OPTIMAL CONTROL FOR STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS AND VISCOSITY SOLUTIONS OF BELLMAN EQUATIONS

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§ 1. Introduction

Recently M. G. Crandall and P. L. Lions developed the viscosity theory on nonlinear equations in infinite dimensions and optimal control in Hilbert spaces, in two series of papers, [1], [4].

In this article we will study of optimal control of stochastic partial differential equations and viscosity solutions of Bellman equation (1.1) below,

$$(1.1) \quad \sup_u \left(-\frac{1}{2}(D^2\nu(\phi)M\phi, M\phi) - (D\nu(\phi), L(u)\phi) + \lambda\nu(\phi) - F(\phi) \right) = 0 \,,$$

where D and D^2 denote the first and second Fréchet differentials and M and $L(u)$ are the first and second order differential operators respectively (see (2.1)).

Let us consider the following stochastic partial differential equation, (SPDE in short)

$$
(1.2) \quad
\begin{aligned}
dq(t) &= \sum_{ij=0}^{d} \frac{\partial}{\partial x_i}\left(a^{ij}(x, U(t))\frac{\partial}{\partial x_j}q(t, x) + f^i(x, U(t)) \right)dt \\
&\quad + \sum_{k=1}^{m}\left(\sum_{i=0}^{d} b_k^i(x)\frac{\partial}{\partial x_i}q(t, x) + g_k(x) \right)dW^k(t) \,,
\end{aligned}
$$

where $W = (W^1, \cdots, W^m)$ is an m-dimensional standard Wiener process and $U(t)$ an admissible control. We will define the criterion J by

$$(1.3) \qquad J(\phi, U) = E\int_0^\infty e^{-\lambda t} F(q(t, \phi, U))dt$$

where $q(t, \phi, U)$ denotes a solution of (1.2) starting at ϕ. The function V defined by

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(1.4) $$V(\phi) = \inf_U J(\phi, U)$$

is called the value function. The SPDE (1.1) describes intuitively a physical object governed by a partial differential equation with random perturbation, which has been investigated from various view points, (see references in [5]). But other important example is the Zakai equation for controlled partially observed diffusions. In this case, inhomogeneous term f^i and g_k are zero and b_k^i arises from the correlation between system and observation noises.

We mainly study regularity of the value function, assuming ellipticity codition (see (A.2)) and smoothness of coefficients. Since the value function becomes semi-concave, we can show its differentiability via convex analysis arguments, (Theorem 4.2). Appealing to this property, we demonstrate the relationship between a viscosity solution of (1.1) and the value function.

We will devote Section 2 to the study of optimal control for (1.2), according to [5]. We introduce a relaxed system and show the existence of an optimal relaxed system, (Theorem 2.1). Section 3 deals with smooth properties of the criterion $J(\phi, U)$ defined by (1.3) and the value function (1.4). From the regularity assumptions on coefficients, we can easily see the differentiability of the criterion, (Proposition 3.1) and semi-concaveness of the value function, (Proposition 3.2). Using these properties, we discuss directional derivatives, via convex analysis, and prove that the value function is densely Gâteux differentiable in Section 4.

Section 5 is devoted to the study of a connection between the value function and the Bellman equation (1.1). We introduce a subsolution (in viscosity sense) according to [4], but we require one condition which says that its super differentials are densely non-empty, (see (5.16)). The value function turns out to be a subsolution of (1.1), (Theorem 5.1) and the maximum subsolution, which satisfies sub-Bellman principle, (Theorem 5.2). When b_k^i and g_k in (1.2) are zero, the SPDE (1.2) can be regarded as a parabolic equation and control problems turns out to be deterministic ones. Section 6 is a supplement and deals with a deterministic control.

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§ 2. Optimal control for SPDE

Let Γ be a convex compact metric space. We call it a control region.

Let us define the operators $L(u)$ and $M = (M_1, \cdots, M_m)$ by

$$(2.1) \qquad L(u)\psi(x) = \sum_{i,j=0}^{d} \frac{\partial}{\partial x_i} \left(a^{ij}(x, u) \frac{\partial}{\partial x_j} \psi(x) + f^i(x, u) \right),$$

$$\text{for} \quad x \in \mathbf{R}^d, \quad u \in \Gamma,$$

and

$$(2.2) \qquad M_k\psi(x) = \sum_{i=0}^{d} b_k^i(x) \frac{\partial}{\partial x_i} \psi(x) + g_k(x), \qquad \text{for } x \in \mathbf{R}^d,$$

respectively, where $\partial/\partial x_0 = $ identity, and a^{ij}, f^i, b_k^i and g_k are bounded and uniformly continuous.

We denote by H^k the Sobolev space of real valued functions defined on $\mathbf{R}^d$, with the following norm,

$$\|\psi\|_k^2 = \sum_{|\alpha| \leq k} \|D^\alpha \psi\|^2$$

where $\| \| = L^2(R^d)$-norm, $\alpha = (\alpha^1, \cdots, \alpha^d) = $ multi-index with non-negative integer α^i, $|\alpha| = \alpha^1 + \cdots + \alpha^d$ and $D^\alpha = (\partial/\partial x_1)^{\alpha^1} \cdots (\partial/\partial x_d)^{\alpha^d}$. $\mathfrak{H}$ denotes the subspace of H^4 with the norm defined by

$$\|\|\psi\|\| = \|\psi\|_4^2 + \|(1 + |x|^2)\psi\|_1^2.$$

For simplicity, we put $H^0 = H(= L^2(\mathbf{R}^d))$, if no confusion occurs.

Now we introduce the following conditions.

(A.1) $D^\alpha a^{ij}$, $D^\alpha b_k^i$ $(0 \leq |\alpha| \leq 4, i, j = 0, 1, \cdots d, k = 1, 2, \cdots, m)$ are bounded and uniformly continuous.

(A.2) ellipticity condition. $a^{ij} = a^{ji}$, $i, j = 1, \cdots d$ and $(a^{ij} - (3/2) b^i \cdot b^j)_{i, j=1, \cdots, d}$ is a non-negative definite matrix, where $b^i = (b_1^i, \cdots, b_m^i)$ and "$\cdot$" means the inner product in $\mathbf{R}^m$.

(A.3) $f^i(\cdot, u)$, $g_k \in \mathfrak{H}$ $(i = 0, \cdots, d, k = 1, \cdots, m)$ and their $\mathfrak{H}$ norms are bounded in u.

Hereafter we always assume (A.1)–(A.3) and for simplicity, we say

$$(2.3) \quad |D^\alpha a^{ij}(x, u)| \leq K, \quad |D^\alpha b_k^i(x)| \leq K, \quad \|\|f^i(\cdot, u)\|\| \leq K, \quad \|\|g_k\|\| \leq K.$$

To study relaxed systems (in wider sense), we need the following spaces. By Λ we denote the set of all Borel measures on $[0, \infty) \times \Gamma$, such that

$$(2.4) \qquad \lambda([0, s] \times \Gamma) = s, \qquad \text{for any } s \geq 0.$$

Endowing with the vague topology, we have the following proposition,

PROPOSIITON 2.1. Λ *is a compact metric space.*

Proof.   Let us see $\Lambda_n = \{\lambda/[0, n] \times \Gamma;\ \lambda \in \Lambda\}$, where λ/X is the restriction of λ on X.   The induced topology on Λ_n turns out to be the weak convergence topology and Λ_n becomes a compact metric space by applying the Prohorov metric, say ρ_n.   Since Λ is metrized by $\rho = \sum 2^{-n} \min(1, \rho_n)$, we complete the proof.   □

Let us set $B(\Gamma) =$ Borel field on Γ, $\sigma_t(\Lambda) =$ the σ-field generated by $\{\lambda([0, s] \times A);\ s \leqq t,\ A \in B(\Gamma)\}$ and $\sigma(\Lambda) =$ the σ-field generated by $\sigma_t(\Lambda)$, $t > 0$.   Let $\mathbf{P} = \mathbf{P}(\Lambda)$ be the space of all probabilities on $(\Lambda, \sigma(\Lambda))$, endowed with the weak convergence topology.   Then Prohorov's theorem asserts

PROPOSITION 2.2. $\mathbf{P}$ *is a compact metric space.*

By virtue of (2.4), λ has a $\sigma_t(\Lambda)$-adapted kernel λ', namely $\lambda(dt, du) = \lambda'(t, du)dt$, and $\lambda'(t, du)$ is a probability on Γ for almost all t.   Moreover, if $\hat{\lambda}$ is a kernel of λ, then $\hat{\lambda}(t, \cdot) = \lambda'(t, \cdot)$ for almost all t.   Let us set

$$\tilde{h}(t, x, \lambda) = \int_\Gamma h(x, u)\lambda'(t, du)\,, \qquad \text{for } h = a^{ij} \text{ and } f^i,$$

and

$$\begin{aligned}
(2.5) \qquad \tilde{L}(t, \lambda)\psi(x)dt &= \int_\Gamma L(u)\psi(x)\lambda'(t, du)dt \\
&= \sum_{ij=0}^{d} \frac{\partial}{\partial x_i}\Big(\tilde{a}^{ij}(t, x, \lambda)\frac{\partial}{\partial x_i}\psi(x) + \tilde{f}^i(t, x, \lambda)\Big)dt\,.
\end{aligned}$$

Now we introduce a relaxed system.

DEFINITION 2.1. $\mathscr{R} = (\Omega, \mathscr{F}, \mathscr{F}_t, \mathscr{P}, W, \mu)$ is called a relaxed system, if

(2.6) $(\Omega, \mathscr{F}, \mathscr{F}_t, \mathscr{P})$ is a probability space with filtration $\mathscr{F}_t$

(2.7) W is an $\mathscr{F}_t$-adapted m-dimensional Wiener process with $W(0) = 0$

and

(2.8) μ is an $\mathscr{F}_t$-adapted Λ-valued random variable (Λ-r.v. in short).

Namely $\mu(A \times B)$ is $\mathscr{F}_t$-measurable, whenever $A \in B[0, t]$ and $B \in B(\Gamma)$.

For simplicity, we put $\mathscr{R} = (W, \mu)$, if no confusion occurs, and sometimes we call μ a relaxed control. $\mathfrak{R}$ denotes the totality of relaxed systems.   Let $\pi(\mathscr{R})$ be the image measure on $C(([0, \infty); \mathbf{R}^d) \times \Lambda$ by $\mathscr{R} = (W, \mu)$.   Again endowing the weak convergence topology on the space $\prod = \{\pi(\mathscr{R});\ \mathscr{R} \in \mathfrak{R}\}$, we have the following proposition.

PROPOSITION 2.3. *Π is a compact metric space.*

Proof. Since W is a Wiener process and Λ is compact, Π is tight by the weak convergence topology. Put $\Omega = C([0, \infty); \mathbf{R}^d) \times \Lambda$ and $(w, \lambda) \in \Omega$. We define the usual σ-field and $\mathscr{F}_t$,

$$\mathscr{F} = \sigma\{w(s), s \in [0, \infty)\} \times \sigma(\Lambda) \quad \text{and} \quad \mathscr{F}_t = \sigma\{w(s), s \leqq t\} \times \sigma_t(\Lambda).$$

Suppose $\pi_n \ (\in \Pi)$ converges to π weakly. Then $\pi(\cdot \times \Lambda)$ is a Wiener measure. Moreover, for any bounded continuous f and g, we see

$$\int_\Omega f(w(t_1 + t) - w(t), \cdots, w(t_k + t) - w(t))g(w(s_1),$$
$$\cdots, w(s_j), \mu(A_1 \times B_1), \cdots, \mu(A_j \times B_j))d\pi_n$$
$$= \int_\Omega f(w(t_1), \cdots, w(t_k))d\pi_n \int_\Omega g(w(s_1), \cdots, w(s_j), \mu(A_1 \times B_1),$$
$$\cdots, \mu(A_j \times B_j))d\pi_n$$

whenever $s_1, \cdots, s_j \leqq t$ and $A_1, \cdots, A_j \subset [0, t]$. As $n \to \infty$, we have the same equality replacing π_n by π. Now setting $W(w, \lambda) = w$ and $\mu(w, \lambda) = \lambda$, W becomes an $\mathscr{F}_t$-Wiener process and $\pi \in \Pi$. This completes the proof. $\square$

DEFINITION 2.2. We say $\mathscr{R}_n$ converges to $\mathscr{R}$ (put $\mathscr{R}_n \to \mathscr{R}$), if $\pi(\mathscr{R}_n) \to \pi(\mathscr{R})$ weakly.

Consider the SPDE (2.9) for $\mathscr{R} = (W, \mu)$,

$$(2.9) \qquad dq(t) = \tilde{L}(t, \mu)q(t)dt + Mq(t)dW(t), \quad q(0) = \phi(\in \mathfrak{H}).$$

An H^1-valued $\mathscr{F}_t$-adapted process $q = q(\cdot, \phi, \mathscr{R})$ is called a solution of (2.9), if (2.10) and (2.11) hold.

$$(2.10) \qquad E\int_0^T \|q(t)\|_1^2 dt < \infty, \quad \text{for any } T < \infty,$$

and for any $\eta \in C_0^\infty$ (smooth function on $\mathbf{R}^d$ with compact support) and almost all t

$$(2.11) \qquad (q(t), \eta) = (\phi, \eta) + \int_0^t \langle \tilde{L}(s, \mu)q(s), \quad \eta > ds$$
$$+ \int_0^t (Mq(s), \eta)dW(s), \quad \text{with probability 1,}$$

holds, where $(\ ,\ ) = L^2(\mathbf{R}^d)$-inner product and $\langle\ ,\ \rangle =$ duality pairing between H^{-1} and H^1 under $H^0 = (H^0)^*$ ($=$ dual space of H^0). Clearly (2.11) does not depend on any special choice of kernel μ'.

According to Krylov-Rozovskii [3] (cf. [5]), there is a unique solution $q = q(\cdot, \phi, \mathscr{R})$ in the function space below

$$(2.12) \qquad q \in L^2((0, T) \times \Omega; \mathfrak{H}) \cap L^2(\Omega; C(0, T; H^3)), \qquad \text{for } T < \infty .$$

Moreover, there are two constants $N = N(T)$ and c, independent of $\mathscr{R}$, such that

$$(2.13) \qquad E(\sup_{t \leq T} \|q(t)\|_k^2) \leq N(\|\phi\|_k^2 + A_k), \qquad \text{for } k = 0, 1, 2, 3 ,$$

$$(2.14) \qquad E(\sup_{t \leq T} \|q(t)\|_k^4) \leq N(\|\phi\|_k^4 + B_k), \qquad \text{for } k = 0, 1, 2, 3 ,$$

$$(2.15) \qquad E\|q(t)\|^2 \leq e^{ct}(\|\phi\|^2 + A_0), \qquad \text{for } t > 0$$

and

$$(2.16) \qquad E(\sup_{s \leq t \leq \theta} \|q(t) - q(s)\|_j^2) \leq N(\theta - s)(\|\psi\|_{j+1}^2 + A_{j+1}),$$
$$\text{for } \theta \leq T \text{ and } j = 0, 1, 2 ,$$

where

$$A_k = \sup_u \sum_{i=0}^d \left\| \frac{\partial}{\partial x_i} f^i(\cdot, u) \right\|_k^2 + \sum_{j=1}^m \|g_j\|_{k+1}^2 ,$$

and

$$B_k = \sup_u \sum_{i=0}^d \left\| \frac{\partial}{\partial x_i} f^i(\cdot, u) \right\|_k^4 + \sum_{j=0}^m \|g_j\|_{k+1}^4 .$$

Now we recall how q depends on $\mathscr{R}$ and ϕ.

PROPOSITION 2.4 ([5]). *Support $\mathscr{R}_n \to \mathscr{R}$. Then, for $\phi \in \mathfrak{H}$*

$$(2.17) \qquad q(t, \phi, \mathscr{R}_n) \to q(t, \phi, \mathscr{R}) \qquad \text{in law as } H^1\text{-r.v.}$$

For $\phi, \psi \in \mathfrak{H}$, we see

$$(2.18) \quad E(\sup_{t \leq T} \|q(t, \phi, \mathscr{R}) - q(t, \psi, \mathscr{R})\|_k^2 \leq N\|\phi - \psi\|_k^2, \text{ for } k = 0, 1, 2, 3,$$

and $p(t, \phi - \psi, \mathscr{R}) = q(t, \phi, \mathscr{R}) - q(t, \psi, \mathscr{R})$ satisfies the linear SPDE (2.19) below

$$(2.19) \qquad \begin{cases} dp(t, x) = \sum_{i,j=0}^d \frac{\partial}{\partial x_i} \left(\tilde{a}^{ij}(t, x, \mu) \frac{\partial}{\partial x_j} p(t, x) \right) dt \\ \qquad\qquad + \sum_{k=1}^m \sum_{i=0}^d b_k^i(x) \frac{\partial}{\partial x_i} p(t, x) dW^k(t) \\ p(0, x) = \phi(x) - \psi(x) . \end{cases}$$

Let $F : H \to \mathbf{R}^1$ be bounded and Lipschitz continuous, say

$$(2.20) \qquad |F(\phi)| \leqq b, \qquad \text{for } \phi \in H$$

and

$$(2.21) \qquad |F(\phi) - F(\psi)| \leqq a \|\phi - \psi\|.$$

For $\mathscr{R} \in \mathfrak{R}$, we will define the cirterion J and the value function V as follows

$$(2.22) \qquad J(\phi, \mathscr{R}) = E \int_0^\infty e^{-\lambda t} F(q(t, \phi, \mathscr{R})) \, dt$$

where $\lambda \ (> 0)$ is a given constant and

$$(2.23) \qquad V(\phi) = \inf_{\mathscr{R}} J(\phi, \mathscr{R}),$$

respectively. Since $J(\phi, \mathscr{R})$ is continuous in $\mathscr{R}$ by Proposition 2.4, Proposition 2.3 asserts the following theorem.

THEOREM 2.1. *There is an optimal relaxed system $\mathscr{R}^* = \mathscr{R}^*(\phi)$, namely*

$$(2.24) \qquad V(\phi) = J(\phi, \mathscr{R}^*).$$

From (2.18) and (2.21), we can easily see that, for $\varepsilon > 0$, there is $\delta = \delta(\varepsilon) > 0$ such that

$$(2.25) \qquad |J(\phi, \mathscr{R}) - J(\psi, \mathscr{R})| < \varepsilon, \qquad \text{for any } \mathscr{R} \in \mathfrak{R},$$

whenever $\phi, \psi \in \mathfrak{H}$ and $\|\phi - \psi\| < \delta$. Therefore

$$(2.26) \qquad |V(\phi) - V(\psi)| < \varepsilon, \qquad \text{for } \phi, \psi \in \mathfrak{H} \text{ and } \|\phi - \psi\| < \delta.$$

Since $\mathfrak{H}$ is dense in H, $J(\cdot, \mathscr{R})$ and V can be extended on H, say $\bar{J}$ and $\bar{V}$. Moreover $\bar{J}$ and $\bar{V}$ satisfy (2.25) and (2.26) respectively, and

$$(2.27) \qquad \bar{V}(\phi) = \inf_{\mathscr{R}} \bar{J}(\phi, \mathscr{R}), \qquad \text{for } \phi \in H$$

still holds. $\bar{J}(\phi, \mathscr{R})$ is also continuous in $\mathscr{R}$, by (2.25). So Proposition 2.3 derives

PROPOSITION 2.5. *There is $\bar{\bar{\mathscr{R}}} = \bar{\bar{\mathscr{R}}}(\phi)$, $\phi \in H$, such that*

$$(2.28) \qquad \bar{V}(\phi) = \bar{J}(\phi, \bar{\bar{\mathscr{R}}}).$$

When a relaxed system $\mathscr{R} = (\Omega, \mathscr{F}, \mathscr{F}_t, \mathscr{P}, \mathscr{W}, \mu)$ satisfies the condition (2.29) below

(2.29) $\mu'(t, \, \cdot \,) = \delta_{U(t)}(\cdot)$

where U is an $\mathscr{F}_t$-adapted process with values in Γ and $\delta_x = \delta$-measure at x, $\mathscr{R}$ is called an admissible system and denoted by $A = (\Omega, \mathscr{F}, \mathscr{F}_t, \mathscr{P}, \mathscr{W}, \mathscr{U})$ or $A = (W, U)$. Let us set

$$a_N = \{A = (W, U); \; U \text{ is } W\text{-adapted and } U(t) = U([2^N t] 2^{-N})\}$$

where [ ] = Gauss symbol. Now using the same arguments as [5. Sect. 5], we obtain the approximation theorem below

THEOREM 2.2.

(2.30) $V(\phi) = \varliminf\limits_{\substack{N \to \infty \\ a_N}} E \int_0^\infty e^{-\lambda t} F(q(t, \phi, A)) dt$

(2.31) $\inf\limits_{\mathscr{R}} E \int_0^T e^{-\lambda t} F(q(t, \phi, \mathscr{R})) dt + e^{-\lambda t} \mathrm{G}(q(T, \phi, \mathscr{R}))$

$$= \varliminf\limits_{\substack{N \to \infty \\ a_N}} E \int_0^T e^{-\lambda t} F(q(t, \phi, A)) dt + e^{-\lambda T} G(q(t, \phi, A))$$

for $G \in \mathrm{BUG}(H)$ (= the space of all bounded and uniformly continuous functions on H, with the supremum norm.)

Thus the Bellman principle is proved by the routine.

THEOREM 2.3 (Bellman principle). *For any T*

$$V(\phi) = \inf\limits_{\mathscr{R}} E \int_0^T e^{-\lambda t} F(q(t, \phi, \mathscr{R})) dt + e^{-\lambda T} V(q(T, \phi, \mathscr{R}))$$

$$= \varliminf\limits_{\substack{N \to \infty \\ a_N}} E \int_0^T e^{-\lambda t} F(q(t, \phi, A)) dt + e^{-\lambda \Gamma} V(q(T, \phi, A))$$

holds.

For the proof we show two inequalities, i.e.

(2.32) $V(\phi) \leqq \varliminf\limits_{\substack{N \to \infty \\ a_N}} E \int_0^T e^{-\lambda t} F(q(t, \phi, A)) dt + e^{-\lambda T} V(q(T, \phi, A))$

and the converse inequality. The converse is easy by the definition of V, but not for (2.32). With this consideration, we will call the following inequality (2.33) a sub Bellman principle

(2.33) $V(\phi) \leqq E \int_0^T e^{-\lambda t} F(q(t, \phi, u)) dt + e^{-\lambda T} V(q(T, \phi, u))$

                                for $T < \infty$ and $u \in \Gamma$.

In Section 5 we will use (2.33), (see Theorem 5.2).

§ 3. Properties of J and V

Hereafter we always assume the following conditions (A.4) and (A.5), besides (A.1)–(A.3).

(A.4) $\lambda > c$, where c is the constant in (2.15).

(A.5) F is the twice Fréchet differentiable and DF and D^2F satisfy (3.1) and (3.2) below

$$(3.1) \qquad \| DF(\phi) \| \leqq \tilde{K}, \quad [\![D^2F]\!] \leqq \tilde{K} \qquad \text{for } \phi \in H,$$

with some constant $\tilde{K}$, and

$$(3.2) \qquad [\![D^2F(\phi) - D^2F(\psi)]\!] \to 0 \text{ uniformly, as } \| \phi - \psi \| \to 0,$$

where we identify $DF(\phi)$ with an element of H and $D^2F(\phi)$ is identified with a bounded self-adjoint operator on H. $L(H)$ denotes the space of all bounded self-adjoint operators on H and $[\![\cdot]\!]$ is its norm.

For example, a taim function F, written by $F(\phi) = f((e_1, \phi), \cdots, (e_n, \phi))$ with a smooth function f on R^n, satisfies (A.5).

First we study differentiability of $\bar{J}$. Putting

$$(3.3) \qquad G(\phi, \mathscr{R}; \psi) = \int_0^\infty e^{-\lambda t} E(DF(q(t, \phi, \mathscr{R})), p(t, \psi, \mathscr{R})) \, dt$$

we see, from (A.5).

$$(3.4)
\begin{aligned}
& |J(\phi + \varepsilon\psi, \mathscr{R}) - J(\phi, \mathscr{R}) - \varepsilon G(\phi, \mathscr{R}; \psi)| \\
&= \Bigg| E \int_0^\infty e^{-\lambda t} F(q(t, \psi + \varepsilon\psi, \mathscr{R})) - F(q(t, \phi, \mathscr{R})) \\
&\qquad\qquad\qquad - \varepsilon(DF(q(t, \phi, \mathscr{R}), p(t, \psi, \mathscr{R}))) \, dt \Bigg| \\
&\leqq \frac{\varepsilon^2}{2} \int_0^\infty e^{-\lambda t} \tilde{K} E \| p(t, \psi, \mathscr{R}) \|^2 dt \leqq \frac{\varepsilon^2 \tilde{K}}{2(\lambda - c)} \| \psi \|^2 .
\end{aligned}$$

Moreover we have

$$(3.5)
\begin{aligned}
& |G(\phi + \xi, \mathscr{R}; \psi) - G(\phi, \mathscr{R}; \psi)| \\
&\leqq \left| \int_0^\infty e^{-\lambda t} E(DF(q(t, \phi + \xi, \mathscr{R})) - DF(q(t, \phi, \mathscr{R})), p(t, \psi, \mathscr{R})) \, dt \right| \\
&\leqq \int_0^\infty e^{-\lambda t} \tilde{K} E \| p(t, \xi, \mathscr{R}) \| \| p(t, \psi, \mathscr{R}) \| \, dt \\
&\leqq \tilde{K} \int_0^\infty e^{-\lambda t} \cdot e^{-ct} \| \xi \| \| \psi \| \, dt = \frac{\tilde{K}}{\lambda - c} \| \xi \| \| \psi \| .
\end{aligned}$$

On the other hand, $G(\phi, \mathscr{R}; \psi)$ is linear in ψ, since $p(t, \psi, \mathscr{R})$ is a unique solution of linear SPDE (2.19). By (3.5), $G(\cdot, \mathscr{R}, \cdot)$ can be extended on $H \times H$, say $\overline{G}(\cdot, \mathscr{R}, \cdot)$, and $\overline{G}(\phi, \mathscr{R}; \psi)$ is linear in $\psi \in H$ and satisfies

$$(3.6) \qquad |\overline{G}(\phi, \Re; \psi)| \leqq \tilde{K}\Big/\Big(\lambda - \frac{c}{2}\Big)$$

and

$$(3.7) \qquad |\overline{G}(\phi + \xi, \mathscr{R}; \psi) - \overline{G}(\phi, \mathscr{R}; \psi)| \leqq \frac{\tilde{K}}{\lambda - c}\|\xi\|\|\psi\|$$

Again (3.4) derives

$$(3.8) \qquad |\overline{J}(\phi + \varepsilon\psi, \mathscr{R}) - \overline{J}(\phi, \mathscr{R}) - \varepsilon\overline{G}(\phi, \mathscr{R}; \psi)| \leqq \frac{\varepsilon^2\tilde{K}}{2(\lambda - c)}\|\psi\|^2.$$

Since $\overline{G}(\phi, \mathscr{R}; \psi)$ is linear in ψ, (3.7) and (3.8) yield the following proposition.

PROPOSITION 3.1. $D\overline{J}(\phi, \mathscr{R}) = \overline{G}(\phi, \mathscr{R}; \cdot)$, and

$$(3.9) \qquad \|D\overline{J}(\phi + \xi, \mathscr{R}) - D\overline{J}(\phi, \mathscr{R})\| \leqq \frac{\tilde{K}}{\lambda - c}\|\xi\|.$$

From (3.6), we can easily see

$$(3.10) \qquad |\overline{V}(\phi) - V(\psi)| \leqq \Big(\tilde{K}\Big/\Big(\lambda - \frac{c}{2}\Big)\Big)\|\phi - \psi\|.$$

For the differentiability of $\overline{V}$, we first show its concavity. A contiunous function h is called semi-concave, if, with some positive c

$$h(\phi + \psi) + h(\phi - \psi) - 2h(\phi) \leqq c\|\psi\|^2, \qquad \text{for any } \phi, \psi \in \mathfrak{H}.$$

Now we first show

PROPOSITION 3.2. $\overline{V}$ is semi-concave.

Proof. It is enough to show (3.11) below.

$$(3.11) \quad V(\phi + \psi) + V(\phi - \psi) - 2V(\phi) \leqq \frac{\tilde{K}}{\lambda - c}\|\psi\|^2, \qquad \text{for } \phi, \psi \in \mathfrak{H}.$$

From (A.5), we see

$$
\begin{aligned}
&J(\phi + \psi, \mathscr{R}) + J(\phi - \psi, \mathscr{R}) - 2J(\phi, \mathscr{R}) \\
(3.12) \quad &= E\int_0^\infty e^{-\lambda t} F(q(t, \phi + \psi, \mathscr{R}) + F(q(t, \phi - \psi, \mathscr{R})) - 2F(q(t, \phi, \mathscr{R})))\,dt \\
&= E\int_0^\infty e^{-\lambda t}\,\tilde{K}\|p(t, \psi, \mathscr{R})\|^2\,dt \leqq \frac{\tilde{K}}{\lambda - c}\|\psi\|^2.
\end{aligned}
$$

Using the inequality "$\inf x_i + \inf y_i - \inf z_i \leqq \sup(x_i + y_i - z_i)$", we obtain (3.11).

Let us set

$$(3.13) \qquad v(\phi) = \overline{V}(\phi) - \frac{\tilde{K}}{2(\lambda - c)} \|\phi\|^2\,, \qquad \text{for } \phi \in H\,.$$

Then v is locally Lipschitz and concave.

§ 4. Differentials of $\overline{V}$

Now we will study differential of v of (3.13), appealing to the arguments of convex analysis. For any ϕ and ψ in H, $v(\phi + \theta\psi) - v(\phi)$ has the right and left derivatives with respect to θ, say $\partial^+ v(\phi; \psi)$ and $\partial^- v(\phi; \psi)$ respectively, namely

$$(4.1) \qquad \partial^+ v(\phi; \psi) = \lim_{\theta \downarrow 0} \frac{v(\phi + \theta\psi) - v(\phi)}{\theta}$$

and

$$(4.2) \qquad \partial^- v(\phi; \psi) = \lim_{\theta \uparrow 0} \frac{v(\phi + \theta\psi) - v(\phi)}{\theta} = -\partial^+ v(\phi; -\psi)\,.$$

Moreover the following properties hold.

$$(4.3) \qquad \partial^+ v(\phi; \psi) = \sup_{\theta > 0} \frac{v(\phi + \theta\psi) - v(\phi)}{\theta}$$

and

$$(4.4) \qquad \partial^+ v(\phi; \psi) \leqq \partial^- v(\phi; \psi)\,.$$

Using the standard arguments of convex analysis [cf. 6], we have following Propositions 4.1–4.3.

PROPOSITION 4.1.

$$(4.5) \qquad \partial^+ v(\phi; \psi) \text{ is lower semi-continuous}$$

$$(4.6) \qquad |\partial^+ v(\phi; \psi) - \partial^+ v(\phi; \eta)| \leqq \frac{\tilde{K}}{\lambda - c}(1 + \|\phi\|)\|\psi - \eta\|$$

$$(4.7) \qquad \varlimsup_{\xi \to \phi} \partial^+ v(\xi; \psi) = \partial^- v(\phi; \psi)$$

Proof. (4.5) is easy by the routine method of convex analysis. Since (4.6) is clear by (3.10) and (3.13), we will only show (4.7).

Since $v(\phi + \theta\psi)$ is concave in $\theta \in (-\infty, \infty)$, $\partial^+ v(\phi + \varepsilon\psi; \psi)$ is decreasing in ε. Moreover

$$(4.9) \qquad \lim_{\varepsilon\uparrow 0} \partial^+ v(\phi + \varepsilon\psi; \psi) \geqq \partial^- v(\phi; \psi).$$

Thus we get

$$(4.10) \qquad \overline{\lim_{\xi\to\theta}} \partial^+ v(\xi; \psi) \geqq \lim_{\varepsilon\uparrow 0} \partial^+ v(\phi + \varepsilon\psi; \psi) \geqq \partial^- v(\phi; \psi).$$

On the other hand, we see from (4.2) (4.4) and (4.5),

$$(4.11) \qquad \begin{aligned} \overline{\lim_{\xi\to\phi}} \partial^+ v(\xi; \psi) &\leqq \overline{\lim_{\xi\to\phi}} \partial^- v(\xi; \psi) = \overline{\lim_{\xi\to\phi}} (-\partial^+ v(\xi; -\psi)) \\ &= -\lim_{\xi\to\phi} \partial^+ v(\xi; -\psi) \leqq -\partial^+(\phi; -\psi) = \partial^- v(\phi; \psi). \end{aligned}$$

Combining (4.11) with (4.10), we have (4.7).

PROPOSITION 4.2. $\partial^+ v(\cdot; \psi)$ *is continuous at ϕ, if and only if*

$$(4.12) \qquad \partial^+ v(\phi; \psi) = \partial^- v(\phi; \psi)$$

holds, namely two-side derivative, say $\partial v(\phi; \psi)$, exists.

Proof. Suppose (4.12) holds. Then (4.5) and (4.7) derive

$$(4.13) \qquad \partial^+ v(\phi; \psi) \leqq \underline{\lim_{\xi\to\phi}} \partial^+ v(\xi; \psi) \leqq \overline{\lim_{\xi\to\phi}} \partial^+ v(\xi; \psi) = \partial^- v(\phi; \psi).$$

Hence $\partial^+ v(\cdot; \psi)$ is continuous at ϕ.

For the converse, we assume

$$(4.14) \qquad \partial^+ v(\phi; \psi) = \overline{\lim_{\xi\to\phi}} \partial^+ v(\xi; \psi) = \underline{\lim_{\xi\to\phi}} \partial^+ v(\xi; \psi).$$

Then (4.7) derives (4.12). □

For $\psi \in H$, we will define $\mathbf{D}_\psi$ and h_ψ by

$$(4.15) \qquad \mathbf{D}_\psi = \{\phi \in H; \partial v(\phi; \psi) \text{ exists}\}$$

and

$$(4.16) \quad h_\psi(\phi) = \partial^+ v(\phi; \psi) + \partial^+ v(\phi; -\psi) \quad (= \partial^+ v(\phi; \psi) - \partial^- v(\phi; \psi)),$$

respectively. Then h_ψ is lower semi-continuous by (4.5) and $h_\psi(\phi) < 0$ for $\phi \notin \mathbf{D}_\psi$. Let us set

$$S_p = \left\{\phi \in H; h_\psi(\phi) \leqq -\frac{1}{p}\right\}, \quad p = 1, 2, \cdots.$$

Then S_p is closed and increasing to $\mathbf{D}_\psi^c$ as $p \uparrow \infty$.

PROPOSITION 4.3. $\{\theta \in (-\infty, \infty); \xi + \theta\psi \notin \mathbf{D}_\psi\}$ *is countable, for any ξ and ψ in H.*

Proof. For the proof we may assume $(\psi, \xi) = 0$. Put

$$g^+(\theta) = \partial^+ v(\xi + \theta\psi; \psi) \quad \text{and} \quad g^-(\theta) = \partial^- v(\xi + \theta\psi; \psi).$$

Then g^+ and g^- are locally bounded and decreasing. Moreover

$$(4.17) \qquad g^-(\theta) \geqq g^+(\theta) \geqq \lim_{\varepsilon \downarrow 0} g^-(\theta + \varepsilon) \quad (\text{say } g^-(\theta +))$$

holds, and $h_\psi(\xi + \theta\psi) = g^+(\theta) - g^-(\theta) \geqq g^-(\theta +) - g^-(\theta)$.

Thus, for $\delta > 0$, $\{\theta \in [\alpha, \beta]; g^-(\theta +) - g^-(\theta) < -\delta\}$ is finite. This implies $\{\theta \in [\alpha, \beta]; h_\psi(\xi + \theta\psi) < -\delta\}$ is finite and $\{\theta \in (-\infty, \infty); \xi + \theta\psi \in S_p\}$ is countable. Since $\mathbf{D}_\psi^c = \bigcup_p S_p$, we complete the proof.

Let e_k, $k = 1, 2, \cdots$ be an orthonormal base of H. Let X_k, $k = 1, 2, \cdots$ be independent $N(0, v_k)$-distributed Gaussian random variables on a probability space $(\Omega, \mathscr{F}, \mathscr{P})$. We assume

$$(\text{A}.6) \qquad v_k > 0 \text{ for } k = 1, 2, \cdots \quad \text{and} \quad \sum v_k < \infty.$$

Hence $P(\sum X_k^2 < \infty) = 1$ and

$$(4.18) \qquad P(\sum X_k e_k \in H) = 1.$$

Q denotes the probability law of $\sum X_k e_k$. So, $(H, \sigma(H), Q)$ is a probability space and $Y_k(w) = (w, e_k)/\sqrt{v_k}$, $k = 1, 2, \cdots$, turn out to be $N(0, 1)$-distributed independent random variables on $(H, \sigma(H), Q)$. For finite sum of e_k, say $\psi = \sum_{j=1}^n r_j e_j \ (\neq 0)$, we can take a new orthogonal base f_k, $k = 1, 2, \cdots$ such that $f_1 = \psi$, linear space of $(f_1 \cdots, f_n) =$ linear space of $(e_1, \cdots e_n)$ and $f_{n+i} = e_{n+i}$, $i = 1, 2, \cdots$ and Z_k; $Z_k(w) = (w, f_k)$, are independent Gaussian random variables. Appealing to Proposition 4.3, we show

PROPOSITION 4.4. $Q(\mathbf{D}_\psi^c) = 0$, *for* $\psi = \sum_{j=1}^n r_j e_j \ (\neq 0)$.

Proof. Let us set

$$Y(w) = w - Z_1(w)\psi = \sum_{k=2}^\infty Z_k(w) f_k, \qquad \text{for } w \in H.$$

Then Y and $Z_1\psi$ are independent. Hence, for any bounded Borel function g on $H \times H$, we have

$$E(g(Y, Z_1\psi)/\sigma(Y)) = \int_H g(Y, u)\mu(du)$$

where μ is the probability distribution of $Z_1\psi$. Since Z_1 has a normal distribution N, we get

$$\int_H g(Y, u)\mu(du) = \int_{-\infty}^{\infty} g(Y, \theta\psi)N(d\theta).$$

This derives, for any $A \in \sigma(H)$

$$Q(A) = \int_H \left(\int_{-\infty}^{\infty} I_A(y + \theta\psi)N(d\theta) \right) \nu(dy)$$

where ν is the probability distribution of Y. Now putting $A = \mathbf{D}_\psi^c$, Proposition 4.3 completes the proof.   □

Let us set

(4.20) $\mathscr{A} = \left\{ \sum_{k=1}^{n} r_k e_k \neq 0; \ r_k = \text{rational}, \ k = 1, 2, \cdots n, \ n = 1, 2, \cdots \right\}$

and

(4.21) $\mathbf{D} = \bigcap_{\psi \in \mathscr{A}} \mathbf{D}_\psi.$

Then $\mathscr{A}$ is a countable dense set of H and $\mathbf{D} \in \sigma(H)$. Now Proposition 4.4 yields

THEOREM 4.1.   *For $\phi \in \mathbf{D}$, $\partial v(\phi; \psi)$ and $\partial v(\phi; -\psi)$ exist for $\psi \in H$.*

Proof.   For $\phi \in \mathbf{D}$ and $\psi \in \mathscr{A}$, (4.15) implies

(4.22) $\partial^+ v(\phi; \psi) = -\partial^+ v(\phi; -\psi).$

But, since $\mathscr{A}$ is dense, (4.22) still holds for $\psi \in H$ by (4.6). This completes the proof.   □

Next we will give one remark on a full probability set of H.

PROPOSITION 4.5.   *If $Q(A) = 1$ for a set A of H, then A is dense in H.*

Proof.   Let us set $X_k(w) = (w, e_k)$, $k = 1, 2, \cdots$. Then X_k is independent and $N(0, v_k)$-distributed. From (A.6), we can choose, for $\varepsilon > 0$, $N(\varepsilon)$ such that $N > n$ and

(4.23) $Q(\sum_{k=N+1}^{\infty} X_k^2 < \varepsilon) > \dfrac{1}{2}$

holds.   On the other hand

$$(4.24) \qquad Q\Big(\sum_{k=1}^{N} |X_k - r_k|^2 < \varepsilon\Big) > 0$$

where $r_k = 0$ for $k \geqq n + 1$, since $X_1, \cdots, X_n$ are independent Gaussian random variables with $EX_k^2 > 0$.   Moreover the independence of X_k implies

$$Q(B(2\varepsilon : \psi)) \geqq Q\Big(\sum_{k=1}^{N} |X_k - r_k|^2 < \varepsilon\Big) Q\Big(\sum_{k=N+1}^{\infty} X_k^2 < \varepsilon\Big) > 0$$

where $B(c; \psi) = \{w \in H; \|w - \psi\|^2 < c\}$.   Therefore

$$(4.25) \qquad Q(A \cap B(2\varepsilon; \psi)) > 0 .$$

Thus, for any $\psi \in \mathscr{A}$ and $\varepsilon > 0$, $A \cap B(2\varepsilon : \psi)$ is non-empty,   This completes the proof.   □

Next we will show that $\partial v(\phi; \psi)$ is linear in ψ.   From the definition, $\partial^+ v(\phi, \psi)$ is clearly positively homogeneous in ψ, i.e. for $\phi, \xi \in H$ and $\theta > 0$,

$$(4.26) \qquad \theta \partial^+ v(\phi; \xi) = \partial^+ v(\phi; \theta\xi) \quad \text{and} \quad \theta \partial^- v(\phi; \xi) = \partial^- v(\phi; \theta\xi) .$$

Therefore, for $\phi \in \mathbf{D}$, $\xi \in H$ and $\theta > 0$, we see

$$(4.27) \qquad \partial^+ v(\phi; -\theta\xi) = -\partial^- v(\phi; \theta\xi) = -\theta \partial^- v(\phi; \xi) = -\theta \partial^+ v(\phi; \xi) .$$

For $\phi, \xi, \eta \in H$, we will prove inequalities (4.28) and (4.29) below

$$(4.28) \qquad \partial^+ v(\phi; \xi + \eta) \leqq \partial^- v(\phi; \xi) + \partial^+ v(\phi; \eta)$$

$$(4.29) \qquad \partial^- v(\phi; \xi + \eta) \leqq \partial^+ v(\phi; \xi) + \partial^- v(\phi; \eta)$$

Proof.   For $\varepsilon > 0$, we have

$$\frac{v(\phi + \varepsilon\xi + \varepsilon\eta) - v(\phi + \varepsilon\eta)}{\varepsilon} \leqq \sup_{\theta > 0} \frac{v(\phi + \varepsilon\xi + \theta\eta) - v(\phi + \varepsilon\eta)}{\theta}$$

$$= \partial^+ v(\phi + \varepsilon\eta; \xi) .$$

Hence we see

$$\frac{v(\phi + \varepsilon\xi + \varepsilon\eta) - v(\phi)}{\varepsilon} \leqq \partial^+ v(\phi + \varepsilon\eta; \xi) + \frac{v(\phi + \varepsilon\eta) - v(\phi)}{\varepsilon} .$$

As $\varepsilon \downarrow 0$, we get (4.28) by virtue of (4.7).

Appealing to "$\partial^- v(\phi; \psi) = -\partial^+ v(\phi; -\psi)$", (4.28) yields (4.29).   □

Now Theorem 4.1 derives, for $\phi \in \mathbf{D}$

$$(4.30) \qquad \partial v(\phi; \xi + \eta) = \partial v(\phi; \xi) + \partial v(\phi; \eta) , \qquad \text{for } \xi, \eta \in H .$$

Combining (4.30) with (4.26) and (4.27), we have

(4.31) $\partial v(\phi; a\xi + b\eta) = av(\phi; \xi) + bv(\phi; \eta)$,   for $\phi \in \mathbf{D}$.

Again (4.6) implies "$\partial v(\phi; \cdot) \in H^*$". Since $\|\phi\|^2$ is differentiable, we get

THEOREM 4.2. $\overline{V}$ *is Gâteux differentiable at* $\phi \in \mathbf{D}$ *and its Gâteux derivative* $\partial\overline{V}(\phi) \in H^*$.

Let $\overline{\mathscr{R}} = \overline{\mathscr{R}}(\phi)$ be an optimal one, which satisfies (2.28). Recalling Proposition 3.1, we have

THEOREM 4.3. $\partial\overline{V}(\phi) = D\overline{J}(\phi; \overline{\mathscr{R}})$, *for* $\phi \in \mathbf{D}$.

Proof. Since $\overline{V}(\phi) = \overline{J}(\phi; \overline{\mathscr{R}})$ and $\overline{V}(\phi + \xi) \leqq \overline{J}(\phi + \xi; \overline{\mathscr{R}})$ for $\xi \in H$, we see from (3.4)

$$\overline{V}(\phi + \varepsilon\psi) - \overline{V}(\phi) \leqq \overline{J}(\phi + \varepsilon\psi; \overline{\mathscr{R}}) - \overline{J}(\phi; \overline{\mathscr{R}})$$
$$\leqq \varepsilon(D\overline{J}(\phi; \overline{\mathscr{R}}), \psi) + \frac{\varepsilon^2 \tilde{K}}{2(\lambda - c)}\|\psi\|^2.$$

Dividing both sides by $\varepsilon > 0$ and tending ε to 0, we get

$$(\partial\overline{V}(\phi), \psi) \leqq (D\overline{J}(\phi; \overline{\mathscr{R}}), \psi),   \text{for } \psi \in H.$$

Since ψ is arbitrary in H, "$(\partial\overline{V}(\phi), -\psi) \leqq (D\overline{J}(\phi; \mathscr{R}), -\psi)$" also holds. This derives

$$(\partial\overline{V}(\phi), \psi) = (D\overline{J}(\phi; \overline{\mathscr{R}}), \psi)   \text{for } \psi \in H.$$

Now we complete the proof.   □

To study the value function V, we are concerned with $\mathbf{D} \cap \mathfrak{H}$. Taking Hermite base and suitable v_k, we can prove that $\mathbf{D} \cap \mathfrak{H}$ is dense in H. Putting

(4.32) $h_n(x) = (-1)^n \sqrt{n!}^{-1} \exp\left(\frac{x^2}{2}\right)\left(\frac{d}{dx}\right)^n\left(\exp\left(\frac{-x^2}{2}\right)\right)$,   for $x \in \mathbf{R}^1$,

the Hermite base $e_{\underline{k}}$ is defined as follows.

(4.33) $e_{\underline{k}}(\underline{x}) = \prod_{i=1}^{d} h_{k_i}(x_i)\exp\left(\frac{-x_i^2}{2}\right)$

for $\underline{x} = (x_1, \cdots x_d) \in \mathbf{R}^d$ and multi index $\underline{k} = (k_1, \cdots k_d)$, $k_i \geqq 0$. Using the following properties of h_n

(4.34) $h_n' = \sqrt{n}\, h_{n-1}$, $h_n' - xh_n = -\sqrt{n+1}\, h_{n+1}$.

we get

$$(4.35) \quad x^2 h_n = \sqrt{n(n-1)} \, h_{n-2} + (2n+1) h_n + \sqrt{(n+1)(n+2)} \, h_{n+2} \, .$$

Therefore we have

$$(4.36) \qquad \frac{\partial e_{\underline{k}}}{\partial x_i} = \frac{1}{2} (\sqrt{k_i} \, e_{\underline{k}-I_i} - \sqrt{k_i+1} \, e_{\underline{k}+I_i})$$

and

$$(4.37) \quad |\underline{x}|^2 e_{\underline{k}} = \sum_{j=1}^{d} \sqrt{k_j(k_j-1)} \, e_{\underline{k}-2I_j} + (2k_j+1) e_{\underline{k}} + \sqrt{(k_j+1)(k_j+2)} \, e_{\underline{k}+2I_j}$$

where I_j is the j th unit vector $(0, \cdots 0, 1, 0, \cdots 0)$ and $\underline{k} + \underline{p} = (k_1 + p_1, \cdots k_d + p_d)$.

For $\phi = \sum c_{\underline{k}} e_{\underline{k}}$, (4.36) yields

$$(4.38) \qquad \frac{\partial \phi}{\partial x_i} = \frac{1}{2} (\sum c_{\underline{k}} \sqrt{k_i} \, e_{\underline{k}-I_i} - c_{\underline{k}} \sqrt{k_i+1} \, e_{\underline{k}+I_i}) \, .$$

So, $\partial \phi / \partial x_i \in H$, if and only if $\sum c_{\underline{k}}^2 k_i < \infty$. Repeating this argument, we have

PROPOSITION 4.6. *If* $\sum c_{\underline{k}}^2 |\underline{k}|^p < \infty$, *then* $\phi \in H^p$.

On the other hand, (4.37) implies that, if $\sum c_{\underline{k}}^2 |\underline{k}|^2 < \infty$, then $(1 + |\underline{x}|^2) \phi \in H$. Hence $(1 + |\underline{x}|^2)(\partial \phi / \partial x_i) \in H$, if $\sum c_{\underline{k}}^2 |\underline{k}|^3 < \infty$. Therefore, if $\sum (\phi, e_{\underline{k}})^2 |\underline{k}|^4 < \infty$, then $\phi \in \mathfrak{H}$.

Putting

$$(4.39) \qquad\qquad v_{\underline{k}} = |\underline{k}|^{-d-5}$$

we will define a probability Q on H in the same way as (4.18) and obtain a full probability set $\mathbf{D}$ of (4.21). Since the number of elements of $\{\underline{k}; |\underline{k}| = n\} = \binom{n+d-1}{d-1} = O(n^{d-1})$, "$\sum v_{\underline{k}} |\underline{k}|^4 < \infty$" holds. Therefore $Q(\mathfrak{H}) = 1$. Hence we get

$$(4.40) \qquad\qquad Q(\mathbf{D} \cap \mathfrak{H}) = 1 \, .$$

Moreover Theorem 4.3 derives that, for $\phi \in \mathfrak{H} \cap \mathbf{D}$,

$$(4.41) \quad
\begin{aligned}
(\partial \overline{V}(\phi), \psi) &= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} (V(\phi + \varepsilon \psi) - V(\phi)) \\
&= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} (J(\phi + \varepsilon \psi; \mathscr{R}^*) - J(\phi; \mathscr{R}^*)) = \partial \overline{J}(\phi; \mathscr{R}^*), \psi)
\end{aligned}$$

holds, for any $\psi \in \mathfrak{H}$, where $\mathscr{R}^* = \mathscr{R}^*(\phi)$ is an optimal relaxed system.

§ 5. Subsolution of Bellman equation

Suppose $\overline{V}$ is smooth, namely satisfies the condition (A.5). Then Ito's formula derives, for $\phi \in \mathfrak{H}$ and an admissible system $A = (W, U)$,

$$(5.1) \quad Ee^{-\lambda t} V(q(t)) - V(\phi) = \int_0^t e^{-\lambda s} E(-\lambda V(q(s)) + (DV(q(s)), L(U(s))q(s)) + \frac{1}{2}(D^2 V(q(s))Mq(s), Mq(s)) ds,$$

where $q = q(\cdot, \cdot, A)$. Appealing to (2.16), the Bellman principle show that $\overline{V}$ satisfies the Bellman equation (4.2) below.

PROPOSITION 5.1. *If $\overline{V}$ satisfies (A.5), then*

$$(5.2) \quad 0 = \sup_{u \in \Gamma}\Big(-\frac{1}{2}(D^2\overline{V}(\phi)M\phi, M\phi) - (D\overline{V}(\phi), L(u)\phi) + \lambda\overline{V}(\phi) - F(\phi)\Big)$$

$$\textit{for } \phi \in \mathfrak{H}.$$

Proof. First we evaluate the integrand of (5.1), for small t. Here K_i denotes a constant independent of A, ϕ and s.

$$(5.3) \quad \begin{aligned} &E|(D\overline{V}(q(s)) - D\overline{V}(\phi), L(U(s))q(s))| \\ &\quad \leqq K_2 E\|q(s) - \phi\|\|q(s)\|_2 \leqq K_2\sqrt{s}\,\sqrt{\|\phi\|_1^2 + A_1}\sqrt{\|\phi\|_2^2 + A_2} \\ &\quad \leqq K_3\sqrt{s}\,(\|\phi\|_2^2 + 1) \end{aligned}$$

$$(5.4) \quad \begin{aligned} E|(D\overline{V}(\phi), L(U(s))q(s) - L(U(s))\phi)| &\leqq K_4 E\|q(s) - \phi\|_2 \\ &\leqq K_5\sqrt{s}\,\sqrt{\|\phi\|_3^2 + 1} \end{aligned}$$

From (5.3) and (5.4), we see

$$(5.5) \quad E|(D\overline{V}(s), L(U(s))q(s)) - (D\overline{V}(\phi), L(U(s))\phi)| \leqq K_6\sqrt{s}\,(\|\phi\|_3 + \|\phi\|_2^2 + 1)$$

Repeating the similar arguments, we have

$$(5.6) \quad E|(D^2\overline{V}(q(s))Mq(s), Mq(s)) - (D^2\overline{V}(q(s))M\phi, M\phi| \leqq K_7\sqrt{s}\,(\|\phi\|_2^2 + 1).$$

For $\varepsilon > 0$, there is $\delta = \delta(\varepsilon) > 0$, such that

$$(5.7) \quad [\![D^2\overline{V}(\xi) - D^2\overline{V}(\eta)]\!] < \varepsilon, \quad \text{whenever } \|\xi - \eta\| < \delta,$$

by virtue of (A.5). Hence we have

$$(5.8) \quad \begin{aligned} E((D^2\overline{V}(q(s)) - D^2\overline{V}(\phi))M\phi, M\phi)| &\leqq K_8\|M\phi\|^2\Big(\varepsilon + \frac{E\|q(s) - \phi\|^2}{\delta^2}\Big) \\ &\leqq K_9\|M\phi\|^2\Big(\varepsilon + \sqrt{s}\,\frac{\|\phi\|_1^2 + 1}{\delta^2}\Big). \end{aligned}$$

Combining (5.6) with (5.8), we have

(5.9)
$$|E(D^2\overline{V}(q(s))Mq(s), Mq(s)) - (D^2\overline{V}(\phi)M\phi, M\phi)|$$
$$\leqq K_{10}(\sqrt{s}\,\|\phi\|_2^2 + 1) + (\|\phi\|_1^2 + 1)\Big(\varepsilon + \frac{\|\phi\|_1^2 + 1}{\delta^2}\sqrt{s}\Big).$$

Now recalling (5.1), we get

(5.10)
$$Ee^{-\lambda t}V(q(t)) - V(\phi)$$
$$= E\int_0^t \frac{1}{2}(D^2\overline{V}(\phi)M\phi, M\phi) + (D\overline{V}(\phi), L(Us)\phi) - \lambda V(\phi)ds + o(t).$$

where $o(t)/t \to 0$, uniformly in A, as $t \to 0$.

On the other hand, the Bellman principle yields

(5.11) $$0 = \inf_A E\frac{1}{t}\Big[\int_0^t e^{-\lambda s}F(q(s, \phi, A))ds + e^{-\lambda t}V(q(t, \phi, A)) - V(\phi)\Big].$$

Combining (5.10) with (5.11), we have

(5.12)
$$0 = \inf_A(F(\phi) - \lambda V(\phi) + \frac{1}{2}(D^2\overline{V}(\phi)M\phi, M\phi)$$
$$+ \frac{1}{t}\int_0^t E(D\overline{V}(\phi), L(U(s))\phi)ds + o(1).$$

Since $(D\overline{V}(\phi), L(U(s))\phi) \geqq \inf_{u \in L}(D\overline{V}(\phi), L(u)\phi)$, we see

$$\inf_A \int_0^t E(D\overline{V}(\phi), L(U(s))\phi)ds \geqq t\inf_{u \in \Gamma}(D\overline{V}(\phi), L(u)\phi)$$
$$= \inf_{u \in \Gamma} t(D\overline{V}(\phi), L(u)\phi) = \inf_{u \in \Gamma}\int_0^t E(D\overline{V}(\phi), L(u)\phi)ds$$
$$\geqq \inf_A \int_0^t E(D\overline{V}(\phi), L(U(s))\phi)ds.$$

Hence

(5.13) $$\inf_A \int_0^t E(D\overline{V}(\phi), L(U(s)\phi)ds = t\inf_{u \in \Gamma}(D\overline{V}(\phi), L(u)\phi)$$

holds.   (5.12) and (5.13) complete the proof.   □

We will give a simple example of smooth $\overline{V}$.

EXAMPLE. Γ denotes the set of all Borel functions on $\mathbf{R}^d$ with values in $[-1, 1]$ and 0 outside the unit ball $\{|x| \leqq 1\}$.   Hence Γ $(\subset L^2(\mathbf{R}^d))$ is a complete separable metric space carring the weak topology.   For $e_1, \cdots, e_k$

$\in H$ and $h \in C_0^\infty(\mathbf{R}^k)$, we will put $c(u) = h((u, e_1), \cdots, (u, e_k))$. Then c is continuous on Γ. Let us assume

$$a^{ij}(x, u) = a^{ij}(x), \qquad \text{for } (i, j) \neq (0, 0), \quad a^{00}(x, u) = c(u)$$

and $f^i(x, u) = 0$, for $i = 0, 1, \cdots d$.

Now we consider the following SPDEs for a relaxed system (W, μ).

$$dq(t) = Aq(t)dt + \tilde{c}(t, \mu)dt + Mq(t)dW(t), \quad q(0) = \phi$$
$$dp(t) = Ap(t)dt + Mp(t)dW(t), \qquad\qquad p(0) = \phi$$

where $A = \sum_{(ij) \neq (00)i} \frac{\partial}{\partial x_i}\left(a^{ij}(x)\left(\frac{\partial}{\partial x_j}\right)\right)$. Since c does not depend on x, Ito's formula yields

$$q(t) = \exp\left(\int_0^t \tilde{c}(s, \mu)ds\right)p(t).$$

On the other hand, there exist u^* such that $\min_{u \in \Gamma} c(u) = c(u^*)$. So $\tilde{c}(s, \mu) \geqq c(u^*)$, and $\|q(t)\| \geqq \exp(c(u^*)t)\|p(t)\|$. If the function F has the form $F(\psi) = f(\|\psi\|^2)$ with smooth and non-increasing f, then $\mu'(t, \cdot) = \delta_{u^*}(\cdot)$ is optimal for any ϕ. Therefore using the same arguments as Proposition 3.1, we obtain the smoothness of $\overline{V}$ by the smooth condition of f.

We will study the connection between $\overline{V}$ and a viscosity solution of (5.2), when $\overline{V}$ is not smooth. Let us define $G : L(H) \times H \times \mathbf{R}^1 \times H^2 \times \Gamma \to \mathbf{R}^1$, by

$$(5.14) \quad G(S, p, a, \phi, u) = -\frac{1}{2}(SM\phi, M\phi) - (p, L(u)\phi) + \lambda a - F(\phi).$$

Then the Bellman equation is denoted by

$$(5.15) \qquad\qquad \sup_{u \in \Gamma} G(D^2 v(\phi), Dv(\phi), v(\phi), \phi, u) = 0.$$

For $v \in C(H, \mathbf{R}^1)$, the superdifferential at ϕ, say $D^+v(\phi)$, (resp. subdifferential at ϕ, $D^-v(\phi)$) is defined by

$$D^+v(\phi) = \left\{(S, p) \in L(H) \times H; \varlimsup_{\xi \to 0} \frac{v(\phi + \xi) - v(\phi) - (p, \xi) - \frac{1}{2}(S\xi, \xi)}{\|\xi\|^2} \leqq 0\right\}$$

$$\left(\text{resp. } D^-v(\phi)\right.$$

$$= \left\{(S, p) \in L(H) \times H; \varliminf_{\xi \to 0} \frac{v(\phi + \xi) - v(\phi) - (p, \xi) - \frac{1}{2}(S\xi, \xi)}{\|\xi\|^2} \geqq 0\right\}\right)$$

DEFINITION 5.1. $v \in \text{BUC}(H)$ is called a subsolution of (5.15), if

(5.16) $\{\phi \in \mathfrak{H} ; \ D^+ v(\phi)$ is not empty$\}$ is dense in H

and, for any $(S, p) \in D^+ v(\phi)$ $(\phi \in H^2)$,

(5.17) $\displaystyle\sup_{u \in \Gamma} G(s, p, v(\phi), \phi, u) \leqq 0$

holds.

DEFINITION 5.2. $v \in \mathrm{BUG}(H)$ is called a supersolution of (5.15), if

(5.18) $\{\phi \in \mathfrak{H} ; \ D^- v(\phi)$ is not empty$\}$ is dense in H

and, for any $(S, p) \in D^-(\phi)$ $(\phi \in H^2)$,

(5.19) $\displaystyle\sup_{u \in \Gamma} G(S, p, v(\phi), \phi, u) \geqq 0$

holds.

If v is both sub and super solution, v is a viscosity solution.

Let us denote by Z the space of all function with sub Bellman principle, namely

$$Z = \{v \in \mathrm{BUC}(H); \ v(\phi) \leqq E \int_0^t e^{-\lambda s} F(q(s, \phi, u)) ds + e^{-\lambda t} v(q(t, \phi, u)),$$

$$\text{for any } \phi \in \mathfrak{H}, \ t \geqq 0 \text{ and } u \in \Gamma\}.$$

Then Z is closed and convex and $\overline{V} \in Z$ by the Bellman principle.

PROPOSITION 5.2.   *If* $v \in Z$ *and satisfies* (5.16), *then* v *is a subsolution of* (5.15).

Proof.   For $(S, p) \in D^+ v(\phi)$, we can take a smooth function h on H, such that

(5.20) $h(\phi) = v(\phi), \quad Dh(\phi) = p, \quad D^2 h(\phi) = S \quad \text{and} \quad h \geqq v \text{ on } H.$

Moreover there is a constant b such that, for ahy $\psi \in H$,

(5.21) $|h(\psi)| \leqq b(1 + \|\psi\|^2), \quad \|Dh(\psi)\| \leqq b(1 + \|\psi\|),$
$[\![D^2 h(\psi)]\!] \leqq b \quad \text{and} \quad \lim_{\xi \to \psi} [\![D^2 h(\xi) - D^2 h(\psi)]\!] = 0.$

For instance, we will define h according to [1]. Assuming $\phi = 0$, we set

$$w(\psi) = \begin{cases} \left[v(\psi) - v(0) - (p, \psi) - \dfrac{1}{2}(S\psi, \psi) \right]^+ \bigg/ \|\psi\|^2 & \text{for } \psi \neq 0 \\ 0 & \text{for } \psi = 0, \end{cases}$$

where $[a]^+ = \max(a, 0)$.   Then w is continuous at 0, since $(S, p) \in D^+ v(\phi)$.

Let ρ be a smooth increasing function on $[0, \infty)$, such that

$$\rho(\theta) = \begin{cases} \theta^3 & \text{on } [0, 1) \\ \theta^2 & \text{on } [2, \infty) \,. \end{cases}$$

Put $k(r) = \sup\{w(\psi); \|\psi\| \leq r\}$ and $f(t) = 2\int_0^{2t}\int_0^s k(r)\,dr\,ds + \rho(t)$. Then $f \in C^2[0, \infty)$. Hence, g, defined by $g(\psi) = f(\|\psi\|)$, satisfies $g(0) = 0$, $Dg(0) = 0$ and $D^2g(0) = 0$. Thus $h(\psi) = g(\psi) + v(0) + (p, \psi) + \frac{1}{2}(S\psi, \psi)$ is a required one.

From the definition of Z, we see

$$0 \leq E\int_0^t e^{-\lambda s}F(q(s, \phi, u))\,ds + e^{-\lambda t}v(q(t, \phi, u)) - v(\phi)$$

$$\leq E\int_0^t e^{-\lambda s}F(q(s, \phi, u))\,ds + e^{-\lambda t}h(q(t, \phi, u)) - h(\phi)$$

$$\leq E\int_0^t e^{-\lambda s}(- G(D^2h(q(s, \phi, u)), Dh(q(s, \phi, u)), h(q(s, \phi, u)),$$

$$q(s, \phi, u), u))\,ds\,.$$

Dividing both side by t and letting t tend to 0, we get

(5.22) $0 \geq G(S, p, v(\phi), \phi, u)\,,$

in the same way as Proposition 5.1. Since $u \in \Gamma$ is arbitrary, we complete the proof. $\qquad\square$

Now we will prove

THEOREM 5.1. $\overline{V}$ *is a subsolution.*

Proof. Since $\overline{V} \in Z$, we will only show (5.16). Using concavity of v of (3.13), Theorem 4.2 implies

$$v(\phi + \psi) - v(\phi) - (\partial v(\phi), \psi) - \frac{1}{2}(0\psi, \psi) \leq 0\,, \quad \text{for } \phi \in \mathbf{D}, \ \psi \in H\,.$$

This derives

(5.23) $\left(\dfrac{\tilde{K}}{\lambda - c}I, \partial\overline{V}(\partial)\right) \in D^+\overline{V}(\phi)\,, \quad \text{for } \phi \in \mathbf{D}\,,$

where $I = $ identity on H. Since $\mathbf{D} \cap \mathfrak{H}$ is dense, (5.16) holds. $\qquad\square$

Next we will show the maximum property.

THEOREM 5.2. $\overline{V}$ *is the maximum subsolution with sub Bellman principle.*

Proof. First we remark that $v \in Z$ satisfies (5.24) below.

$$(5.24) \qquad v(\phi) \leqq E \int_0^t e^{-\lambda s} F(q(s, \phi, A)) ds + e^{-\lambda t} v(q(t, \phi, A)),$$

for $A = (W, U) \in a_N$. Put $\Delta = 2^{-N}$. For $s \in [k\Delta, k\Delta + \Delta)$, $U(s)$ can be regarded as a constant ($\in \Gamma$), under the conditional probability $P(\cdot / \sigma_{k\Delta}(W))$, and $q(s, \phi, U) = q(s - k\Delta, q(k\Delta, \phi, U), U(k\Delta))$ holds. This argument yields

$$(5.25) \qquad \begin{aligned} & E \int_{k\Delta}^t e^{-\lambda s} F(q(s, \phi, A)) ds + e^{-\lambda t} v(q(t, \phi, A)) / \sigma_{k\Delta}(W) \\ & \qquad \geqq e^{-\lambda k\Delta} v(q(k\Delta, \phi, A)) . \end{aligned}$$

Repeating this evaluation, we can easily obtain (5.24) by (2.12):

As $t \to \infty$, (5.24) derives

$$(5.26) \qquad v(\phi) \leqq \liminf_{\substack{N \to \infty \\ a_N}} E \int_0^\infty e^{-\lambda s} F(q(s, \phi, A)) ds , \qquad \text{for } \phi \in \mathfrak{H} .$$

Thus (2.30) yields "$v(\phi) \leqq V(\phi)$ for $\phi \in \mathfrak{H}$". Since $\mathfrak{H}$ is dense, we can complete the proof. $\qquad \square$

The proposition 5.3 below is a partial converse of Proposition 5.2.

PROPOSITION 5.3. *Let ν be a twice differentiable. If ν is a subsolution, then $\nu \in Z$.*

Proof. Since $(D^2\nu(\psi), D\nu(\psi)) \in D^+\nu(\psi)$ for any $\psi \in H$ and $q(\cdot, \phi, u) \in C([0, \infty), H^3)$ with probability 1 for $\phi \in \mathfrak{H}$, we see

$$(5.27) \quad G(D^2\nu(q(s, \phi, u)), D\nu(q(s, \phi, u)), \nu(q(s, \phi, u)), q(s, \phi, u), u) \leqq 0 .$$

Hence Ito's formula concludes $\nu \in Z$.

In the customary version of admissible control for SPDE, we use Γ-valued adapted process, where $\Gamma \subset \{u; \mathbf{R}^d \to \mathbf{R}^k, \text{ Borel measurable}\}$. In some cases Γ admits only constant functions, for instance, this control is appropriate for the Zakai equation.

§ 6. Remarks on deterministic control

When $M = 0$, our stochastic control turns out to be a deterministic control for PDE. In this case we usually take Γ-valued Borel function on $[0, \infty)$ as an admissible control. For an admissible control $u(\cdot)$, we consider PDE

$$(6.1) \qquad \frac{\partial q}{\partial t} = L(u(t))q(t) , \qquad \text{with } q(0) = \phi \ (\in \mathfrak{H}) .$$

The value function $w(\phi)$ is defined by

$$(6.2) \qquad w(\phi) = \inf_{u(\cdot)} \int_0^\infty e^{-\lambda t} F(q(t, \phi, u(\cdot))) dt .$$

So this control can be studied in two ways of viewing, either as a stochastic control with $M = 0$, or as a deterministic control. First we remark that both value functions coincide.

PROPOSITION 6.1. $w(\phi) = V(\phi)$ for $\phi \in \mathfrak{H}$.

Proof. Since an admissible control $u(\cdot)$ can be regarded as a relaxed system, clearly

$$(6.3) \qquad\qquad\qquad V(\phi) \leqq w(\phi) .$$

On the other hand, for $A(W, U) \in a_N$, U turns out to be a Γ-valued Borel function under the conditional probability $P(\cdot/\sigma(W))$. Hence with probability 1,

$$(6.4) \qquad E\!\left(\int_0^\infty e^{-\lambda t} F(q(t, \phi, A)) dt / \sigma(W) \right) \geqq w(\phi)$$

holds. Now (2.30) completes the proof. $\qquad\qquad\qquad\square$

In the same way we can easily see the Bellman principle

$$(6.5) \qquad w(\phi) = \inf_{u(\cdot)} \int_0^T e^{-\lambda t} F(q(t, \phi, u(\cdot))) dt + e^{-\lambda T} w(q(T, \phi, u(\cdot))) .$$

When $M = 0$, the Bellman equation (5.15) turns out to be the first order equation (6.6) below.

$$(6.6) \qquad \sup_{u \in \Gamma} \left(-(Dv(\phi), L(u)\phi) + \lambda v(\phi) - F(\phi) \right) = 0 .$$

We will give the definition of viscosity subsolution of (6.6) from the point of first order equation [1].

$$D_1^+ v(\phi) = \left\{ p \in H; \varlimsup_{\xi \to 0} \frac{v(\phi + \xi) - v(\phi) - (p, \xi)}{\|\xi\|} \leqq 0 \right\}$$

is called superdifferential.

DEFINITION 6.1. $v \in \mathrm{BUC}(H)$ is called a subsolution of (6.6), if

$$(6.7) \qquad \{\phi \in \mathfrak{H}; D_1^+ v(\phi) \text{ is not empty}\} \text{ is dense in } H$$

and, for any $p \in D_1^+ v(\phi)$, $(\phi \in H^2)$,

$$(6.8) \qquad \sup_{u \in \Gamma} \left(- (p, L(u)\phi) + \lambda v(\phi) - F(\phi) \right) \leqq 0$$

holds.

If v is a subsolution of Definition 5.1, then v turns out to be a subsolution of Definition 6.1, because $p \in D_1^+ v(\phi)$ whenever $(S, p) \in D^+ v(\phi)$. Hence, Theorem 5.1 concludes that $\overline{V}$ is a subsolution of (6.6).

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