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Linear homeomorphisms of function spaces and the position of a space in its compactification

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Abstract. An old question of Arhangel'skii asks if the Menger property of a Tychonoff space X is preserved by homeomorphisms of the space $C_p(X)$ of continuous real-valued functions on X endowed with the pointwise topology. We provide affirmative answer in the case of linear homeomorphisms. To this end, we develop a method of studying invariants of linear homeomorphisms of function spaces $C_p(X)$ by looking at the way X is positioned in its (Čech–Stone) compactification.

1 Introduction

The present paper is concerned mainly with two classical covering-type properties of a topological space X, the Menger property and the Hurewicz property, and their connections with the linear-topological structure of the space $C_p(X)$ of continuous real-valued functions on X equipped with the pointwise topology. All spaces under consideration are assumed to be Tychonoff.

An old question of Arhangel'skii (cf. [2, Problem II.2.8] or [24, Problem 4.2.12]) asks if the Menger property[1] of a space X is preserved by homeomorphisms of its function space $C_p(X)$. One of the main results of this paper is the following theorem, which settles this question in the case of linear homeomorphisms.

Theorem 1.1 *Suppose that $C_p(X)$ and $C_p(Y)$ are linearly homeomorphic. Then X is a Menger space if and only if Y is a Menger space.*

Let us recall that a topological space X is *Menger* (resp., *Hurewicz*) if for every sequence $(\mathscr{U}_n)_{n\in\mathbb{N}}$ of open covers of X, there is a sequence $(\mathscr{V}_n)_{n\in\mathbb{N}}$ such that for every n, $\mathscr{V}_n$ is a finite subfamily of $\mathscr{U}_n$ and the family $\bigcup_{n\in\mathbb{N}}\mathscr{V}_n$ covers X (resp., every point of X is contained in $\bigcup\mathscr{V}_n$ for all but finitely many n's). These classical notions go back to early works of Witold Hurewicz and Karl Menger. Since then, they were studied by many authors and found numerous applications (see [19] and the references therein).

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[1]There is inconsistency in the terminology that is used in the literature. What we (and most of the modern authors) call the Menger property by some authors is called the Hurewicz property. In this paper, by the Hurewicz property, we mean something else.

Clearly,

$$\sigma\text{-compact} \Rightarrow \text{Hurewicz} \Rightarrow \text{Menger} \Rightarrow \text{Lindelöf}.$$

It is known that no implication above is reversible. There has been a lot of work done on the interplay between the linear topological structure of a function space $C_p(X)$ and topological properties of underlying space X; we refer the interested reader to the monograph [24]. One of the major results in this area of research is the following deep theorem of Velichko [28] (the theorem below was further generalized by Bouziad [7] to arbitrary Lindelöf numbers).

Theorem 1.2 (Velichko) *Suppose that $C_p(X)$ and $C_p(Y)$ are linearly homeomorphic. Then X is Lindelöf if and only if Y is Lindelöf.*

At the other extreme, it is relatively easy to show that σ-compactness of X is determined by the linear-topological structure of the function space $C_p(X)$ (see, e.g., [27, Theorem 6.9.1]) (actually more is true: σ-compactness of X can be characterized by a certain topological property of $C_p(X)$; see [15] or [2, Section III.2]). For the Hurewicz property, the following theorem was proved by Zdomskyy [29, Corollary 7].

Theorem 1.3 (Zdomskyy) *Suppose that $C_p(X)$ and $C_p(Y)$ are linearly homeomorphic. Then X is a Hurewicz space if and only if Y is a Hurewicz space.*

Regarding the Menger property, analogous assertion is provided by our Theorem 1.1. Though, some partial results were known before. In [29], Zdomskyy showed that in the linear case, the answer to Arhangel'skii's question mentioned above, is affirmative under an additional set-theoretic assumption $\mathfrak{u} < \mathfrak{g}$ (see [29, Corollary 7]). More recently, Sakai [18] gave a partial solution in ZFC[2] (see [18, Theorem 2.5]).

In the proof of Theorem 1.1, Velichko's Theorem 1.2 plays an important role. This is because of the following observation essentially due to Telgársky (see [23, Proposition 2]; cf. [5, Proposition 8]).

Proposition 1.4 *A space X is Menger if and only if X is Lindelöf and every separable metrizable continuous image of X is Menger.*

Analogous fact is also true for the Hurewicz property (see [11, Theorem 3.2] or [5, Proposition 31]). A space X is called *projectively Menger* (resp., *projectively Hurewicz*) provided every separable metrizable continuous image of X is Menger (resp., Hurewicz). According to Theorem 1.2 and Proposition 1.4, Theorem 1.1 reduces to the following result, which we prove in this paper.

Theorem 1.5 *Suppose that $C_p(X)$ and $C_p(Y)$ are linearly homeomorphic. Then X is projectively Menger if and only if Y is projectively Menger.*

We also prove a similar theorem for the projective Hurewicz property.

Theorem 1.6 *Suppose that $C_p(X)$ and $C_p(Y)$ are linearly homeomorphic. Then X is projectively Hurewicz if and only if Y is projectively Hurewicz.*

[2] The abbreviation ZFC stands for "Zermelo–Fraenkel set theory with the axiom of choice."

From Theorem 1.6, we immediately get Zdomskyy's Theorem 1.3 as a corollary. The above results answer questions asked in [18] by Sakai.

Our approach relies on the fact that the (projective) properties of Menger and Hurewicz of a space X can be conveniently expressed in terms of the Čech–Stone compactification βX of X. We develop a method of studying invariants of linear homeomorphisms of function spaces $C_p(X)$ by looking at the way X is positioned in its Čech–Stone compactification.

2 Notation and auxiliary results

In this section, we collect some notation and auxiliary results that we shall use throughout the paper.

2.1 Hyperspaces and set-valued maps

For a topological space X, by $\mathcal{K}(X)$, we denote the set of all nonempty compact subsets of X. We endow $\mathcal{K}(X)$ with the *Vietoris topology*, i.e., the topology generated by basic open sets of the form

$$\langle \mathcal{U} \rangle = \{K \in \mathcal{K}(X) : \forall U \in \mathcal{U} \quad K \cap U \neq \varnothing \text{ and } K \subseteq \bigcup \mathcal{U}\},$$

where $\mathcal{U} = \{U_1, \ldots, U_n\}$ is a finite collection of open subsets of X.

For an integer $n \geq 1$, we put $[X]^{\leq n} = \{K \in \mathcal{K}(X) : |K| \leq n\}$ and $[X]^n = [X]^{\leq n} \setminus [X]^{\leq n-1}$, i.e., $[X]^{\leq n}$ ($[X]^n$) is the subspace of $\mathcal{K}(X)$ consisting of all at most (precisely) n-element subsets of X.

A set-valued map $\phi : X \to \mathcal{K}(Y)$ is *lower semi-continuous* if the set

$$\phi^{-1}(U) = \{x \in X : \phi(x) \cap U \neq \varnothing\}$$

is open, for every open $U \subseteq Y$.

Let us note the following simple fact.

Lemma 2.1 *For a space X and a compact space Z, let $\phi : X \to \mathcal{K}(Z)$ be a lower semi-continuous map. If K is a compact G_δ-subset of Z, then the set $\phi^{-1}(K) = \{x \in X : \phi(x) \cap K \neq \varnothing\}$ is G_δ in X.*

Proof The set K is compact G_δ, so there is a sequence $U_1, U_2, \ldots$ of open subsets of Z such that $U_{n+1} \subseteq \overline{U_{n+1}} \subseteq U_n$, for every $n \geq 1$, and $K = \bigcap_{n=1}^{\infty} U_n = \bigcap_{n=1}^{\infty} \overline{U_n}$. Since the map ϕ is lower semi-continuous, it suffices to check that

$$\phi^{-1}(K) = \bigcap_{n=1}^{\infty} \phi^{-1}(U_n).$$

Pick $x \in \bigcap_{n=1}^{\infty} \phi^{-1}(U_n)$, i.e., for every $n \geq 1$, we have $\phi(x) \cap U_n \neq \varnothing$. The map ϕ has compact values and Z is compact. Hence, the intersection of the (decreasing) family $\{\phi(x) \cap \overline{U_n} : n = 1, 2, \ldots\}$ of nonempty closed subsets of Z must be nonempty, i.e., we have $\phi(x) \cap \bigcap_{n=1}^{\infty} \overline{U_n} = \phi(x) \cap K \neq \varnothing$. This gives $x \in \phi^{-1}(K)$. The converse inclusion is obvious. ∎

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If $\phi : X \to \mathcal{K}(Y)$ is a set-valued map and $A \subseteq X$, then we define the image $\phi(A)$ of A under ϕ as

$$\phi(A) = \bigcup\{\phi(x) : x \in A\}.$$

A subset S of Y that meets all values of $\phi : X \to \mathcal{K}(Y)$ is called a *section* of ϕ. The following theorem can be attributed to Bouziad (cf. [6, Theorem 6]).

Theorem 2.2 *Suppose that X is a G_δ subspace of a compact space. If C is compact, then every lower semi-continuous function $\phi : C \to \mathcal{K}(X)$ admits a compact section.*

Proof This is a direct consequence of [6, Theorem 2] and [8, Theorem 4.1]. ■

2.2 The *k*-Porada game

Let us recall the description of a certain topological game that will be of great importance in the proof of Theorem 1.1. The game defined below is so-called k-modification (instead of points one considers compact sets) of a game introduced in [17]. It was studied in [23] and, more recently, in [13]. Our terminology follows [23]. Let Z be a space, and let $X \subseteq Z$ be a subspace of Z.

The *k-Porada game* on Z with values in X is a game with ω-many innings, played alternately by two players: I and II. Player I begins the game and makes the first move by choosing a pair (K_0, U_0), where $K_0 \subseteq X$ is nonempty compact and U_0 is an open set in Z that contains K_0. Player II responds by choosing an open (in Z) set V_0 such that $K_0 \subseteq V_0 \subseteq U_0$. In the second round of the game, player I picks a pair (K_1, U_1), where K_1 is a nonempty compact subset of V_0 and U_1 is an open subset of Z with $K_1 \subseteq U_1 \subseteq V_0$. Player II responds by picking an open (in Z) set V_1 such that $K_1 \subseteq V_1 \subseteq U_1$. The game continues in this way and stops after ω many rounds. Player II wins the game if $\varnothing \neq \bigcap_{n\in\mathbb{N}} U_n (= \bigcap_{n\in\mathbb{N}} V_n) \subseteq X$. Otherwise, player I wins.

The game described above is denoted by $kP(Z, X)$.

2.3 Strategies

Denote by $\mathcal{T}_Y$ the collection of all nonempty open subsets of the space Y. A *strategy* of player I in the game $kP(Z, X)$ is a map σ defined inductively as follows: $\sigma(\varnothing) \in \mathcal{K}(X) \times \mathcal{T}_Z$. If the strategy σ is defined for the first n moves, then an n-tuple $(V_0, V_1, \dots, V_{n-1}) \in \mathcal{T}_Z^n$ is called *admissible* if $K_0 \subseteq V_0 \subseteq U_0$ and $K_i \subseteq V_i \subseteq U_i$, and $(K_i, U_i) = \sigma(V_0, \dots V_{i-1})$ for $i \in \{1, \dots, n-1\}$. For any admissible n-tuple $(V_0, \dots, V_{n-1})$, we choose a pair $(K_n, U_n) \in \mathcal{K}(V_{n-1}) \times \mathcal{T}_{V_{n-1}}$ with $K_n \subseteq U_n$ and we set

$$\sigma(V_0, \dots, V_{n-1}) = (K_n, U_n).$$

A strategy σ of player I in the game $kP(Z, X)$ is called *winning* if player I wins every run of the game $kP(Z, X)$ in which she plays according to the strategy σ.

We will need the following simple, though a little technical lemma concerning the game $kP(Z, X)$.

Lemma 2.3 *Suppose that $X \subseteq Z$ where Z is compact. Assume that there is a countable family $\{F_i : i = 1, 2, \dots\}$ consisting of compact subsets of Z and satisfying $\bigcup_{i=1}^{\infty} F_i \supseteq X$.*

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If σ is a winning strategy of player I in the game $kP(Z, Z\backslash X)$, then for every $k \in \mathbb{N}$ and every admissible tuple $(V_0, \dots, V_k)$, there exists $m > k$ and open sets $V_{k+1}, \dots, V_m$ such that the tuple $(V_0, \dots, V_m)$ is admissible and $\sigma(V_0, \dots, V_m) = (K_{m+1}, U_{m+1})$ satisfies $K_{m+1} \cap \bigcup_{i=1}^{\infty} F_i \neq \varnothing$.

Proof Striving for a contradiction, suppose that for some admissible $(k+1)$-tuple $(V_0, \dots, V_k)$, we have

$$(*) \qquad \begin{array}{l} \text{For every } m > k, \text{ if } (V_0, \dots, V_k, \dots, V_m) \text{ is admissible and} \\ \sigma(V_0, \dots, V_m) = (K_{m+1}, U_{m+1}), \text{ then } K_{m+1} \cap \bigcup\{F_i : i \geq 1\} = \varnothing. \end{array}$$

We recursively define sets V_m, for $m > k$, as follows: If the sets $V_0, \dots, V_{m-1}$ are already defined in such a way that the tuple $(V_0, \dots, V_{m-1})$ is admissible, we consider the pair $(K_m, U_m) = \sigma(V_0, \dots, V_{m-1})$. Let

$$V'_m = U_m \cap (Z \backslash F_{(m-k)}).$$

By (*), $K_m \subseteq V'_m$. Let V_m be an open set in Z satisfying

$$K_m \subseteq V_m \subseteq \overline{V_m} \subseteq V'_m.$$

It is clear that the tuple $(V_0, \dots, V_m)$ is admissible and we can proceed with our recursive construction.

In this way, we define a play in the game $kP(Z, Z\backslash X)$ in which player I applies her strategy and fails. Indeed, we have $\bigcap_{m=0}^{\infty} V_m \neq \varnothing$, because $V_{m+1} \subseteq \overline{V_{m+1}} \subseteq V_m \subseteq Z$ and Z is compact. Moreover, $X \cap \bigcap_{m=0}^{\infty} V_m = \varnothing$, because $V_{k+i} \cap F_i = \varnothing$ and $X \subseteq \bigcup_{i=1}^{\infty} F_i$. ∎

From the previous lemma, we can easily deduce the following proposition.

Proposition 2.4 *Suppose that $X \subseteq Z$ where Z is compact. Assume that there is a countable family $\{F_i : i = 1, 2, \dots\}$ consisting of compact subsets of Z and satisfying $\bigcup_{i=1}^{\infty} F_i \supseteq X$.*

If player I has a winning strategy in the game $kP(Z, Z\backslash X)$, then player I has a winning strategy σ' in this game such that every compact set played by player I according to the strategy σ', meets $\bigcup_{i=1}^{\infty} F_i$.

Proof Let σ be an arbitrary winning strategy of player I in the game $kP(Z, Z\backslash X)$. We will define a strategy σ' recursively. Consider $(K_0, U_0) = \sigma(\varnothing)$. Let V_0 be an arbitrary open set in Z such that $K_0 \subseteq V_0 \subseteq U_0$. By Lemma 2.3, there is m_0 and sets $V_1, \dots, V_{m_0}$ such that the tuple $(V_0, \dots, V_{m_0})$ is admissible and if $\sigma(V_0, \dots, V_{m_0}) = (K_{m_0}, U_{m_0})$, then $K_{m_0} \cap \bigcup_{i=1}^{\infty} F_i \neq \varnothing$. We define

$$\sigma'(\varnothing) = (K_{m_0}, V_{m_0}).$$

If $V^0 = V_{m_0+1}$ is an open set in Z with $K_{m_0} \subseteq V^0 \subseteq U_{m_0}$, then the tuple $(V_0, \dots, V_{m_0}, V_{m_0+1})$ is admissible for σ. Hence, by Lemma 2.3, there is $m_1 > m_0 + 1$ and sets $V_{m_0+2}, \dots, V_{m_1}$ such that the tuple $(V_0, \dots, V_{m_1})$ is admissible and if $\sigma(V_0, \dots, V_{m_1}) = (K_{m_1}, U_{m_1})$, then $K_{m_1} \cap \bigcup_{i=1}^{\infty} F_i \neq \varnothing$. We define

$$\sigma'(V^0) = (K_{m_0}, V_{m_0})$$

and so on. ∎

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2.4 Position of a space in its compactification

It was already observed by Smirnov [21] that the Lindelöf property of a Tychonoff space X can be conveniently characterized by the way X is placed in its compactification bX of X (cf. [9, Problem 3.12.25]).

A similar characterization of the Hurewicz property were obtained by Just *et al.* [10] (for the subsets of the real line), Banakh and Zdomskyy [4] (for separable metrizable spaces), and Tall [22] (the general case). We have the following theorem (see [22, Theorem 6]).

Theorem 2.5 *For any Tychonoff space X, the following conditions are equivalent:*

(1) *X has the Hurewicz property.*
(2) *For every compactification bX of X and every σ-compact subset F of the remainder $bX\setminus X$, there exists a G_δ-subset G of bX such that $F \subseteq G \subseteq bX\setminus X$.*
(3) *There exists a compactification bX of X such that for every σ-compact subset F of the remainder $bX\setminus X$, there exists a G_δ-subset G of bX such that $F \subseteq G \subseteq bX\setminus X$.*

Let X be a space, and let bX be a compactification of X. It was proved in [23] that the k-Porada game on bX with values in $bX\setminus X$, characterizes the Menger property of X (cf. Remark 2.7). We have the following theorem.

Theorem 2.6 [23, Theorem 2] *If $X \subseteq Z$, where Z is compact, then the following two conditions are equivalent:*

(1) *X has the Menger property.*
(2) *Player I has no winning strategy in the k-Porada game $kP(Z, Z\setminus X)$.*

Remark 2.7 It is perhaps worth mentioning here that Theorem 2 in [23] asserts actually that the game $kP(Z, Z\setminus X)$ is equivalent to the Menger game. It is well known, however (see, e.g., [20, Theorem 13] or [3, Theorem 2.32]), that a topological space X is Menger if and only if player I has no winning strategy in the Menger game.

It was recently observed by Krupski and Kucharski [14] that one can obtain similar characterizations for the projective properties of Hurewicz and Menger. As usual, by βX, we denote the Čech–Stone compactification of X. A subset A of a topological space Z is called *zero-set* if $A = f^{-1}(0)$ for some continuous function $f : Z \to [0,1]$. Vedenissov's lemma (see [9, Corollary 1.5.12]) asserts that if Z is a normal space (in particular compact), then A is a zero-set if and only if A is closed G_δ-subset of Z.

The proof of the following assertion is quite easy to derive from Theorem 2.5.

Proposition 2.8 *For any Tychonoff space X, the following conditions are equivalent:*

(1) *X is projectively Hurewicz.*
(2) *For every subset F of $\beta X\setminus X$ being a countable union of zero-sets in βX, there exists a G_δ subset G of βX such that $F \subseteq G \subseteq \beta X\setminus X$.*

In order to formulate a respective result for the projective Menger property, we need the following modification of the k-Porada game. Let Z be a compact space, and let $X \subseteq Z$ be a subspace of Z. The *z-Porada game* on Z with values in X (denoted by $zP(Z, X)$) is played as $kP(Z, X)$ with the only difference that compact sets played by player II are required to be additionally zero-sets in Z. A strategy for player I in the game $zP(Z, X)$ is defined analogously with obvious modifications. We have the following proposition (see [14]).

Proposition 2.9 *The following two conditions are equivalent:*

(1) *X has the projective Menger property.*
(2) *Player I has no winning strategy in the z-Porada game $zP(\beta X, \beta X \setminus X)$.*

Let us also note the following simple fact.

Lemma 2.10 *Let Z be a compact space. If $L \subseteq G$ where L is compact and G is a G_δ-subset of Z, then there exists a zero-set L' such that $L \subseteq L' \subseteq G$.*

We omit the obvious proof of the above lemma.

3 The support map

Let $\varphi : C_p(X) \to C_p(Y)$ be continuous and linear. For $y \in Y$, we define *the support of y with respect to φ* as the set $\operatorname{supp}_\varphi(y)$ of all $x \in X$ satisfying the condition that for every neighborhood U of x, there is $f \in C_p(X)$ such that $f[X \setminus U] \subseteq \{0\}$ and $\varphi(f)(y) \neq 0$ (see [1] and [27, Section 6.8]).

The following fact is well known (see [27, Lemmas 6.8.1 and 6.8.2]).

Lemma 3.1 *Let $\varphi : C_p(X) \to C_p(Y)$ be continuous and linear. Then:*

(1) $\operatorname{supp}_\varphi(y)$ *a finite subset of X.*
(2) *If $f \in C_p(X)$ satisfies $f[\operatorname{supp}_\varphi(y)] \subseteq \{0\}$, then $\varphi(f)(y) = 0$.*
(3) *If φ is surjective, then $\operatorname{supp}_\varphi(y) \neq \varnothing$ for every $y \in Y$.*
(4) *The multivalued map $y \mapsto \operatorname{supp}_\varphi(y)$ is lower semi-continuous.*

Let $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ be the one-point compactification of $\mathbb{R}$, and let Z be a Tychonoff space. For a function $f \in C_p(Z)$, the function $\widetilde{f} : \beta Z \to \overline{\mathbb{R}}$ is the continuous extension of f over the Čech–Stone compactification βZ of Z (i.e., $\widetilde{f}$ is continuous and $\widetilde{f} \restriction Z = f$). Since the addition is not defined for all pairs of points in $\overline{\mathbb{R}}$, the sum of two functions $\widetilde{f}$ and $\widetilde{g}$ may not be well defined. However, we have the following lemma.

Lemma 3.2 *Let $r_1, \dots, r_n \in C_p(Z)$ be a finite collection of continuous functions, and let $z \in \beta Z$. Let $r = r_1 + \cdots + r_n$. If for every $i \le n$, $\widetilde{r_i}(z) \in \mathbb{R}$, then $\widetilde{r}(z) = \widetilde{r_1}(z) + \cdots + \widetilde{r_n}(z)$.*

Proof For $i \le n$, define $W_i = \{x \in \beta Z : |\widetilde{r_i}(x) - \widetilde{r_i}(z)| < 1\}$. Note that this set is well defined because $\widetilde{r_i}(z) \in \mathbb{R}$. It is also open by continuity of $\widetilde{r_i}$. The set $W = \bigcap_{i=1}^n W_i$ is an open neighborhood of z in βZ, and for every $x \in W$, the quantity $\widetilde{r_1}(x) + \cdots + \widetilde{r_n}(x)$ is a well-defined real number. Thus, $(\widetilde{r_1} + \cdots + \widetilde{r_n}) \restriction W$ is a well-defined continuous function on W. Since $r(x) = r_1(x) + \cdots + r_n(x)$ for $x \in Z$ and Z is dense in W, we must have $\widetilde{r}(x) = \widetilde{r_1}(x) + \cdots + \widetilde{r_n}(x)$, for $x \in W$. In particular, $\widetilde{r}(z) = \widetilde{r_1}(z) + \cdots + \widetilde{r_n}(z)$. ∎

It will be convenient to introduce the following definition.

Definition 3.1 Let $\varphi : C_p(X) \to C_p(Y)$ be a linear continuous map, and let $y \in Y$. An open set $U \subseteq \beta X$ is called *y-effective* if every function $f \in C_p(X)$ such that

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$f[X\setminus U] \subseteq \{0\}$ satisfies $\widetilde{\varphi(f)}(y) = 0$. An open set $U \subseteq \beta X$ is called *y-ineffective* if it is not y-effective.

For $y \in \beta Y$, we set

$$s_\varphi(y) = \{x \in \beta X : \text{every open neighborhood of } x \text{ is } y\text{-ineffective}\}.$$

Remark 3.3 We should point out that the idea of considering the set $s_\varphi(y)$ is not new. The same concept (for spaces of bounded continuous functions) was used, e.g., by Valov in [25, 26].

Directly from the definition, we get the following.

Lemma 3.4 *The set $s_\varphi(y)$ is closed in βX; hence, it is compact.*

Lemma 3.5 *If $y \in Y$, then $s_\varphi(y) = \operatorname{supp}_\varphi(y)$.*

Proof The inclusion $\supseteq$ is clear. Suppose that there is $x \in s_\varphi(y)\setminus \operatorname{supp}_\varphi(y)$. Since $\operatorname{supp}_\varphi(y)$ is finite (see Lemma 3.1), there is an open neighborhood U of x in βX such that $U \cap \operatorname{supp}_\varphi(y) = \varnothing$. Let $f \in C_p(X)$ be such that $f[X\setminus U] \subseteq \{0\}$. Then $f[\operatorname{supp}_\varphi(y)] \subseteq \{0\}$ and hence $\varphi(f)(y) = 0$, by Lemma 3.1. This means that $U \ni x$ is y-effective, contradicting $x \in s_\varphi(y)$. ∎

Lemma 3.6 *Let $\varphi : C_p(X) \to C_p(Y)$ be a linear continuous map. Let $y \in \beta Y$, and let U be an open set in βX such that $s_\varphi(y) \subseteq U$. If $f \in C_p(X)$ satisfies $f[U \cap X] \subseteq \{0\}$, then $\widetilde{\varphi(f)}(y) = 0$.*

Proof Fix $f \in C_p(X)$ with $f[U \cap X] \subseteq \{0\}$. We have $s_\varphi(y) \subseteq U$, so if $x \in \beta X\setminus U$, then there exists an open neighborhood U_x of x in βX which is y-effective. For each $x \in \beta X\setminus U$, let V_x be an open neighborhood of x in βX satisfying $V_x \subseteq \overline{V_x} \subseteq U_x$. The family $\{V_x : x \in \beta X\setminus U\}$ covers the compact set $\beta X\setminus U$. Let $\{V_{x_1}, \ldots, V_{x_n}\}$ be its finite subcover. Let

$$F = \overline{V_{x_1}} \cup \cdots \cup \overline{V_{x_n}}.$$

For $i = 1, \ldots, n$, let $g_i : \beta X \to [0,1]$ be a continuous function that satisfies

$$g_i[\overline{V_{x_i}}] = \{1\} \quad \text{and} \quad g_i[\beta X\setminus U_{x_i}] = \{0\}.$$

For $i = 1, \ldots, n$, there exists a function $f_i \in C_p(\beta X)$ such that

$$f_i(x) = \begin{cases} \frac{g_i(x)}{g_1(x)+\cdots+g_n(x)}, & \text{for } x \in F,\\ 0, & \text{for } x \in \beta X\setminus(U_{x_1} \cup \cdots \cup U_{x_n}). \end{cases}$$

Let $h_i = f \cdot (f_i \upharpoonright X)$ be the product of the functions f and $f_i \upharpoonright X$. The function h_i has the following property:

(*) If $x \in X\setminus U_{x_i}$, then $h_i(x) = 0$.

Indeed, if $x \in F\setminus U_{x_i}$, then $f_i(x) = g_i(x)/(g_1(x) + \cdots + g_n(x)) = 0$, because $g_i(x) = 0$ for $x \notin U_{x_i}$. Hence, $h_i(x) = f(x) \cdot f_i(x) = 0$. On the other hand, if $x \notin F$, then $x \in U$, so $f(x) = 0$. Hence, $h_i(x) = f(x) \cdot f_i(x) = 0$ too.

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Each set U_{x_i} is y-effective; thus, $\widetilde{\varphi(h_i)}(y) = 0$, for every $i = 1, \dots, n$, by (*). Let $h = h_1 + \cdots + h_n$. We can apply Lemma 3.2 with $r = \varphi(h)$ and $r_i = \varphi(h_i)$, obtaining

$$\widetilde{\varphi(h)}(y) = 0.$$

We claim that $h = f$. Indeed, if $x \notin F$, then $x \in U$. Thus, $f(x) = 0$, by our assumption on f. It follows that for such x and for all $i \in \{1, \dots, n\}$, we have $h_i(x) = f(x) \cdot f_i(x) = 0$. Thus, $h(x) = h_1(x) + \cdots + h_n(x) = 0 = f(x)$ for $x \notin F$. Now, suppose that $x \in F$. We have

$$\begin{aligned} h(x) &= h_1(x) + \cdots + h_n(x) = f(x) \cdot (f_1(x) + \cdots + f_n(x)) \\ &= f(x) \cdot \sum_{i=1}^{n} \frac{g_i(x)}{g_1(x) + \cdots + g_n(x)} = f(x). \end{aligned}$$

■

Corollary 3.7 *If $\varphi : C_p(X) \to C_p(Y)$ is a linear continuous surjection, then for every $y \in \beta Y$, the set $s_\varphi(y)$ is nonempty.*

Proof Take $y \in \beta Y$ and suppose that $s_\varphi(y) = \varnothing$. Then $\varnothing$ is an open set containing $s_\varphi(y)$. Let $g \in C_p(Y)$ be such that $\widetilde{g}(y) = 1$. Since φ is onto, there is $f \in C_p(X)$ with $\varphi(f) = g$. Clearly, $f[\varnothing] \subseteq \{0\}$, so

$$\widetilde{g}(y) = \widetilde{\varphi(f)}(y) = 0,$$

by Lemma 3.6, which is a contradiction. ■

Proposition 3.8 *Let $\varphi : C_p(X) \to C_p(Y)$ be a continuous surjection. The set-valued map $s_\varphi : \beta Y \to \mathcal{K}(\beta X)$, given by the assignment $y \mapsto s_\varphi(y)$, is lower semi-continuous.*

Proof By Lemma 3.4 and Corollary 3.7, the map s is well defined. Let $U \subseteq \beta X$ be open. We need to show that the set

$$s_\varphi^{-1}(U) = \{y \in \beta Y : s_\varphi(y) \cap U \neq \varnothing\}$$

is open in βY. Pick $y_0 \in s_\varphi^{-1}(U)$ and take $x_0 \in s_\varphi(y_0) \cap U$ witnessing $s_\varphi(y_0) \cap U \neq \varnothing$. Let V be an open neighborhood of x_0 such that $\overline{V} \subseteq U$. Since $x_0 \in s_\varphi(y_0)$, the set $V \ni x_0$ is y_0-ineffective. Therefore, there is $f \in C_p(X)$ such that $f[X \setminus V] \subseteq \{0\}$ and $\widetilde{\varphi(f)}(y_0) \neq 0$. Consider the open set

$$W = \{y \in \beta Y : \widetilde{\varphi(f)}(y) \neq 0\}.$$

Clearly, $y_0 \in W$. We claim that $W \subseteq s_\varphi^{-1}(U)$. Take $y \in W$. If $s_\varphi(y) \cap U = \varnothing$, then $s_\varphi(y) \subseteq \beta X \setminus \overline{V}$, the set $\beta X \setminus \overline{V}$ is open in βX, and $f[X \setminus \overline{V}] \subseteq \{0\}$. Hence, $\widetilde{\varphi(f)}(y) = 0$, by Lemma 3.6. A contradiction with $y \in W$.

It follows that $y_0 \in W \subseteq s_\varphi^{-1}(U)$, where W is open. Since y_0 was chosen arbitrarily, the set $s_\varphi^{-1}(U)$ is open. ■

Corollary 3.9 *For every integer $n \geq 1$, the set $\widetilde{Y}_n = \{y \in \beta Y : |s_\varphi(y)| \leq n\}$ is closed in βY.*

Proof Let $y \in \beta Y \setminus \widetilde{Y}_n$. Then $s_\varphi(y)$ has at least $n+1$ elements, so there are distinct $x_1, \dots, x_{n+1} \subseteq s_\varphi(y)$. Let $V_1, \dots, V_{n+1}$ be pairwise disjoint open subsets of βX such

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that $x_i \in V_i$, for $i = 1, \ldots, n+1$. By lower semi-continuity of s_φ (cf. Proposition 3.8), the set $W = \bigcap_{i=1}^{n+1} s_\varphi^{-1}(V_i)$ is open and clearly $y \in W$. For any $z \in W$, the set $s_\varphi(z)$ meets $n+1$ pairwise disjoint sets V_i. Hence, $y \in W \subseteq \beta Y \setminus \widetilde{Y}_n$. ■

Similarly, from lower semi-continuity of the support map $y \mapsto \operatorname{supp}_\varphi(y)$ (see Lemma 3.1), it follows that the set

$$Y_n = \{y \in Y : |\operatorname{supp}_\varphi(y)| \le n\}$$

is closed in Y. Also, since $s_\varphi(y) = \operatorname{supp}_\varphi(y)$ for $y \in Y$ (cf. Lemma 3.5), we have $Y_n \subseteq \widetilde{Y}_n$.

4 Technical lemmata

In this section, we will use some ideas from Okunev [16] (see also [12]). Let Z be a Tychonoff space. For $\varepsilon > 0$ and a finite set $F = \{z_1, \ldots, z_k\} \subseteq \beta Z$, we set

$$O_Z(F, \varepsilon) = \{f \in C_p(Z) : |\widetilde{f}(z_i)| < \varepsilon, \ i = 1, \ldots, k\}.$$

For a point $z \in \beta Z$ and $\varepsilon > 0$, let

$$\bar{O}_Z(z, \varepsilon) = \{f \in C_p(Z) : |\widetilde{f}(z)| \le \varepsilon\}.$$

Note that for $z \in Z$, the set $\bar{O}_Z(z, \varepsilon)$ is closed in $C_p(Z)$, whereas for $z \in \beta Z \setminus Z$, it is dense and has empty interior in $C_p(Z)$.

Let $\varphi : C_p(X) \to C_p(Y)$ be a linear homeomorphism. By linearity, $\varphi(\underline{0}) = \underline{0}$, where $\underline{0}$ is the constant function equal to 0 in the respective space.

For positive integers k and m, we define the set

$$Z_{m,k} = \left\{(y, F) \in Y \times [X]^{\le k} : \varphi\left(O_X\left(F, \tfrac{1}{m}\right)\right) \subseteq \bar{O}_Y(y, 1)\right\}.$$

We should remark that, in [16], sets $Z_{m,k}$ are defined in a slightly different way, i.e., the product X^k is used instead of the hyperspace $[X]^{\le k}$. However, our (cosmetic) change does not affect the arguments from [16].

Now, for positive integers k and m, let $S_{m,k}$ be the closure of $Z_{m,k}$ in the (compact) space $\beta Y \times [\beta X]^{\le k}$. Recall that $[\beta X]^{\le k}$ is endowed with the Vietoris topology (cf. Section 2.1). We have the following (cf. [16, Lemma 1.4]) (we reproduce the proof here for the convenience of the reader).

Lemma 4.1 *If $(y, F) \in S_{m,k}$, then $\varphi\left(O_X\left(F, \tfrac{1}{m}\right)\right) \subseteq \bar{O}_Y(y, 1)$.*

Proof Otherwise, there is $f \in C_p(X)$ with $|\widetilde{f}(x)| < \frac{1}{m}$ for each $x \in F$ and $|\widetilde{\varphi(f)}(y)| > 1$. The set

$$U = \left\{A \in [\beta X]^{\le k} : \widetilde{f}(A) \subseteq \left(-\tfrac{1}{m}, \tfrac{1}{m}\right)\right\}$$

is open in $[\beta X]^{\le k}$ and $F \in U$. Similarly, the set

$$V = \left\{z \in \beta Y : |\widetilde{\varphi(f)}(z)| > 1\right\}$$

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is an open neighborhood of y in βY. Since $(y, F) \in S_{m,k}$, the open set $V \times U$ has a nonempty intersection with $Z_{m,k}$. This, however, contradicts the definition of $Z_{m,k}$. ∎

For $k, m \geq 1$, define

$$C_{m,k} = \pi_{\beta Y}(S_{m,k}),$$

where $\pi_{\beta Y} : \beta Y \times [\beta X]^{\leq k} \to \beta Y$ is the projection onto the first factor. Clearly, $C_{m,k}$ is closed in βY.

Recall that $Y_n = \{y \in Y : |\operatorname{supp}_\varphi(y)| \leq n\}$. We set

$$A_{m,n} = Y_n \cap C_{m,n}.$$

Lemma 4.2 *If $y \in Y_n$, then for some $m \geq 1$, $(y, \operatorname{supp}_\varphi(y)) \in Z_{m,n}$ and thus $y \in A_{m,n}$.*

Proof Take $y \in Y_n$. By continuity of φ, there is a finite set $F = \{x_1, \dots, x_k\} \subseteq X$ and $m \geq 1$ with

$$O_X\left(F, \tfrac{1}{m}\right) \subseteq \varphi^{-1}\left(\bar{O}_Y(y, 1)\right).$$

We will check that the number m does the job. To this end, consider a function $f \in C_p(X)$ satisfying

$$|f(x)| < \tfrac{1}{m}, \quad \text{for every } x \in \operatorname{supp}_\varphi(y).$$

Striving for a contradiction, suppose that $|\varphi(f)(y)| > 1$, and let $g \in C_p(X)$ be such that $g \upharpoonright \operatorname{supp}_\varphi(y) = f \upharpoonright \operatorname{supp}_\varphi(y)$ and $g(x) = 0$, for every $x \in F \setminus \operatorname{supp}_\varphi(y)$. Since g and f agree on $\operatorname{supp}_\varphi(y)$, we have $\varphi(g)(y) = \varphi(f)(y) > 1$ (see Lemma 3.1). On the other hand,

$$g \in O_X\left(F, \tfrac{1}{m}\right) \subseteq \varphi^{-1}\left(\bar{O}_Y(y, 1)\right),$$

a contradiction. ∎

Proposition 4.3 *$Y = \bigcup_{n=1}^{\infty} A_{n,n}$ and $A_{n,n} \subseteq A_{m,m}$ for $m \geq n$.*

Proof Let $y \in Y$. By Lemma 3.1, $Y = \bigcup_{n=1}^{\infty} Y_n$, so $y \in Y_n$, for some $n \geq 1$. From Lemma 4.2, we infer that $y \in C_{m,n}$ for some m. Note that if $m \leq k$, then $C_{m,n} \subseteq C_{k,n}$, so we can assume that $m > n$, for otherwise $y \in C_{n,n}$ and we are done. Since $Y_n \subseteq Y_k$ for $k \geq n$, we have $y \in Y_k$ for all $k \geq n$. In particular, $y \in Y_m$. So $y \in Y_m \cap C_{m,n}$, where $m > n$. But clearly, $m > n$ implies $C_{m,n} \subseteq C_{m,m}$, whence $y \in A_{m,m}$. This gives the equality $Y = \bigcup_{n=1}^{\infty} A_{n,n}$. The inclusion $A_{n,n} \subseteq A_{m,m}$, for $m \geq n$, is clear. ∎

Let $B_{m,n}$ be the closure of $A_{m,n}$ in βY. Since $Y_n \subseteq \widetilde{Y}_n$ and both $\widetilde{Y}_n$ and $C_{m,n}$ are closed in βY, we infer that

$$B_{m,n} \subseteq \widetilde{Y}_n \cap C_{m,n}.$$

In particular, if $y \in B_{m,n}$, then the set $s_\varphi(y)$ is at most n-element subset of βX.

Lemma 4.4 *If $y \in B_{m,n}$, then $\varphi\left(O_X\left(s_\varphi(y), \tfrac{1}{m}\right)\right) \subseteq \bar{O}_Y(y, 1)$.*

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Proof Pick $f \in C_p(X)$ such that $|\widetilde{f}(x)| < \frac{1}{m}$ for $x \in s_\varphi(y)$. The set $U = \{x \in \beta X : |\widetilde{f}(x)| < \frac{1}{m}\}$ is open in βX and $s_\varphi(y) \subseteq U$. Let V be an open subset of βX satisfying

$$s_\varphi(y) \subseteq V \subseteq \overline{V} \subseteq U. \tag{4.1}$$

Since $y \in B_{m,n} \subseteq C_{m,n}$, there is $F \in [\beta X]^{\leq n}$ with $(y, F) \in S_{m,n}$. Let $\widetilde{g} \in C_p(\beta X)$ be a function satisfying

$$\widetilde{g} \restriction \overline{V} = \widetilde{f} \restriction \overline{V} \quad \text{and} \quad \widetilde{g}(x) = 0 \text{ for each } x \in F \backslash \overline{V}.$$

Denote by g the function $\widetilde{g} \restriction X$, i.e., the restriction of $\widetilde{g}$ to X. Clearly, $g \in O_X\left(F, \frac{1}{m}\right)$, so by Lemma 4.1, we have

$$|\widetilde{\varphi(g)}(y)| \leq 1. \tag{4.2}$$

Further, $(\widetilde{f} - \widetilde{g}) \restriction V = 0$. So from (4.1) and Lemma 3.6, we infer that

$$\widetilde{\varphi(f-g)}(y) = 0. \tag{4.3}$$

By linearity of φ, we get

$$\varphi(f-g) + \varphi(g) = \varphi(f).$$

Inequality (4.2) and equation (4.3) ensure that Lemma 3.2 can be applied with $r = \varphi(f)$, $r_1 = \varphi(f-g)$, and $r_2 = \varphi(g)$, whence

$$|\widetilde{\varphi(f)}(y)| = |\widetilde{\varphi(f-g)}(y) + \widetilde{\varphi(g)}(y)| \leq |\widetilde{\varphi(f-g)}(y)| + |\widetilde{\varphi(g)}(y)| \leq 1,$$

by (4.2) and (4.3). ∎

From the previous lemma, we get the following.

Proposition 4.5 *For every $y \in B_{m,n} \backslash A_{m,n}$, the set $s_\varphi(y) \cap (\beta X \backslash X)$ is nonempty.*

Proof Let $y \in B_{m,n} \backslash A_{m,n}$. Since $A_{m,n}$ is closed in Y and $B_{m,n}$ is the closure of $A_{m,n}$ in βY, we have $y \in \beta Y \backslash Y$. So the set $\bar{O}_Y\left(y, \frac{1}{m}\right)$ has empty interior in $C_p(Y)$. By Lemma 4.4, we have

$$\varphi\left(O_X\left(s_\varphi(y), \tfrac{1}{m}\right)\right) \subseteq \bar{O}_Y(y, 1).$$

Now, if $s_\varphi(y)$ were a subset of X, then the set $\varphi\left(O_X\left(s_\varphi(y), \frac{1}{m}\right)\right)$ would be open, contradicting emptiness of the interior of $\bar{O}_Y(y, 1)$. ∎

Proposition 4.6 *Let $\varphi : C_p(X) \to C_p(Y)$ be a linear homeomorphism. If $y \in B_{m,n}$, then there exists $x \in s_\varphi(y)$ with $y \in s_{\varphi^{-1}}(x)$.*

Proof Since $B_{m,n} \subseteq \widetilde{Y}_n$, the set $s_\varphi(y)$ is at most n-element. Thus, $K = \bigcup\{s_{\varphi^{-1}}(x) : x \in s_\varphi(y)\}$ is compact, being a finite union of compact sets $s_{\varphi^{-1}}(x)$.

Striving for a contradiction, suppose that $y \notin K$. Let U be an open set in βY with $K \subseteq U$ and $y \notin U$. Let V be an open set in βY with

$$K \subseteq V \subseteq \overline{V} \subseteq U.$$

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Let $f \in C_p(\beta Y)$ satisfies $f(\overline{V}) \subseteq \{0\}$ and $f(y) = 2$. From Lemma 3.6 (applied to the map φ^{-1}), we get

$$\varphi^{-1}\widetilde{(f \upharpoonright Y)}(x) = 0, \quad \text{for every } x \in s_\varphi(y).$$

Combining this with Lemma 4.4, we get

$$|\varphi(\varphi^{-1}\widetilde{(f \upharpoonright Y)})(y)| = |f(y)| \leq 1,$$

which contradicts $f(y) = 2$. ∎

5 The main results

Let M be a separable metrizable space, and let $h : Y \to M$ be a continuous surjection. Since M is separable metrizable, it has a metrizable compactification bM. Let $\widetilde{h} : \beta Y \to bM$ be a continuous extension of h. Denote by d a metric on bM that generates the topology of bM.

For a natural number $k \geq 1$, we define sets

$$E_k = \{y \in Y : \big(\forall a, b \in h(s_{\varphi^{-1}}(s_\varphi(y)))\big)\ a \neq b \Rightarrow d(a,b) \geq \tfrac{1}{k}\},$$
$$F_k = \{y \in \beta Y : \big(\forall a, b \in \widetilde{h}(s_{\varphi^{-1}}(s_\varphi(y)))\big)\ a \neq b \Rightarrow d(a,b) \geq \tfrac{1}{k}\}.$$

It is easy to prove the following.

Lemma 5.1 *The sets E_k and F_k have the following properties:*

(i) *The set E_k is closed in Y, for every $k \geq 1$.*
(ii) *The set F_k is closed in βY, for every $k \geq 1$.*
(iii) $\bigcup_{k=1}^\infty E_k = Y$.
(iv) *If $k \leq r$, then $E_k \subseteq E_r$.*

Proof Let $y \in Y \setminus E_k$. By Lemma 3.5, $s_\varphi(y) = \mathrm{supp}_\varphi(y)$, so $y \notin E_k$ means that there are distinct $a, b \in h(\mathrm{supp}_{\varphi^{-1}}(\mathrm{supp}_\varphi(y)))$ with $d(a,b) < \frac{1}{k}$. Let $\varepsilon > 0$ be such that

$$\varepsilon < \frac{1}{2k} - \frac{d(a,b)}{2} \quad \text{and} \quad \varepsilon < \frac{d(a,b)}{2}. \tag{5.1}$$

For $x \in \{a, b\}$, let B_x be an ε-ball in the space M, centered at x.

The set

$$V_x = h^{-1}(B_x)$$

is open in Y and

$$V_x \cap \mathrm{supp}_{\varphi^{-1}}(\mathrm{supp}_\varphi(y)) \neq \varnothing, \tag{5.2}$$

for $x \in \{a, b\}$. Since the map $\mathrm{supp}_{\varphi^{-1}}$ is lower semi-continuous, the set

$$W_x = \mathrm{supp}_{\varphi^{-1}}^{-1}(V_x)$$

is open in X and, by (5.2),

$$\mathrm{supp}_\varphi(y) \cap W_x \neq \varnothing.$$

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It follows that, for $x \in \{a, b\}$, the set

$$U_x = \operatorname{supp}_\varphi^{-1}(W_x)$$

is an open neighborhood of y in Y. Put

$$U = U_a \cap U_b.$$

If $z \in U$, then

$$V_x \cap \operatorname{supp}_{\varphi^{-1}}(\operatorname{supp}_\varphi(z)) \neq \varnothing,$$

and hence there is

$$\xi_x \in B_x \cap h(\operatorname{supp}_{\varphi^{-1}}(\operatorname{supp}_\varphi(z))),$$

for $x \in \{a, b\}$. By (5.1), the balls B_a and B_b are disjoint, so $\xi_a \neq \xi_b$ and

$$d(\xi_a, \xi_b) < 2\varepsilon + d(a, b) < \frac{1}{k},$$

by (5.1). This shows that $U \cap E_k = \varnothing$ and finishes the proof of (i). The proof of (ii) is analogous.

Assertion (iii) follows from the fact that for $y \in Y$ we have $s_\varphi(y) = \operatorname{supp}_\varphi(y) \subseteq X$ (Lemma 3.5) and thus the set $s_{\varphi^{-1}}(s_\varphi(y)) = \operatorname{supp}_{\varphi^{-1}}(\operatorname{supp}_\varphi(y))$ is finite, by Lemma 3.1.

Assertion (iv) is clear. ■

Now, for $n \geq 1$, let

$$H_n = A_{n,n} \cap E_n$$

and let $\overline{H_n}$ be the closure of H_n in βY. The sets H_n and $\overline{H_n}$ have the following properties.

Observation 5.2 *$\bigcup_{n=1}^{\infty} H_n = Y$ and $H_n \subseteq H_{n+1}$ for all $n \geq 1$.*

Proof This follows immediately from Proposition 4.3 and Lemma 5.1(iii) and (iv). ■

Observation 5.3 *If $y \in \overline{H_n} \setminus H_n$, then $s_\varphi(y) \cap (\beta X \setminus X) \neq \varnothing$.*

Proof Since H_n is closed in Y, we have $\overline{H_n} \setminus H_n \subseteq B_{n,n} \setminus A_{n,n}$. So it is enough to apply Proposition 4.5. ■

Observation 5.4 *For every $y \in \overline{H_n}$, we have $y \in s_{\varphi^{-1}}(s_\varphi(y))$.*

Proof According to Lemma 5.1, we have

$$\overline{H_n} = \overline{A_{n,n} \cap E_n} \subseteq B_{n,n} \cap F_n.$$

Hence, our assertion follows from Proposition 4.6. ■

Observation 5.5 *For every $y \in \overline{H_n}$ and for all distinct $a, b \in \widetilde{h}(s_{\varphi^{-1}}(s_\varphi(y)))$, we have $d(a, b) \geq \frac{1}{n}$.*

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Proof Again, by Lemma 5.1, we have $\overline{H_n} \subseteq B_{n,n} \cap F_n$. So the assertion follows from the definition of the set F_n. ■

For each $n \geq 1$, we define a set-valued mapping $e_n : \overline{H_n} \to \mathcal{K}(\beta X)$ by the formula

$$e_n(y) = \{x \in s_\varphi(y) : \widetilde{h}(y) \in \widetilde{h}(s_{\varphi^{-1}}(x))\}.$$

Note that the set $e_n(y)$ is finite because the set $s_\varphi(y)$ is finite for $y \in \overline{H_n}$. Also, $e_n(y)$ is nonempty, by Observation 5.4. So the map e_n is well defined.

Lemma 5.6 *For every $n \geq 1$, the map $e_n : \overline{H_n} \to \mathcal{K}(\beta X)$ is lower semi-continuous.*

Proof Take an open set $U \subseteq \beta X$. Pick $y \in e_n^{-1}(U)$ and take $x_0 \in e_n(y) \cap U$ witnessing $e_n(y) \cap U \neq \varnothing$.

We need to show that there is an open set W in βY with

$$y \in W \cap \overline{H_n} \subseteq e_n^{-1}(U).$$

Denote by B the ball in bM of radius $\frac{1}{2n}$ centered at $\widetilde{h}(y)$. Since the map $s_{\varphi^{-1}} : \beta X \to \mathcal{K}(\beta Y)$ is lower semi-continuous (see Proposition 3.8), the set

$$V = \{x \in \beta X : s_{\varphi^{-1}}(x) \cap \widetilde{h}^{-1}(B) \neq \varnothing\} \cap U$$

is open in βX and $x_0 \in V$.

We set

$$W = \{z \in \beta Y : s_\varphi(z) \cap V \neq \varnothing\} \cap \widetilde{h}^{-1}(B).$$

Note that W is open in βY (by Proposition 3.8) and $y \in W$ because $x_0 \in V \cap e_n(y) \subseteq V \cap s_\varphi(y)$.

We claim that W is as required. Indeed, pick $z \in W \cap \overline{H_n}$. We have

$$\widetilde{h}(z) \in B \qquad \text{and} \tag{5.3}$$

$$s_\varphi(z) \cap V \neq \varnothing. \tag{5.4}$$

Let x_1 be a witness for (5.4), i.e.,

$$x_1 \in s_\varphi(z) \cap U \qquad \text{and} \tag{5.5}$$

$$s_{\varphi^{-1}}(x_1) \cap \widetilde{h}^{-1}(B) \neq \varnothing. \tag{5.6}$$

By (5.6), there is $z' \in s_{\varphi^{-1}}(x_1)$ such that

$$\widetilde{h}(z') \in B. \tag{5.7}$$

On the other hand, since $z \in \overline{H_n}$, we infer from Observation 5.4 that there is $x_2 \in s_\varphi(z)$ (possibly $x_2 = x_1$) such that $z \in s_{\varphi^{-1}}(x_2)$. By (5.3) and (5.7), we must have

$$\widetilde{h}(z) = \widetilde{h}(z'). \tag{5.8}$$

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For otherwise $a = \widetilde{h}(z)$ and $b = \widetilde{h}(z')$ would be distinct elements of $\widetilde{h}(s_{\varphi^{-1}}(s_\varphi(z))$ satisfying

$$d(a,b) \leq d\left(a, \widetilde{h}(y)\right) + d\left(\widetilde{h}(y), b\right) < 1/2n + 1/2n = 1/n,$$

by definition of B. However, this would contradict $z \in \overline{H_n}$, by Observation 5.5.

Now, (5.8) gives $\widetilde{h}(z) \in \widetilde{h}(s_{\varphi^{-1}}(x_1))$. But this means that $x_1 \in e_n(z)$ and thus by (5.5), $x_1 \in e_n(z) \cap U$. In particular, the latter set is nonempty. ∎

Remark 5.7 Clearly, the sets H_n and $\overline{H_n}$ and the map e_n depend on the function $h : Y \to M$. In what follows, we will always be given a function $h : Y \to M$. The sets H_n, $\overline{H_n}$ and the map e_n will be associated with the given function h.

It will be convenient to use the following notation. For a continuous map $f : S \to T$ between topological spaces S and T and a set $A \subseteq S$, we denote by $f^\#(A)$ the set $T \setminus f(S \setminus A)$. It is straightforward to verify the following.

Proposition 5.8 *Suppose that $f : S \to T$ is a continuous map between topological spaces S and T. Then:*

(a) *If S is compact and $U \subseteq S$ is open, then $f^\#(U)$ is open in T.*
(b) *For any $t \in T$ and $A \subseteq S$, if $t \in f^\#(A)$, then $f^{-1}(t) \subseteq A$.*
(c) *For any $A \subseteq S$ and $B \subseteq T$, if $f^{-1}(B) \subseteq A$, then $B \subseteq f^\#(A)$.*

We are ready now to present proofs of the results announced in the Introduction.

Proof of Theorem 1.6 By symmetry, it is enough to show that the projective Hurewicz property of X implies the projective Hurewicz property of Y. Suppose that X is projectively Hurewicz and fix a continuous surjection $h : Y \to M$ that maps Y onto a separable metrizable space M. Let bM be a metrizable compactification of M, and let $\widetilde{h} : \beta Y \to bM$ be a continuous extension of h. Denote by d a metric on bM that generates the topology of bM. Note that

$$\text{If } A \subseteq bM \setminus M, \text{ then } \widetilde{h}^{-1}(A) \subseteq \beta Y \setminus Y. \tag{5.9}$$

In order to prove that M is Hurewicz, we will employ Theorem 2.5. For this purpose, take a σ-compact set $F \subseteq bM \setminus M$. Write $F = \bigcup_{i=1}^\infty K_i$, where each K_i is compact and $K_i \subseteq K_{i+1}$. We need to show that there is a G_δ-subset G of bM with $F \subseteq G \subseteq bM \setminus M$.

If no K_i intersects $\bigcup_{n=1}^\infty \widetilde{h}(\overline{H_n})$, then we are done because the complement of the latter union in bM is a G_δ-subset of $bM \setminus M$ (by Observation 5.2 and surjectivity of $h : Y \to M$). So suppose that, for some i, the set K_i meets $\bigcup_{n=1}^\infty \widetilde{h}(\overline{H_n})$ and let i_0 be the first such i. Since the family $\{K_i : i = 1, 2, \ldots\}$ is increasing, K_i meets $\bigcup_{n=1}^\infty \widetilde{h}(\overline{H_n})$, for every $i \geq i_0$. In order to find the required G_δ-set G, it suffices to find such set for the family $\{K_i : i \geq i_0\}$, i.e., it is enough to find a G_δ-subset G' of bM such that $\bigcup_{i=i_0}^\infty K_i \subseteq G' \subseteq bM \setminus M$. This is because the set $\bigcup_{i=1}^{i_0-1} K_i$ is contained in a G_δ-subset of $bM \setminus M$ (the complement of $\bigcup_{n=1}^\infty \widetilde{h}(\overline{H_n})$ in bM) and the union of two G_δ-sets is G_δ.

For each $i \geq i_0$, there is a positive integer n_i such that the compact set

$$K'_{n_i} = \widetilde{h}^{-1}(K_i) \cap \overline{H_{n_i}}$$

is nonempty. By Observation 5.2, we can additionally require that $n_{i_0} < n_{i_0+1} < \cdots$.

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Let $i \geq i_0$. Since bM is metrizable, the set K_i is G_δ in bM. It follows from Proposition 3.8 and Lemma 2.1 that the set

$$G_i = s_{\varphi^{-1}}^{-1}\left(\widetilde{h}^{-1}(K_i)\right) = \{x \in \beta X : s_{\varphi^{-1}}(x) \cap \widetilde{h}^{-1}(K_i) \neq \varnothing\}$$

is a G_δ-set in βX. In addition, by (5.9) and Lemmas 3.5 and 3.1, we have $G_i \subseteq \beta X \backslash X$.

The map e_{n_i} restricted to K'_{n_i} is lower semi-continuous (by Lemma 5.6), and note that if $y \in K'_{n_i}$, then $e_{n_i}(y) \subseteq G_i \subseteq \beta X \backslash X$ (by definition of e_{n_i}). Thus, we may consider the map $e_{n_i} \upharpoonright K'_{n_i}$ as a (lower semi-continuous) map into $\mathcal{K}(G_i)$. By Theorem 2.2, this map admits a compact section, i.e., there is a compact set $L_i \subseteq G_i$ such that

$$L_i \cap e_{n_i}(y) \neq \varnothing, \text{ for every } y \in K'_{n_i}.$$

In particular, since $e_{n_i}(y) \subseteq s_\varphi(y)$, we have

$$L_i \cap s_\varphi(y) \neq \varnothing, \text{ for every } y \in K'_{n_i}. \tag{5.10}$$

Using Lemma 2.10, we can enlarge the set L_i to a zero-set in βX contained in G_i. Clearly, this is still a section of e_{n_i}, so without loss of generality we can assume that each L_i is a zero-set.

The space X is projectively Hurewicz and $L = \bigcup_{i=i_0}^{\infty} L_i \subseteq \beta X \backslash X$, where all L_i's are zero-sets in βX. Hence, by Proposition 2.8, there is a G_δ-set P in βX with

$$L \subseteq P \subseteq \beta X \backslash X. \tag{5.11}$$

We can write

$$P = \bigcap_{i=i_0}^{\infty} P_i,$$

where the sets P_i are open in βX and form a decreasing sequence, i.e., $P_i \supseteq P_{i+1}$.

For $i \geq i_0$, we infer from the lower semi-continuity of the map s_φ, that the set

$$V_i = s_\varphi^{-1}(P_i) = \{y \in \beta Y : s_\varphi(y) \cap P_i \neq \varnothing\}$$

is open in βY and $V_i \supseteq V_{i+1}$. For each $i \geq i_0$, we set

$$W_i = V_i \cup (\beta Y \backslash \overline{H_{n_i}}).$$

Clearly, W_i is open in βY and $W_i \supseteq W_j$ for $i \leq j$ (because $V_i \supseteq V_j$ and $H_{n_i} \subseteq H_{n_j}$). Moreover, by (5.10) and (5.11), we have $\widetilde{h}^{-1}(K_i) \subseteq W_i$, for every $i \geq i_0$. Fix an arbitrary $i \geq i_0$. If $j \geq i$, then $K_i \subseteq K_j$, so $\widetilde{h}^{-1}(K_i) \subseteq \widetilde{h}^{-1}(K_j) \subseteq W_j$. If $i_0 \leq j < i$, then $\widetilde{h}^{-1}(K_i) \subseteq W_i \subseteq W_j$. Therefore, for every $i \geq i_0$, we have

$$\widetilde{h}^{-1}(K_i) \subseteq \bigcap_{j=i_0}^{\infty} W_j. \tag{5.12}$$

We claim that the set $G' = \bigcap_{i=i_0}^{\infty} \widetilde{h}^{\#}(W_i)$ is the G_δ-set we are looking for. First, note that G' is indeed a G_δ-set in bM, by Proposition 5.8(a). From (5.12) and Proposition 5.8(c), we get

$$\bigcup_{i=i_0}^{\infty} K_i \subseteq \bigcap_{i=i_0}^{\infty} \widetilde{h}^{\#}(W_i).$$

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It remains to show that $\bigcap_{i=i_0}^{\infty}\widetilde{h}^{\#}(W_i)\subseteq bM\backslash M$. Suppose that this is not the case and fix $a\in M\cap\bigcap_{i=i_0}^{\infty}\widetilde{h}^{\#}(W_i)$. Since the map $h:Y\to M$ is surjective, there is $y\in Y$ such that $h(y)=\widetilde{h}(y)=a$. By Proposition 5.8 (b), $\widetilde{h}^{-1}(a)\subseteq\bigcap_{i=i_0}^{\infty}W_i$. Thus,

$$y\in Y\cap\bigcap_{i=i_0}^{\infty}W_i=Y\cap\bigcap_{i=i_0}^{\infty}\left(V_i\cup(\beta Y\backslash\overline{H_{n_i}})\right).$$

On the other hand, it follows from Observation 5.2 that $y\in H_{n_i}$ for all but finitely many i's. Hence, we must have

$$y\in V_i=s_{\varphi}^{-1}(P_i),\ \text{for all but finitely many } i\text{'s}.$$

In addition, $y\in Y$, so the set $s_{\varphi}(y)=\operatorname{supp}_{\varphi}(y)$ is a finite subset of X (cf. Lemmata 3.8 and 3.1). Therefore, there must be $x\in\operatorname{supp}_{\varphi}(y)\subseteq X$ such that the set $\{i:x\in P_i\}$ is infinite. Since $P_{i_0}\supseteq P_{i_0+1}\supseteq\cdots$, we get $x\in X\cap P$, which contradicts (5.11). ∎

Let us remark that Theorem 1.3 follows immediately from Theorems 1.6 and 1.2 and [11, Theorem 3.2] (cf. [5, Proposition 31]).

Now, we present a proof of Theorem 1.5. Conceptually, the proof is virtually the same as the previous one. It is more technical though. This is because in place of Theorem 2.5, we need to use Theorem 2.6, i.e., instead of dealing with σ-compact subsets of the remainder $bM\backslash M$, we need to work with strategies in the game $kP(bM,bM\backslash M)$, which is a more complicated task.

Proof of Theorem 1.5 By symmetry, it is enough to show that the projective Menger property of X implies that Y is projectively Menger. To this end, suppose that X is projectively Menger and let us fix a continuous surjection $h:Y\to M$ that maps Y onto a separable metrizable space M. Let bM be a metrizable compactification of M, and let $\widetilde{h}:\beta Y\to bM$ be a continuous extension of h. Denote by d a metric on bM that generates the topology of bM. Note that

$$\text{(5.13)}\qquad \text{if } A\subseteq bM\backslash M,\ \text{then } \widetilde{h}^{-1}(A)\subseteq\beta Y\backslash Y.$$

In order to prove that M is Menger, we will employ Theorem 2.6. For this purpose, suppose that σ is a strategy for player I in the k-Porada game $kP(bM,bM\backslash M)$. We need to show that the strategy σ is not winning. Since h is surjective, we have $M\subseteq\bigcup_{n=1}^{\infty}\widetilde{h}(\overline{H_n})$, by Observation 5.2. So applying Proposition 2.4, we may, without loss of generality, assume that

$$\text{(5.14)}\qquad \text{every compact set played according to } \sigma \text{ meets } \bigcup_{n=1}^{\infty}\widetilde{h}(\overline{H_n}).$$

Using σ, we will recursively define a strategy τ for player I in the z-Porada game $zP(\beta X,\beta X\backslash X)$ (cf. Proposition 2.9). In addition, with each open set V_i played by player II in his $(i+1)$st move in the game $zP(\beta X,\beta X\backslash X)$ (where the strategy τ is applied by player I), we will associate a set V_i' played by player II in his $(i+1)$st move in $kP(bM,bM\backslash M)$ (where the strategy σ is applied by player I).

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Let $(K_0, U_0) = \sigma(\varnothing)$ be the first move played by player I according to σ. By (5.14), there exists n_0 such that the compact set

$$K'_{n_0} = \widetilde{h}^{-1}(K_0) \cap \overline{H_{n_0}}$$

is nonempty. Since bM is metrizable, the set K_0 being compact is G_δ in bM. It follows from Proposition 3.8 and Lemma 2.1 that the set

$$G_0 = s^{-1}_{\varphi^{-1}}\left(\widetilde{h}^{-1}(K_0)\right) = \{x \in \beta X : s_{\varphi^{-1}}(x) \cap \widetilde{h}^{-1}(K_0) \neq \varnothing\}$$

is a G_δ-set in βX. In addition, by (5.13) and Lemmas 3.5 and 3.1, we have $G_0 \subseteq \beta X \backslash X$.

The map e_{n_0} restricted to K'_{n_0} is lower semi-continuous (by Lemma 5.6), and note that if $y \in K'_{n_0}$, then $e_{n_0}(y) \subseteq G_0 \subseteq \beta X \backslash X$ (by definition of e_{n_0}). Thus, we may consider the map $e_{n_0} \restriction K'_{n_0}$ as a (lower semi-continuous) map into $\mathcal{K}(G_0)$. By Theorem 2.2, this map admits a compact section, i.e., there is a compact set $L_0 \subseteq G_0$ such that

$$L_0 \cap e_{n_0}(y) \neq \varnothing, \text{ for every } y \in K'_{n_0}.$$

In particular, since $e_{n_0}(y) \subseteq s_\varphi(y)$, we have

$$L_0 \cap s_\varphi(y) \neq \varnothing, \text{ for every } y \in K'_{n_0}. \tag{5.15}$$

Using Lemma 2.10, we can enlarge the set L_0 to a zero-set in βX contained in G_0. Clearly, this is still a section of e_{n_0}, so without loss of generality we can assume that each L_0 is a zero-set in βX.

We define

$$\tau(\varnothing) = (L_0, \beta X).$$

Let V_0 be the first move of player II in $zP(\beta X, \beta X \backslash X)$, i.e., V_0 is an arbitrary open set in βX containing L_0. Consider the following subset W_0 of bM:

$$W_0 = \widetilde{h}^{\#}\left(s^{-1}_\varphi(V_0) \cup (\beta Y \backslash \overline{H_{n_0}})\right).$$

Since s_φ is lower semi-continuous, it follows from Proposition 5.8(a) that W_0 is open in bM. Moreover, since $L_0 \subseteq V_0$, we infer from (5.15) that $K'_{n_0} = \widetilde{h}^{-1}(K_0) \cap \overline{H_{n_0}} \subseteq s^{-1}_\varphi(V_0)$ and thus $\widetilde{h}^{-1}(K_0) \subseteq s^{-1}_\varphi(V_0) \cup (\beta Y \backslash \overline{H_{n_0}})$. Hence, by Proposition 5.8(c), we get $K_0 \subseteq W_0$. Define

$$V'_0 = W_0 \cap U_0.$$

Clearly, $K_0 \subseteq V'_0 \subseteq U_0$, so V'_0 is a legal move of player II in $kP(bM, bM \backslash M)$. Let $(K_1, U_1) = \sigma(V'_0)$ be the response of player I, consistent with her strategy. By (5.14) and Observation 5.2, there is $n_1 > n_0$ such that the compact set

$$K'_{n_1} = \widetilde{h}^{-1}(K_1) \cap \overline{H_{n_1}}.$$

is nonempty. Arguing as before, we note that the set

$$G_1 = s^{-1}_{\varphi^{-1}}\left(\widetilde{h}^{-1}(K_1)\right)$$

is G_δ in βX and $G_1 \subseteq \beta X \backslash X$. Again, since $e_{n_1}(y) \subseteq G_1$, for $y \in K'_{n_1}$, we infer that the map $e_{n_1} \restriction K'_{n_1}$ (i.e., e_{n_1} restricted to K_{n_1}) maps the compact set K'_{n_1} lower

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semi-continuously into $\mathcal{K}(G_1)$. By Theorem 2.2, this map admits a compact section L_1. Again, using Lemma 2.10, we can assume that L_1 is a zero-set in βX.

We define

$$\tau(V_0) = (L_1, V_0).$$

Let V_1 be an arbitrary open set in βX satisfying $L_1 \subseteq V_1 \subseteq V_0$ (the next move of player II in $zP(\beta X, \beta X\backslash X)$). We set

$$W_1 = \widetilde{h}^{\#}\left(s_\varphi^{-1}(V_1) \cup (\beta Y\backslash\overline{H_{n_1}})\right).$$

The lower semi-continuity of s_φ and Proposition 5.8(a) imply that W_0 is open in bM. Arguing as before, we get that $K_1 \subseteq W_1$. Let V_1' be an open set in bM satisfying

$$K_1 \subseteq V_1' \subseteq \overline{V_1'} \subseteq W_1 \cap U_1.$$

We continue our construction following this pattern. In this way, we define a strategy τ for player I in the game $zP(\beta X, \beta X\backslash X)$. Moreover, a play

$$\tau(\varnothing),\ V_0,\ \tau(V_0),\ V_1,\ \tau(V_0, V_1), \ldots$$

in $zP(\beta X, \beta X\backslash X)$ generates the play

$$\sigma(\varnothing),\ V_0',\ \sigma(V_0'),\ V_1',\ \sigma(V_0', V_1'), \ldots$$

in $kP(bM, bM\backslash M)$, where

$$V_k' \subseteq \widetilde{h}^{\#}\left(s_\varphi^{-1}(V_k) \cup (\beta Y\backslash\overline{H_{n_k}}\right) \text{ and} \tag{5.16}$$

$$\overline{V_{k+1}'} \subseteq V_k'. \tag{5.17}$$

The numbers $n_0 < n_1 < \cdots < n_k < \cdots$ form an increasing sequence.

By our assumption, the space X is projectively Menger; hence, by Proposition 2.9, there is a play

$$\tau(\varnothing),\ V_0,\ \tau(V_0),\ V_1,\ \tau(V_0, V_1), \ldots$$

in which player I applies her strategy τ and fails, i.e.,

$$\varnothing \neq \bigcap_{k=0}^{\infty} V_k \subseteq \beta X\backslash X. \tag{5.18}$$

The above play generates the play

$$\sigma(\varnothing),\ V_0',\ \sigma(V_0'),\ V_1',\ \sigma(V_0', V_1'), \ldots$$

in $kP(bM, bM\backslash M)$. We claim that player II wins this run of the game and thus σ is not winning for player I.

Indeed, otherwise $M \cap \bigcap_{k=0}^{\infty} V_k' \neq \varnothing$ (note that (5.17) guarantees that the intersection of the family $\{V_k' : k = 0, 1, \ldots\}$ is nonempty by compactness). Fix $a \in M \cap \bigcap_{k=0}^{\infty} V_k'$. Since $h : Y \to M$ is surjective, there is $y \in Y$ with $h(y) = a$. Applying (5.16) and Proposition 5.8(b), we get

$$\widetilde{h}^{-1}(a) \subseteq s_\varphi^{-1}(V_k) \cup \left(\beta Y\backslash\overline{H_{n_k}}\right),$$

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for every k. On the other hand, it follows from Observation 5.2 that $y \in H_{n_k}$ for all but finitely many k's. Hence, we must have

$$y \in s_\varphi^{-1}(V_k), \text{ for all but finitely many } k\text{'s}.$$

In addition, $y \in Y$, so the set $s_\varphi(y) = \operatorname{supp}_\varphi(y)$ is a finite subset of X (cf. Lemmata 3.8 and 3.1). Therefore, there must be $x \in \operatorname{supp}_\varphi(y) \subseteq X$ such that the set $\{k : x \in V_k\}$ is infinite. Since $V_0 \supseteq V_1 \supseteq \cdots$, we get $x \in X \cap \bigcap_{k=0}^{\infty} V_k$, which contradicts (5.18). ∎

Proof of Theorem 1.1 By symmetry, it is enough to show that the Menger property of X implies the Menger property of Y. Suppose that X is Menger. Then X is Lindelöf, so by Velichko's Theorem 1.2, the space Y is Lindelöf too. Moreover, since X is Menger, it is projectively Menger (cf. Proposition 1.4), so according to Theorem 1.5, the space Y is projectively Menger. The result follows now from Proposition 1.4. ∎

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