a^r)\|_p < \varepsilon \tag{2.3}$$

and

$$\|T_l^{m+r} f - (h_{l,m} * \delta_a^r)(\varphi_{l,r} * \delta_a^r)(\chi_K * \delta_a^r)\|_p < \varepsilon \qquad (2 \le l \le N). \tag{2.4}$$

Hence, by (2.3), we have

$$\begin{aligned}
\varepsilon^p &> \|T_1^{m+r} f - (\varphi_{1,r} * \delta_a^r)(\chi_K * \delta_a^r)\|_p^p \\
&= \sum_G |T_1^{m+r} f(x) - \varphi_{1,r}(xa^{-r})\chi_K(xa^{-r})|^p \\
&= \sum_G \Big| \prod_{j=0}^{m-1} w_1(xa^{-j})(T_1^r f)(xa^{-m}) - \varphi_{1,r}(xa^{-r})\chi_K(xa^{-r})\Big|^p \\
&= \sum_G \Big| \prod_{j=1}^{m} w_1(xa^{j})(T_1^r f)(x) - \varphi_{1,r}(xa^{m-r})\chi_K(xa^{m-r})\Big|^p \\
&= \sum_G |\varphi_{1,m}(x)\widetilde{\varphi}_{1,r}(x)^{-1} f(xa^{-r}) - \varphi_{1,r}(xa^{m-r})\chi_K(xa^{m-r})|^p,
\end{aligned}$$

which says

$$|\varphi_{1,m}(x)\widetilde{\varphi}_{1,r}(x)^{-1} f(xa^{-r})| < \varepsilon \qquad (x \in K)$$

by $K \cap Ka^{m-r} = \varnothing$. Combining the inequality above with (2.2), we have, for $x \in K$,

$$|\varphi_{1,m}(x)| < \frac{\varepsilon}{|\widetilde{\varphi}_{1,r}(x)^{-1} f(xa^{-r})|} < \frac{\varepsilon}{1 - \varepsilon}. \tag{2.5}$$

Similarly, by (2.4), we have

$$|\varphi_{l,m}(x)\widetilde{\varphi}_{l,r}(x)^{-1} f(xa^{-r})| < \varepsilon \qquad (x \in K).$$

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Hence,

$$|\varphi_{l,m}(x)| < \frac{\varepsilon}{1-\varepsilon} \qquad (x \in K, 2 \le l \le N). \tag{2.6}$$

On the other hand, by (2.3), one has the following inequality:

$$\begin{aligned}
\varepsilon^p &> \|T_1^{m+r} f - (\varphi_{1,r} * \delta_a^r)(\chi_K * \delta_a^r)\|_p^p \\
&= \sum_G |T_1^{r+m} f(x) - \varphi_{1,r}(xa^{-r})\chi_K(xa^{-r})|^p \\
&= \sum_G |\prod_{j=0}^{r-1} w_1(xa^{-j})(T_1^m f)(xa^{-r}) - \varphi_{1,r}(xa^{-r})\chi_K(xa^{-r})|^p \\
&= \sum_G |\prod_{j=1}^{r} w_1(xa^{j})(T_1^m f)(x) - \varphi_{1,r}(x)\chi_K(x)|^p \\
&= \sum_G |\varphi_{1,r}(x)\widetilde{\varphi}_{1,m}(x)^{-1} f(xa^{-m}) - \varphi_{1,r}(x)\chi_K(x)|^p.
\end{aligned}$$

Therefore, for $x \in K$, one has

$$|\varphi_{1,r}(x)||\widetilde{\varphi}_{1,m}(x)^{-1} f(xa^{-m}) - 1| < \varepsilon.$$

Hence,

$$|\widetilde{\varphi}_{1,m}(x)^{-1} f(xa^{-m}) - 1| < \frac{\varepsilon}{|\varphi_{1,r}(x)|} \le \frac{\varepsilon}{M} \qquad (x \in K), \tag{2.7}$$

where $M := \min\{|\varphi_{1,r}(x)| : x \in K\} > 0$. By (2.1) and (2.7), we arrive at

$$|\widetilde{\varphi}_{1,m}(x)| < \frac{|f(xa^{-m})|}{1-\frac{\varepsilon}{M}} < \frac{M\varepsilon}{M-\varepsilon} \qquad (x \in K). \tag{2.8}$$

Similarly, by (2.4), we have

$$|\widetilde{\varphi}_{l,m}(x)^{-1} f(xa^{-m}) - h_{l,m}(x)| < \frac{\varepsilon}{M} \qquad (x \in K, 2 \le l \le N). \tag{2.9}$$

Finally, combining (2.8), (2.9), and (2.7), we have, for $x \in K$,

$$\begin{aligned}
|\phi_{l,m}(x) - h_{l,m}(x)| &= |\widetilde{\varphi}_{1,m}(x)\widetilde{\varphi}_{l,m}(x)^{-1} - h_{l,m}(x)| \\
&= |\widetilde{\varphi}_{1,m}(x)||\widetilde{\varphi}_{l,m}(x)^{-1} - h_{l,m}(x)\widetilde{\varphi}_{1,m}(x)^{-1}| \\
&< \frac{|f(xa^{-m})|}{1-\frac{\varepsilon}{M}}|\widetilde{\varphi}_{l,m}(x)^{-1} - h_{l,m}(x)\widetilde{\varphi}_{1,m}(x)^{-1}| \\
&= \frac{1}{1-\frac{\varepsilon}{M}}|\widetilde{\varphi}_{l,m}(x)^{-1} f(xa^{-m}) - h_{l,m}(x)\widetilde{\varphi}_{1,m}(x)^{-1} f(xa^{-m})| \\
&\le \frac{1}{1-\frac{\varepsilon}{M}}|\widetilde{\varphi}_{l,m}(x)^{-1} f(xa^{-m}) - h_{l,m}(x)| \\
&\quad + \frac{1}{1-\frac{\varepsilon}{M}}|h_{l,m}(x)||1 - \widetilde{\varphi}_{1,m}(x)^{-1} f(xa^{-m})| \\
&< \frac{1}{1-\frac{\varepsilon}{M}}\Big(\frac{\varepsilon}{M} + \|h_{l,m}\|_\infty \frac{\varepsilon}{M}\Big) = \varepsilon\Big(\frac{1+\|h_{l,m}\|_\infty}{M-\varepsilon}\Big).
\end{aligned}$$

From the last inequality as well as from (2.5), (2.6), and (2.8), we observe that $(|\varphi_{l,n}|)$, $(|\widetilde{\varphi}_{1,n}|)$, and $(|\phi_{l,n} - h_{l,n}|)$ satisfy Lemma 2.1(i) for each finite set $K \subset G$. It follows from Lemma 2.1(ii) that these sequences admit a subsequence $(n_k) \subset \mathbb{N}$ such that $(|\varphi_{l,n_k}|)$ and $(|\widetilde{\varphi}_{1,n_k}|)$ converge to 0 pointwise in G, and the set

$$\big\{(\phi_{2,n_k}(x), \phi_{3,n_k}(x), \dots, \phi_{N,n_k}(x)) : k \in \mathbb{N}\big\}$$

is dense in $\mathbb{R}^{N-1}$ for all $x \in G$. ∎

Example 2.3 Let $G = \mathbb{Z}$ and $a = -1 \in \mathbb{Z}$. Given $N \geq 2$, let $w_l * \delta_{-1}$ be a weight on $\mathbb{Z}$ for $l = 1, 2, \dots, N$. Then the weighted translation operator $T_{-1, w_l * \delta_{-1}}$ is defined by

$$T_{-1, w_l * \delta_{-1}} f(i) = w_l(i+1) f(i+1) \qquad (f \in \ell^p(\mathbb{Z})).$$

Hence, the operator $T_{-1, w_l * \delta_{-1}}$ is just the bilateral weighted backward shift T_l, given by $T_l e_i = w_{l,i} e_{i-1}$ with $w_{l,i} = w_l(i)$. That is, $T_l = T_{-1, w_l * \delta_{-1}}$ for $1 \leq l \leq N$. Here $(e_i)_{i \in \mathbb{Z}}$ is the canonical basis of $\ell^p(\mathbb{Z})$ and $(w_{l,i})_{i \in \mathbb{Z}}$ is a sequence of nonzero real numbers. By Theorem 2.2, the operators $T_1, T_2, \dots, T_N$ are densely disjoint hypercyclic if there exists a strictly increasing sequence $(n_k)_{k \in \mathbb{N}}$ of positive integers such that

$$|\varphi_{l,n_k}(i)| = \Big| \prod_{j=1}^{n_k} (w_l * \delta_{-1}) * \delta_1^j(i) \Big| = \Big| \prod_{j=0}^{n_k - 1} w_l(i-j) \Big| \longrightarrow 0 \qquad (1 \leq l \leq N)$$

and

$$|\widetilde{\varphi}_{1,n_k}(i)|^{-1} = \Big| \prod_{j=0}^{n_k-1} (w_1 * \delta_{-1}) * \delta_{-1}^j(i) \Big| = \Big| \prod_{j=1}^{n_k} w_1(i+j) \Big| \longrightarrow \infty,$$

as $k \to \infty$ for all integers i, and the set $\{(\phi_{2,n_k}(i), \phi_{3,n_k}(i), \dots, \phi_{N,n_k}(i)) : k \in \mathbb{N}\}$ is dense in $\mathbb{R}^{N-1}$ for all $i \in \mathbb{Z}$, where

$$\phi_{l,n_k}(i) = \frac{\widetilde{\varphi}_{1,n_k}(i)}{\widetilde{\varphi}_{l,n_k}(i)} = \prod_{j=1}^{n_k} \frac{w_l(i+j)}{w_1(i+j)}$$

for $l = 2, 3, \dots, N$.

Finally, we conclude the paper with a discussion of dual disjoint hypercyclicity. Recall that a hypercyclic operator T whose dual T^* is also hypercyclic is called a *dual hypercyclic* operator. In [20], Salas constructed a dual hypercyclic weighted shift on a Hilbert space. Similarly, the d-hypercyclic operators $T_1, T_2, \dots, T_N$ are said to be *dual d-hypercyclic* in [22] if $T_1^*, T_2^*, \dots, T_N^*$ are also d-hypercyclic where T_l^* is the dual of T_l for $1 \leq l \leq N$. It was shown in [22, 24] independently that a separable Banach space supports dual d-hypercyclic operators. We note that Bès and Peris considered a weighted bilateral forward shift A with dual A^* on $\ell^2(\mathbb{Z})$ and showed that both the operators $A, A^2, \dots, A^N$ and $A^*, A^{*2}, \dots, A^{*N}$ are d-hypercyclic under some weight sequences in [7, Theorem 4.11]. Here, we will characterize disjoint hypercyclicity for the dual of weighted translation operators by applying Theorem 2.2 directly.

Let $p \in [1, \infty)$ with conjugate exponent q, and let $\langle \cdot, \cdot \rangle : \ell^p(G) \times \ell^q(G) \to \mathbb{C}$ be the duality. A simple computation gives

$$\langle T_{a,w} f, g \rangle = \langle f, T_{a^{-1}}(wg) \rangle \qquad \big(f \in \ell^p(G), g \in \ell^q(G)\big).$$

Therefore, the dual map $T_{a,w}^* : \ell^q(G) \to \ell^q(G)$ is given by

$$T_{a,w}^*(g) = T_{a^{-1}}(wg) = T_{a^{-1}, w * \delta_{a^{-1}}}(g) \qquad \big(g \in \ell^q(G)\big),$$

which is also a weighted translation operator on $\ell^q(G)$.

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The following result reveals that for certain weight functions, both the weighted translation operators $T_1, T_2, \dots, T_N$ and their dual operators $T_1^*, T_2^*, \dots, T_N^*$ can be d-hypercyclic at the same time.

Corollary 2.4 *Let G be a discrete group and let a be non-torsion element in G. Given some $N \geq 2$, let $T_l = T_{a,w_l}$ be a weighted translation on $\ell^p(G)$, generated by a and a weight w_l for $1 \leq l \leq N$. Let $T_l^* = T_{a^{-1}, w_l * \delta_{a^{-1}}}$ be the dual of T_l. Then the operators $T_1^*, T_2^*, \dots, T_N^*$ have a dense set of d-hypercyclic vectors if each weight $w_l * \delta_{a^{-1}}$ satisfies condition* (iii) *for a^{-1} in Theorem 2.2; that is, these sequences*

$$\varphi_{l,n}^* := \prod_{j=1}^{n} (w_l * \delta_{a^{-1}}) * \delta_{(a^{-1})^{-1}}^j \quad and \quad \widetilde{\varphi}_{l,n}^* := \Big(\prod_{j=0}^{n-1} (w_l * \delta_{a^{-1}}) * \delta_{a^{-1}}^j \Big)^{-1},$$

and

$$\phi_{l,n}^* := \frac{\widetilde{\varphi}_{1,n}^*}{\widetilde{\varphi}_{l,n}^*} = \frac{\prod_{j=0}^{n-1} (w_l * \delta_{a^{-1}}) * \delta_{a^{-1}}^j}{\prod_{j=0}^{n-1} (w_1 * \delta_{a^{-1}}) * \delta_{a^{-1}}^j}$$

satisfy the weight conditions in Theorem 2.2.

Example 2.5 As in Example 2.3, let $G = \mathbb{Z}$ and $a = -1 \in \mathbb{Z}$. Given $N \geq 2$, let $w_l * \delta_{-1}$ be a weight on $\mathbb{Z}$ for $l = 1, 2, \dots, N$. Then the weighted translation operator $T_l := T_{-1, w_l * \delta_{-1}}$ given by

$$T_{-1, w_l * \delta_{-1}} f(i) = w_l(i+1) f(i+1) \qquad \big(f \in \ell^p(\mathbb{Z}) \big)$$

is the bilateral weighted backward shift on $\ell^p(\mathbb{Z})$. Let T_l^* be the dual of T_l for $1 \leq l \leq N$. By Corollary 2.4, the operators $T_1^*, T_2^*, \dots, T_N^*$ are densely disjoint hypercyclic if there exists a strictly increasing sequence $(n_k)_{k \in \mathbb{N}}$ of positive integers such that

$$|\varphi_{l,n_k}^*(i)| = \Big| \prod_{j=1}^{n_k} ((w_l * \delta_{-1}) * \delta_1) * \delta_{-1}^j(i) \Big| = \Big| \prod_{j=1}^{n_k} w_l(i+j) \Big| \longrightarrow 0 \qquad (1 \leq l \leq N)$$

and

$$|\widetilde{\varphi}_{1,n_k}^*(i)|^{-1} = \Big| \prod_{j=0}^{n_k-1} ((w_1 * \delta_{-1}) * \delta_1) * \delta_1^j(i) \Big| = \Big| \prod_{j=0}^{n_k-1} w_1(i-j) \Big| \longrightarrow \infty,$$

as $k \to \infty$ for all integer i, and the set $\{(\phi_{2,n_k}^*(i), \phi_{3,n_k}^*(i), \dots, \phi_{N,n_k}^*(i)) : k \in \mathbb{N}\}$ is dense in $\mathbb{R}^{N-1}$ for all $i \in \mathbb{Z}$, in which

$$\phi_{l,n_k}^*(i) = \frac{\widetilde{\varphi}_{1,n_k}^*(i)}{\widetilde{\varphi}_{l,n_k}^*(i)} = \prod_{j=0}^{n_k-1} \frac{w_l(i-j)}{w_1(i-j)} \qquad (2 \leq l \leq N).$$

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