Can. J. Math., Vol. XXV, No. 2, 1973, pp. 261–272

ANGLE BISECTION AND ORTHOAUTOMORPHISMS IN HILBERT LATTICES

RONALD P. MORASH

1. Introduction. The lattices of all closed subspaces of separable, infinite-dimensional Hilbert space (real, complex, and quaternionic) share the following purely lattice-theoretic properties. Each is complete, orthocomplemented, atomistic, irreducible, separable, M-symmetric, and orthomodular [**2**]. We will call any lattice possessing these seven properties a Hilbert lattice. The general situation which motivates the investigations of this paper concerns infinite-dimensional Hilbert lattices (the dimension of a Hilbert lattice being the cardinality of any maximal family of orthogonal atoms). There are several lattice theoretic properties, possessed by the three canonical lattices, whose only known proofs involve the analytic properties of the underlying Hilbert space, that is, there is no known purely lattice-theoretic proof of these properties. For example, it is known that, in each of these three lattices, there exists an orthoisomorphism between any two orthogonal intervals of the same height. Also, it is known that each of the lattices is O-symmetric, a property that could be proved abstractly if certain information were known about the orthoautomorphisms of the lattice. This paper provides what we hope is a step toward settling the validity of properties such as these in an arbitrary infinite-dimensional Hilbert lattice. We might note that a more far-reaching problem in this vein is whether there are, in fact, any other infinite-dimensional Hilbert lattices (i.e. lattices which are not orthoisomorphic to any of the known examples) than the canonical three. None is known and we, in [**2**; **3**; **4**], presented some evidence in support of the possibility that there is no other. In this paper, most of the results are based on the coordinatization theorem of Birkhoff and von Neumann for finite dimensional Hilbert lattices. In section 2, we present a purely lattice-theoretic definition, which corresponds, in the three examples, to the notion of the bisection of an angle between two orthogonal vectors by a third vector. We relate the existence of angle-bisecting atoms, in the general case, to a property of the division ring which coordinatizes the finite intervals of the lattice (it is a consequence of the uniqueness property of the Fundamental Theorem of Projective Geometry that one division ring coordinatizes all finite intervals). In section 3, we prove, under the assumption of the existence of angle-bisectors, the existence of certain orthoautomorphisms of a Hilbert lattice.

This paper assumes basic results of the theory of orthomodular lattices;

Received December 8, 1971.

https://doi.org/10.4153/CJM-1973-026-2 Published online by Cambridge University Press

we use freely facts about Sasaki projections and the theory of commutativity. Also, we use results from the theory of atomistic lattices with the covering property and results about the notion of perspectivity. We suggest [**1**] as a reference for this material.

2. Angle bisectors. In this section, we give a lattice-theoretic formulation of the concept of the bisection of the angle between two orthogonal atoms by a third atom lying beneath their join. Then, we relate the existence of angle-bisectors to an algebraic property of the division ring which coordinatizes L. Throughout this section, L represents a Hilbert lattice (either finite or infinite-dimensional). We denote by $d(a)$ the dimension of an element $a \in L$ and by φ_b the Sasaki projection on the element $b \in L$. We begin with some preliminary material which will lead to the definition of angle-bisection.

2.1 FACT. *Let p, z, r be distinct atoms in a Hilbert lattice with $(p \vee z) \nleqq r^\perp$. Then $(p \vee z) \wedge r^\perp$ is an atom.*

Proof. Since $(p \vee z) \nleqq r^\perp$, then $(p \vee z) \wedge r^\perp < p \vee z$ so that $d((p \vee z) \wedge r^\perp) < 2$, so $(p \vee z) \wedge r^\perp$ is at most an atom. Thus, we need only show that $(p \vee z) \wedge r^\perp \neq 0$. But if $(p \vee z) \wedge r^\perp = 0$, then since $(p \vee z) \vee r^\perp = 1$, we would have that $(p \vee z)$ is a complement of $r^\perp$, so $p \vee z$ is perspective to r. Hence, since perspectivity implies equal dimension, $2 = d(p \vee z) = d(r) = 1$, a contradiction. Thus, $(p \vee z) \wedge r^\perp \neq 0$.

2.2 COROLLARY. *If p, z are distinct atoms, then $(p \vee z) \wedge z^\perp (= \varphi_{z^\perp}(p))$ is an atom.*

2.3 LEMMA. *Let p, q be orthogonal atoms. Let r be a third atom such that $r < p \vee q$. Let x be any atom such that $x \perp r$, but $x \not< p \vee q$. Let y be an atom, distinct from p, q, r, x such that $y < p \vee r \vee x$ and $y \perp x$. Then the following are equivalent:*

(i) $y = \varphi_{x^\perp}\varphi_{r^\perp}(p)$ $(= \varphi_{r^\perp}\varphi_{x^\perp}(p)$, *since* $xCr)$.

(ii) $y \perp r$.

(iii) $r = (p \vee q) \wedge x^\perp \wedge y^\perp$.

(iv) *There exists an atom s such that $x \vee y = \varphi_{r^\perp}(p) \vee s$, and $s \perp (p \vee q)$. This atom s is necessarily unique and distinct from the atoms x, y, and $\varphi_{r^\perp}(p)$.*

Proof. (i) $\Rightarrow$ (ii). $y = \varphi_{r^\perp}\varphi_{x^\perp}(p) < r^\perp$ so $y \perp r$.

(ii) $\Rightarrow$ (iii). Since $y \perp r$, then $r \leqq (p \vee q) \wedge x^\perp \wedge y^\perp$. Since $(p \vee q) \wedge x^\perp \wedge y^\perp$ is at most an atom (by 2.1), we must have equality.

(iii) $\Rightarrow$ (iv). Let $s = (x \vee y) \wedge p^\perp \wedge q^\perp$. We claim s is an atom, i.e., $(x \vee y) \wedge p^\perp \wedge q^\perp \neq 0$. By assumption, $(p \vee q) \wedge x^\perp \wedge y^\perp \neq \mathbf{0}$ so, since $(p \vee q) \wedge x^\perp$ is an atom, by 2.1, we must have $(p \vee q) \wedge x^\perp \wedge y^\perp = (p \vee q) \wedge x^\perp$ and therefore $(p \vee q) \wedge x^\perp \leqq y^\perp$. But then $y \leqq x \vee (p^\perp \wedge q^\perp)$

https://doi.org/10.4153/CJM-1973-026-2 Published online by Cambridge University Press

so $(x \vee y) \leqq x \vee (p^\perp \wedge q^\perp)$; hence

$$\begin{aligned}(x \vee y) \wedge p^\perp &\leqq [(p^\perp \wedge q^\perp) \vee x] \wedge p^\perp \\ &= (p^\perp \wedge q^\perp) \vee (x \wedge p^\perp) \quad (\text{since } m(x, p^\perp)) \\ &= p^\perp \wedge q^\perp \quad (\text{since } x \nleqq p^\perp) \\ &\leqq q^\perp.\end{aligned}$$

Thus $(x \vee y) \wedge p^\perp \wedge q^\perp = (x \vee y) \wedge p^\perp$ which is an atom, by 2.1. Hence s is an atom. Clearly $s \perp (p \vee q)$ so s is automatically distinct from x, y, and $\varphi_{r^\perp}(p)$ (since it is easy to show that $y \not\subset p \vee q$). Clearly $s < x \vee y$ and, since $y = \varphi_{x^\perp}\varphi_{r^\perp}(p) = (\varphi_{r^\perp}(p) \vee x) \wedge x^\perp < \varphi_{r^\perp}(p) \vee x$, then $\varphi_{r^\perp}(p) < x \vee y$ so $s \vee \varphi_{r^\perp}(p) = x \vee y$. Finally s is unique since a desired atom must be $\leqq x \vee y$ and orthogonal to $p \vee q$. But $(x \vee y) \wedge (p \vee c)^\perp$, as we have already seen, is an atom.

(iv) $\Rightarrow$ (i). We claim $y = (\varphi_{r^\perp}(p) \vee x) \wedge x^\perp$. By assumption, $y \leqq x^\perp$. By (iv) $\varphi_{r^\perp}(p) < x \vee y$ so $y < \varphi_{r^\perp}(p) \vee x$, and therefore $y \leqq (\varphi_{r^\perp}(p) \vee x) \wedge x^\perp$, which is an atom, by 2.1, and so we have equality.

2.4 *Definition.* Let p, q, x, y, r be distinct atoms such that $p \perp q$, $r < p \vee q$, $x \perp r$, $x \not\subset p \vee q$, $y \perp x$, and $y < p \vee q \vee x$. We write $(p, q)\ H\ (x, y)$ via r if the four equivalent statements of 2.3 are true.

2.5 *Remark.* Given three distinct atoms p, q, r with $p \perp q$ and $r < p \vee q$, there always exist pairs (x, y) such that $(p, q)\ H\ (x, y)$ via r, namely there exists one pair for each atom x such that $x \perp r$ and $x \not\subset p \vee q$, y being then uniquely determined by p, r, and x (by 2.3(i)).

2.6 *Definition.* Let p, q, r be distinct atoms with $p \perp q$ and $r < p \vee q$. We say that r *bisects the angle between p and q*, to be denoted $rB(p, q)$ if and only if, for any pair (x, y) of orthogonal atoms with $(p, q)\ H\ (x, y)$ via r, we have $r \leqq [(p \vee x) \wedge (q \vee y)] \vee [p \vee y) \wedge (q \vee x)]$.

2.7 *Remark.* The remainder of this section is devoted to relating Definition 2.6 to an algebraic property of the division ring which coordinatizes the lattice L. In the process, we will show that the words "for any pair (x, y) . . ." in Definition 2.6 can be replaced by "for some pair (x, y) . . .".

The way to these results is cleared by a series of rather technical lemmas. We list here, for the sake of brevity, the assumptions and notation which prevail in 2.8, 2.9, and 2.10. We assume throughout that (p, q) is a pair of orthogonal atoms in L and that $r < p \vee q$ is a third atom. We let (x, y) be a pair of atoms such that $(p, q)\ H\ (x, y)$ via r. We let L' be a finite interval in L such that $p, q, x \in L'$ and $n = \dim L' \geqq 4$. Then, we let $(D, *, (,))$ be the triple, whose existence is given by the theorem of Birkhoff and von Neumann such that L' is orthoisomorphic to the lattice $L(V_n(D, *, (,)))$ of subspaces of V_n. We write $p = De_1$ and $q = De_2$ where $e_1, e_2 \in V_n$ are such that $(e_1, e_2) = 0$. Since $r < p \vee q$, and r is an atom, we can write $r = D(e_1 + \alpha e_2)$ for some

$\alpha \in D$. In the following lemmas, we let ω represent the element $(e_2, e_2)\alpha^*(e_1, e_1)^{-1}$ of D. The goal of 2.8 and 2.9 is to represent the atoms which occur in 2.6 as the space spanned by specific vectors in V_n.

2.8 LEMMA. (a) $\varphi_{r^\perp}(p) = D(\omega e_1 - e_2)$.

(b) $x = D(\omega e_1 - e_2 + f)$, *where* f *is a vector in* V_n *such that* $s = Df$, s *being the atom whose existence is given in* 2.3(iv).

(c) $y = D(\omega e_1 - e_2 - ((1 + \omega\alpha)(e_2, e_2)(f, f)^{-1})f)$.

Proof. (a) $\varphi_{r^\perp}(p) < p \vee q$ so there exist $\tau, \rho \in D$ such that $\varphi_{r^\perp}(p) = D(\tau e_1 + \rho e_2)$. Since $\varphi_{r^\perp}(p) \perp r$, then $(\tau e_1 + \rho e_2, e_1 + \alpha e_2) = 0$. Letting $\rho = -1$ gives $\tau = (e_2, e_2)\alpha^*(e_1, e_1)^{-1} = \omega$ so $\varphi_{r^\perp}(p) = D(\omega e_1 - e_2)$.

(b) Let s be the atom whose existence is given in 2.3(iv), such that $x \vee y = \varphi_{r^\perp}(p) \vee s$. Since $x < \varphi_{r^\perp}(p) \vee s$ and $\varphi_{r^\perp}(p) = D(\omega e_1 - e_2)$, then we can write $x = D(\omega e_1 - e_2 + f)$, where f is a vector in V_n such that $s = Df$.

(c) As in (b), $y < \varphi_{r^\perp}(p) \vee s$ so $y = D(\omega e_1 - e_2 + \tau f)$, for some $\tau \in D$. τ is determined by the fact that $y \perp s$, namely

$$(\omega e_1 - e_2 + f, \omega e_1 - e_2 + \tau f) = 0.$$

Computation yields $\tau = -(1 + \omega\alpha)(e_2, e_2)(f, f)^{-1}$.

2.9 LEMMA.

$$\text{(a)}\quad (p \vee x) \wedge (q \vee y) = D[((1 + \omega\alpha)^{-1}\omega)e_1 + ((e_2, e_2)(f, f)^{-1})e_2 - ((e_2, e_2)(f, f)^{-1})f].$$

$$\text{(b)}\quad (p \vee y) \wedge (q \vee x) = D[((e_2, e_2)(f, f)^{-1}\omega)e_1 + ((1 + \omega\alpha)^{-1})e_2 + ((e_2, e_2)(f, f)^{-1})f].$$

Proof. First we note that $(p \vee x) \wedge (q \vee y)$ and $(p \vee y) \wedge (q \vee x)$ are easily seen to be atoms by a dimension argument. To represent the atom $(p \vee x) \wedge (q \vee y)$ as the space spanned by a vector in V_n, we must find scalars $W, X, Y, Z \in D$ such that $We_1 + X(\omega e_1 - e_2 + f) - Ye_2 - Z(\omega e_1 - e_2 - ((1 + \omega\alpha)(e_2, e_2)(f, f)^{-1})f) = 0$.

This equation implies the three equations.

(i) $W + (X - Z)\omega = 0$

(ii) $X + Y - Z = 0$, and

(iii) $X + Z((1 + \omega\alpha)(e_2, e_2)(f, f)^{-1}) = 0$.

Letting $Z = (1 + \omega\alpha)^{-1}$, we get from (iii) that $X = -(e_2, e_2)(f, f)^{-1}$. Hence $Y = Z - X = (1 + \omega\alpha)^{-1} + (e_2, e_2)(f, f)^{-1}$. Thus, $W = Y\omega = (1 + \omega\alpha)^{-1}\omega + (e_2, e_2)(f, f)^{-1}\omega$. Since $(p \vee z) \wedge (p \vee y) = D[We_1 + X(\omega e_1 - e_2 + f)]$, for the scalars W, X specified above, we conclude that

$$\begin{aligned}(p \vee x) \wedge (q \vee y) &= D[(W + X\omega)e_1 - Xe_2 + Xf] \\ &= D[((1 + \omega\alpha)^{-1}\omega)e_1 + ((e_2, e_2)(f, f)^{-1})e_2 \\ &\qquad - ((e_2, e_2)(f, f)^{-1})f].\end{aligned}$$

To compute $(p \vee y) \wedge (q \vee x)$, simply exchange the roles of W and $-Y$ in the above computation.

2.10 THEOREM. *$rB(p, q)$ if and only if $r = D[e_1 + \alpha e_2]$ where the scalar $\alpha \in D$ has the property that*

$$\alpha(e_2, e_2)\alpha^* = (e_1, e_1).$$

Proof. Since $r < p \vee q$, then $r = D[e_1 + \alpha e_2]$ for some $\alpha \in D$. We wish to show that $r \leqq [(p \vee x) \wedge (q \vee y)] \vee [(p \vee y) \wedge (q \vee x)]$ if and only if $\alpha(e_2, e_2)\alpha^* = (e_1, e_1)$. In view of 2.9, we will have $r < [(p \vee x) \wedge (q \vee y)] \vee [(p \vee y) \wedge (q \vee x)]$ if and only if there exist scalars $\mu, \nu \in D$ such that

(i) $1 = \mu[(1 + \omega\alpha)^{-1}\omega] + \nu[(e_2, e_2)(f, f)^{-1}\omega]$,

(ii) $\alpha = \mu[(e_2, e_2)(f, f)^{-1}] + \nu[(1 + \omega\alpha)^{-1}]$ and

(iii) $0 = -\mu + \nu$, or $\mu = \nu$.

By (iii), (i) and (ii) become

(i)′ $1 = \mu[(1 + \omega\alpha)^{-1}\omega + (e_2, e_2)(f, f)^{-1}\omega]$
$= \mu[(1 + \omega\alpha)^{-1} + (e_2, e_2)(f, f)^{-1}]\omega$ and

(ii)″ $\alpha = \mu[(e_2, e_2)(f, f)^{-1} + (1 + \omega\alpha)^{-1}]$.

We note that the quantity in brackets in (i)′ and (ii)′, namely $(1 + \omega\alpha)^{-1} + (e_2, e_2)(f, f)^{-1}$ (to be denoted henceforth as Q) is nonzero, otherwise we could prove $y = x$, thus contradicting $y \perp x$. Thus we have

(i)′ $1 = \mu Q\omega$,

(ii)″ $\alpha = \mu Q$ and $Q \neq 0$.

From this we conclude $\alpha\omega = 1$, so that $\alpha(e_2, e_2)\alpha^* = (e_1, e_1)$, as desired.

2.11 COROLLARY. *Suppose that (p, q) H (x, y) via r and (p, q) H (w, z) via r. Then*

$$r \leqq [(p \vee x) \wedge (q \vee y)] \vee [(p \vee y) \wedge (q \vee x)] \Leftrightarrow$$
$$r \leqq [(p \vee w) \wedge (q \vee z)] \vee [(p \vee z) \wedge (q \vee w)].$$

Proof. Coordinatize the finite interval $L(0, p \vee q \vee x \vee w)$ and note that, by 2.10, $r < [(p \vee x) \wedge (q \vee y) \vee [(p \vee y) \wedge (q \vee x)]$ if and only if $r = D(e_1 + \alpha e_2)$ for some $\alpha \in D$ such that $(e_1, e_1) = \alpha(e_2, e_2)\alpha^*$, which holds if and only if $r < [(p \vee w) \wedge (q \vee z)] \vee [(p \vee z) \wedge (q \vee w)]$.

2.12 *Definition.* A Hilbert lattice L will be called a *bisecting Hilbert lattice* if every pair of orthogonal atoms in L has an angle bisecting atom.

2.13 *Remarks.* We pause here to refer to some examples. Loosely speaking, the algebraic requirement of Theorem 2.10 is that, given two orthogonal vectors e_1, e_2, we must be able to find a vector in the direction of e, having the same "length" as e_2. In real, complex, and quaternionic Hilbert space (finite-dimensional or separable infinite-dimensional), vectors can be orthonormalized (due to the existence of square roots of positive elements in the division ring) so the condition is met. On the other hand, in the 3-dimensional vector space over the field of rational numbers (with the canonical form), the orthogonal vectors, (1, 1, 1) and (−2, 1, 1) do not have an angle-bisecting vector. If D is commutative, $*$ = identity, and the condition of 2.10 is that each field element in the range of (,) have a square root. Also, we point out the non-uniqueness of angle-bisectors, where they exist (e.g. in complex Hilbert space, each pair of orthogonal vectors has an infinite number of angle-bisecting vectors).

3. Orthoautomorphisms of a bisecting Hilbert lattice. In this section, we show that the existence of an angle bisector for each pair of orthogonal atoms in a Hilbert lattice L is enough to insure the existence of certain ortho-automorphisms of L. Specifically, the main result of this section is:

3.1 THEOREM. *Let L be a bisecting Hilbert lattice, let a, b be elements of L such that $a \perp b$ and $d(a) = d(b) < \infty$. Then there exists an involutory orthoautomorphism Θ of L such that $\Theta(a) = b$.*

Throughout this section, L will represent a bisecting Hilbert lattice. We construct the map Θ, just referred to, by first defining a mapping on the atoms of L.

3.2 *Definition.* Let p, q, r be three distinct atoms in L such that $r < p \vee q$, $p \nleqq r^\perp$, and $q \nleqq r^\perp$. We define a map, to be denoted θ_{pqr}, on the atoms of L which are either $\nleqq p \vee q$ or are equal to r, by the rules

$$\theta_{pqr}(z) = \begin{cases} z \text{ if } zCr, \text{ that is, if } z = r \text{ or } z \perp r \\ \{[(p \vee z) \wedge r^\perp] \vee q\} \wedge (z \vee r) \text{ if } z \nleqq r^\perp \text{ and } z \nleqq p \vee q. \end{cases}$$

An easy dimension argument will convince the reader that $\theta_{pqr}(z)$ is an atom for each atom z in the domain of θ_{pqr}. Note that, at this point, the atoms z which are $< p \vee q$ (except for r) have been excluded from the domain of θ_{pqr}. The value of θ_{pqr}, on these atoms, will be defined later. The properties of these maps that we wish to derive concern the case $p \perp q$ and $rB(p, q)$. Under these assumptions, we can prove that θ_{pqr} is involutory and preserves orthogonality. In 3.3, we give a criterion for θ_{pqr} to be involutory.

3.3 LEMMA. *Let p, q, r, z be four distinct atoms in L such that $r < p \vee q$, $p \nleqq r^\perp$, $q \nleqq r^\perp$, $z \nleqq p \vee q$, and $z \nleqq r^\perp$. Suppose that $(p \vee \theta_{pqr}(z)) \wedge r^\perp \leqq q \vee z$. Then $\theta_{pqr}{}^2(z) = z$.*

Proof. (In this proof, and henceforth, where clarity does not suffer, we shorten θ_{pqr} to θ.) If $(p \vee \theta(z)) \wedge r^\perp \leqq q \vee z$, then $(p \vee \theta(z)) \wedge r^\perp = (q \vee z) \wedge r^\perp$. Hence

$$\begin{aligned}
\theta^2(z) &= \{[(p \vee \theta(z)) \wedge r^\perp] \vee q\} \wedge (\theta(z) \vee r) \\
&= \{[(q \vee z) \wedge r^\perp] \vee q\} \wedge (\theta(z) \vee r) \\
&\quad \text{(since } m(r^\perp, q \vee z) \text{ by finite modularity)} \\
&= (q \vee z) \wedge (r^\perp \vee q) \wedge (\theta(z) \vee r) \\
&\quad \text{(since } q \vee r^\perp = 1, \text{ because } q \nleqq r^\perp) \\
&= (q \vee z) \wedge (\theta(z) \vee r) \\
&\quad \text{(since } r < z \vee \theta(z), \text{ because } \theta(z) < r \vee z) \\
&= (q \vee z) \wedge (z \vee r) \\
&\quad (z \vee r \neq q \vee z, \text{ because } r \nleqq q \vee z. \text{ Otherwise } z \leqq p \vee q). \\
&= z.
\end{aligned}$$

3.4 THEOREM. *Let p, q, r be three distinct atoms as in Definition* 3.2. *Also, assume $p \perp q$, $z \nleqq p \vee q$, and $rB(p, q)$. Then $\theta_{pqr}{}^2(z) = z$.*

Proof. Note first that $z \nleqq p \vee q$. But $z < 1 = (p \vee q) \vee (p \vee q)^\perp$, so by [**1**, Theorem 9.2(e)], there exists an atom $s < (p \vee q)^\perp$ such that $z < p \vee q \vee s$.

Let L' be any finite interval in L of dimension $n \geqq 4$, containing p, q, and s (and thus z). Next, apply the coordinatization theorem as before to find $(D, *, (,))$ such that L' is orthoisomorphic to $L(V_n(D), *, (,))$. Let $p = De_1$, $q = De_2$, where $e_1, e_2 \in V_n$ with $(e_1, e_2) = 0$. Since $rB(p, q)$, we can, by 2.10, let $r = D(e_1 + \alpha e_2)$ where $\alpha(e_2, e_2)\alpha^* = (e_1, e_1)$. Now $z < p \vee q \vee s$ means that $z = D(\gamma e_1 + \delta e_2 + f)$, f a vector in V_n such that $Df = s$. By 3.3, we need prove only that $(p \vee \theta(z)) \wedge r^\perp < q \vee z$. We do this by expressing the atom $(p \vee \theta(z)) \wedge r^\perp$ as the span of a vector in V_n. To do this, we must express $\theta(z)$ in V_n.

For this, in turn, we first compute $(p \vee z) \wedge r^\perp$. Since $z = D(\gamma e_1 + \delta e_2 + f)$, we must find scalars $\sigma, \tau \in D$ such that $(\sigma e_1 + \tau(\gamma e_1 + \delta e_2 + f), e_1 + \alpha e_2) = 0$. Setting $\tau = -1$ and noting that $(e_2, e_2)\alpha^* = \alpha^{-1}(e_1, e_1)$ (since $rB(p, q)$), the quantity above becomes $(\sigma - \gamma - \delta\alpha^{-1})(e_1, e_1)$ which is zero if and only if $\sigma - \gamma - \delta\alpha^{-1} = 0$, or $\sigma = \gamma + \delta\alpha^{-1}$. Thus $(p \vee z) \wedge r^\perp = D[(\gamma + \delta\alpha^{-1})e_1 - (\gamma e_1 + \delta e_2 + f)) = D(\delta\alpha^{-1}e_1 - \delta e_2 + f)$.

Next, we compute $\theta(z)$. For this, we need to find scalars $W, X, Y, Z \in D$ such that

$$W(\delta\alpha^{-1}e_1 - \delta e_2 - f) + Xe_2 - Y(\gamma e_1 + \delta e_2 + f) - Z(e_1 + \alpha e_2) = 0.$$

Computation, together with the assumption $W = 1$, yields $X = \delta + \gamma\alpha$, so $\theta(z) = D[\delta\alpha^{-1}e_1 + \gamma\alpha e_2 - f]$.

Finally, we compute $(p \vee \theta(z)) \wedge r^\perp$. Proceeding as before, we arrive at $(p \vee \theta(z)) \wedge r^\perp = D(\gamma e_1 - \gamma\alpha e_2 + f)$.

But $\gamma e_1 - \gamma\alpha e_2 + f = (\gamma e_1 + \delta e_2 + f) + (-\delta - \gamma\alpha)e_2$ so $[p \vee \theta(z)] \wedge r^\perp < q \vee z$, as desired, and hence $\theta^2(z) = z$ by 3.3.

Thus if $p \perp q$ and $rB(p, q)$, the map θ_{pqr} is involutory on its domain. We next show that it preserves orthogonality. As in 3.4, the proof uses the coordinatization.

3.5 THEOREM. *Suppose* $p \perp q$, $rB(p, q)$, $z \nleqq p \vee q$, $w \nleqq p \vee q$, *and* $z \perp w$ (z, w *are atoms*). *Then* $\theta_{pqr}(z) \perp \theta_{pqr}(w)$.

Proof. There are two cases:

(Case 1) Neither z nor w is orthogonal to r: Since neither z nor w is $< p \vee q$, then as in the proof of 3.4, we can find atoms f', g' in L such that $z < p \vee q \vee f'$ and $w < p \vee q \vee g'$. Coordinatizing, as in that proof, we can write $z = D(\gamma e_1 + \delta e_2 + f)$ and $w = D(\sigma e_1 + \tau e_2 + g)$ where $f' = Df$ and $g' = Dg$. By assumption, $z \perp w$ so we must have $\gamma(e_1, e_1)\sigma^* + \delta(e_2, e_2)\tau^* + (f, g) = 0$. In the proof of 3.4, we obtained $\theta(z) = \delta\alpha^{-1}e_1 + \gamma\alpha e_2 - f$ and $\theta(w) = \tau\alpha^{-1}e_1 + \sigma\alpha e_2 - g$. Observe that

$$\begin{aligned}(\theta(z), \theta(w)) &= \delta\alpha^{-1}(e_1, e_1)(\alpha^*)^{-1}\tau^* + \gamma\alpha(e_2, e_2)\alpha^*\sigma^* \\ &\quad + (f, g) \\ &= \delta(e_2, e_2)\tau^* + \gamma(e_1, e_1)\sigma^* + (f, g) = 0.\end{aligned}$$

Thus $\theta(z) \perp \theta(w)$.

(Case 2) One of z or w is orthogonal to r, say $w \leqq r^\perp$ and $z \nleqq r^\perp$: As in case 1, we can let $z = D(\gamma e_1 + \delta e_2 + f)$, but $w \perp r$ means that

$$w = D(\alpha^{-1}e_1 - e_2 + g).$$

Since $w \leqq r^\perp$, then $\theta(w) = w$ while $\theta(z) = \delta\alpha^{-1}e_1 + \gamma\alpha e_2 - f$, as in case 1. Now $z \perp w$ means

$$\begin{aligned}0 &= (\gamma e_1 + \delta e_2 + f, \alpha^{-1}e_1 - e_2 + g) \\ &= \gamma(e_1, e_1)(\alpha^*)^{-1} - \delta(e_2, e_2) + (f, g).\end{aligned}$$

But

$$\begin{aligned}(\theta(z), w) &= (\delta\alpha^{-1}e_1 + \gamma\alpha e_2 - f, \alpha^{-1}e_1 - e_2 + g) \\ &= \delta\alpha^{-1}(e_1, e_1)(\alpha^*)^{-1} - \gamma\alpha(e_2, e_2) - (f, g) \\ &= \delta(e_2, e_2) - \gamma(e_1, e_1)(\alpha^*)^{-1} - (f, g) \\ &= -0 = 0.\end{aligned}$$

Thus $\theta(z) \perp w$.

3.6 *Remark*. Thus, the map θ_{pqr}, where $p \perp q$ and $rB(p, q)$, is involutory and preserves orthogonality where it is defined. Our next task is to extend θ_{pqr} to atoms $z < p \vee q$, other than r. We do this by means of:

3.7 THEOREM. *Let* p, q, r *be atoms in* L *such that* $p \perp q$ *and* $rB(p, q)$. *Then there exists an orthogonal pair of atoms* (x, y) *such that* (p, q) H (x, y) *via* r *and such that the atoms* $\bar{p} = (p \vee x) \wedge (p \vee y)$ *and* $\bar{q} = (p \vee y) \wedge (q \vee x)$ *have the following properties:*

(i) $\bar{p} \nleqq p \vee q$ *and* $\bar{q} \nleqq p \vee q$.

(ii) $\bar{p} \perp \bar{q}$.

(iii) $rB(\bar{p}, \bar{q})$.
(iv) $\theta_{pqr}(\bar{p}) = \bar{q}$.
(v) *For any atom z with $z \nleq p \vee q$ and $z \nleq \bar{p} \vee \bar{q}$, $\theta_{pqr}(z) = \theta_{\overline{pq}r}(z)$.*

Proof. We construct the orthogonal pair of atoms (x, y) here and leave the computational proof of (i)–(iv) (via coordinatization) to the reader. Let s be any atom in L orthogonal to $p \vee q$. Using coordinatization, as in the proofs of 3.4 and 3.5, let $p = De_1$ and $q = De_2$, noting that, since $rB(p, q)$, then $r = D(e_1 + \alpha e_2)$, where $\alpha(e_2, e_2)\alpha^* = (e_1, e_1)$. Now, by 2.3, an orthogonal pair of atoms (x, y) has the property $(p, q)\ H\ (x, y)$ via r if and only if $x \not\subset p \vee q$, $y < p \vee q \vee x$, and $(x \vee y) = \varphi_{r^\perp}(p) \vee s$, for some atom s orthogonal to $p \vee q$. To obtain the desired x, we use the fact that L is a bisecting Hilbert lattice and choose an atom x such that $xB(\varphi_{r^\perp}(p), s)$. Necessarily, then, since $\varphi_{r^\perp}(p) = D(\omega e_1 - e_2)$, where $\omega = (e_2, e_2)\alpha^*(e_1, e_1)^{-1}$, we must have $x = D(\omega e_1 - e_2 + f)$ for some vector f with $Df = s$. Also, since $xB(\varphi_{r^\perp}(p), s)$, we must have, by 2.10,

$$\begin{aligned}(f, f) = (\omega e_1 - e_2, \omega e_1 - e_2) &= \omega(e_1, e_1)\omega^* + (e_2, e_2)\\ &= (e_2, e_2) + (e_2, e_2)\\ &= 2(e_2, e_2),\end{aligned}$$

which is nonzero since $f \neq 0$ (thus D cannot have characteristic 2). With x having been chosen, y is of course determined. Using the formulas derived in 2.9 for $(p \vee x) \wedge (q \vee y)$ and $(p \vee y) \wedge (q \vee x)$ (together with the facts that $\omega\alpha = (e_2, e_2)\alpha^*(e_1, e_1)^{-1}\alpha = (e_2, e_2)(e_2, e_2)^{-1} = 1$ and $(f, f) = 2(e_2, e_2)$), we get

$$\bar{p} = (p \vee x) \wedge (q \vee y) = D[\alpha^{-1}e_1 - e_2 - f] \text{ and}$$
$$\bar{q} = (p \vee y) \wedge (q \vee x) = D[\alpha^{-1}e_1 - e_2 + f].$$

The verification of (i)–(iv) is now a matter of computation.

3.8 *Definition.* Let p, q be atoms with $p \perp q$ and let r be an atom with the property $rB(p, q)$. Choose atoms $\bar{p}, \bar{q}$ having the five properties of 3.7. Extend the map θ_{pqr} to all the atoms of L by defining, for atoms z with $z < p \vee q$ and $z \neq r$, $\theta_{pqr}(z)$ to equal $\theta_{\overline{pq}r}(z)$. We observe that the latter expression has a well-defined value, because $z \nleq \bar{p} \vee \bar{q}$.

3.9 Theorem. *Let p, q be atoms in L such that $p \perp q$ and let r be an atom such that $rB(p, q)$. The map θ_{pqr} has the following properties:*

(i) *θ_{pqr} is an everywhere-defined, one-to-one, involutory map of the set of atoms of L onto itself, which preserves orthogonality.*

(ii) $\theta_{pqr}(p) = q$.

Proof. (i) We already know that θ_{pqr} is everywhere-defined, involutory, and preserves orthogonality. Let $\theta(z_1) = \theta(z_2)$. Then $z_1 = \theta^2(z_1) = \theta^2(z_2) = z_2$, so θ is one-to-one. Finally, for any atom z in L, $z = \theta(\theta(z))$, so θ is onto.

(ii) Choose $\bar{p}$, $\bar{q}$ having the five properties of 3.7. By 3.8, $\theta_{pqr}(p) =$

https://doi.org/10.4153/CJM-1973-026-2 Published online by Cambridge University Press

$\{[(\bar{p} \vee p) \wedge r^{\perp}] \vee \bar{q}\} \wedge (p \vee r)$. Thus, $\bar{q} \leqq [(p \vee \bar{p}) \wedge r^{\perp}] \vee q$ so that $q \leqq [(p \vee \bar{p}) \wedge r^{\perp}] \vee \bar{q}$. Hence, $q \leqq \{[(p \vee \bar{p}) \wedge r^{\perp}] \vee \bar{q}\} \wedge (p \vee r)$, which is an atom by 3.2. Thus $q = \theta_{\bar{p}\bar{q}r}(p) = \theta_{pqr}(p)$.

We now wish to extend these maps from the atoms of L to all of L. For this purpose, the following lemma is needed. We omit the computational proof.

3.10 LEMMA. *Suppose that $p_1 \perp q_1$, $p_2 \perp q_2$, $r_1B(p_1, q_1)$, $r_2B(p_2, q_2)$ and $(p_1 \vee q_1) \perp (p_2 \vee q_2)$, where all the symbols just used represent atoms in L. Let $\theta_1 = \theta_{p_1q_1r_1}$ and $\theta_2 = \theta_{p_2q_2r_2}$. Then, the maps $\theta_2\theta_1$ and $\theta_1\theta_2$ are equal.*

3.11 *Definition.* Let $a \in L$. Define a map Θ_{pqr} on L by the rule $\Theta_{pqr}(a) = \bigvee \{\theta_{pqr}(z) \mid z \text{ an atom}, z \leqq a\}$.

3.12 THEOREM. *Let $a \in L$. Let $\{p_1, p_2, \ldots\}$ be any orthogonal family of atoms such that $\bigvee \{p_1, p_2, \ldots\} = a$. Then $\Theta_{pqr}(a) = \bigvee \{\theta_{pqr}(p_1), \theta_{pqr}(p_2), \ldots\}$.*

Proof. Let I be an indexing set for $\{p_1, p_2, \ldots\}$ so that this set can be written $\{p_i | i \in I\}$. Now since $\Theta_{pqr}(a) = \bigvee \{\theta_{pqr}(z) | z \text{ an atom with } z \leqq a\}$, we have $\bigvee \{\theta_{pqr}(p_k) | k \in I\} \leqq \Theta_{pqr}(a)$ clearly. Now suppose that strict inequality held. Then, by orthomodularity, we could find an atom t such that $t \perp \bigvee \{\theta_{pqr}(p_k) | k \in I\}$ and $t \leqq \Theta_{pqr}(a)$. Now, $t \perp \theta_{pqr}(p_k)$ for each $k \in I$ so $\theta_{pqr}(t) \perp \theta_{pqr}^2(p_k) = p_k$ for each $k \in I$. Thus $\theta_{pqr}(t) \perp \bigvee \{p_k | k \in I\} = a = \bigvee \{z | z \text{ an atom and } z \leqq a\}$. Therefore $\theta_{pqr}(t) \perp z$ for each atom $z \leqq a$. Hence, $t \perp \bigvee \{\theta_{pqr}(z) | z \text{ an atom}, z \leqq a\} = \Theta_{pqr}(a)$, a contradiction of the fact that $t \leqq \Theta_{pqr}(a)$. Thus, we must have that $\bigvee \{\theta_{pqr}(p_k) | k \in I\} = \Theta_{pqr}(a)$, as claimed. (Note that the last step before the contradiction is obtained by applying θ_{pqr} to both sides of the statement $\theta_{pqr}(t) \perp z$, for each atom $z \leqq a$. The statement then becomes $t \perp \theta_{pqr}(z)$ for each atom $z \leqq a$.)

3.13 COROLLARY. *If $rB(p, q)$, then the map Θ_{pqr} is an involutory orthoautomorphism of L, which maps p onto q.*

Proof. (i) Θ_{pqr} is involutory. Let $a \in L$, say $a = \bigvee \{p_k | k \in I\}$. Then $\Theta_{pqr}(a) = \bigvee \{\theta_{pqr}(p_k) | k \in I\}$ by 3.12 so

$$\Theta_{pqr}^2(a) = \Theta_{pqr}(\bigvee \{\theta_{pqr}(p_k) | k \in I\}) = \bigvee \{\theta_{pqr}^2(p_k) | k \in I\},$$

again by 3.12, and this last expression equals $\bigvee \{p_k | k \in I\} = a$. Thus Θ_{pqr} is involutory because θ_{pqr} is and because of 3.12.

(ii) Θ_{pqr} is one-to-one and order preserving. For suppose that $a \leqq b$. Then, by orthomodularity, we can write $b = a \vee c$, where the element c of L is orthogonal to a. Since Θ_{pqr} obviously preserves joins by 3.12, then $\Theta_{pqr}(a) \leqq \Theta_{pqr}(b)$. On the other hand, if $\Theta_{pqr}(a) \leqq \Theta_{pqr}(b)$, then $a = \Theta_{pqr}^2(a) \leqq \Theta_{pqr}^2(b) = b$. Hence, Θ_{pqr} is order preserving in both direction and so is one-to-one.

(iii) Θ_{pqr} preserves orthogonality, for suppose $a \perp b$. Let $a = \bigvee\{p_k | k \in I\}$ and $b = \bigvee \{q_j | j \in J\}$. Note that $p_k \perp q_j$ for all j, k. Hence, by 3.5,

$\theta_{pqr}(p_k) \perp \theta_{pqr}(q_j)$ for all k, j. Thus, since $\Theta_{pqr}(a) = \bigvee \{\theta_{pqr}(p_k) | k \in I\}$ and $\Theta_{pqr}(b) = \bigvee \{\theta_{pqr}(q_j) | j \in J\}$, we have $\Theta_{pqr}(a) \perp \Theta_{pqr}(b)$.

(iv) Obviously, $\Theta_{pqr}(p) = q$, since $\theta_{pqr}(p) = q$, by 3.9(ii).

A restatement of 3.13, with a slight change in emphasis, is given by the following

3.14 Corollary. *Given orthogonal atoms $p, q \in L$, there exists an involutory orthoautomorphism of L mapping p onto q.*

Proof. Θ_{pqr} has these properties.

3.15 Corollary. *Given two finite elements a, b in L such that $a \perp b$ and $d(a) = d(b)$, there exists an involutory orthoautomorphism Θ of L such that $\Theta(a) = b$.*

Proof. Let $a = \bigvee \{p_i | i = 1, 2, \ldots, n\}$ and $b = \bigvee \{q_j | j = 1, 2, \ldots, n\}$ (where $n = d(a) = d(b)$) and note that $p_i \perp q_j$ for all i, $j = 1, 2, \ldots, n$. Now, for each $i = 1, 2, \ldots, n$, let r_i be an atom such that $r_i B(p_i, q_i)$ and form $\Theta_i = \Theta_{p_i q_i r_i}$. Note that, if $j \neq i$, then $\Theta_j(p_i) = p_i$, because $p_i \perp p_j \vee q_j$ which forces $p_i \perp r_j$. Thus, if we let $\Theta = \Theta_1 \circ \Theta_2 \circ \ldots \circ \Theta_n$, we have that

$$\begin{aligned}\Theta(a) &= (\Theta_1 \circ \Theta_2 \circ \ldots \circ \Theta_n)(p_1 \vee P_2 \vee \ldots \vee p_n)\\ &= (\Theta_1 \circ \Theta_2 \circ \ldots \circ \Theta_{n-1})(p_1 \vee p_2 \vee \ldots \vee p_{n-1} \vee q_n)\end{aligned}$$

which, after n steps, equals $q_1 \vee q_2 \vee \ldots \vee q_n = b$. Thus, $\Theta(a) = b$ as claimed. Clearly, Θ is an orthoautomorphism of L, since each Θ_i is. Finally, Θ is involutory because

$$\begin{aligned}&(\Theta_1 \circ \Theta_2 \circ \ldots \circ \Theta_n) \circ (\Theta_1 \circ \Theta_2 \circ \ldots \circ \Theta_n)\\ &\qquad = \Theta_1^2 \circ \Theta_2^2 \circ \ldots \circ \Theta_n^2 = \text{identity} \qquad \text{(by 3.10)}.\end{aligned}$$

3.16 Corollary. *If a, b are finite elements of L with $d(a) = d(b)$, then the interval $L(0, a)$ is orthoisomorphic to the interval $L(0, b)$.*

Concluding remarks. We conclude with a list of questions which present themselves. Let L be an infinite dimensional Hilbert lattice:

(1) Need every pair of orthogonal atoms in L have an angle-bisecting atom?

(2) Given $a, b \in L$ with $a \perp b$ and $d(a) = d(b) \leqq \infty$, need there exist an orthoautomorphism θ of L such that $\theta(a) = b$?

(3) If $a \wedge b = a^\perp \wedge b^\perp = 0$, need there exist an involutory orthoautomorphism θ of L such that $\theta(a) = b^\perp$? (If so, then we could derive 0-symmetry [**1**, Corollary 36.14].)

References

1. F. Maeda and S. Maeda, *Theory of symmetric lattices* (Springer-Verlag, Berlin, 1970).

2. R. P. Morash, *The orthomodular identity and metric completeness of the coordinatizing division ring*, Proc. Amer. Math. Soc. *27* (1971), 446–448.
3. ——— *supplement to* [**2**], Proc. Amer. Math. Soc. *29* (1971), 627.
4. ——— *Orthomodularity and the direct sum of division subrings of the quaternions*, Proc. Amer. Math. Soc. *36* (1972), 63–68.

University of Massachusetts,
Amherst, Massachusetts;
University of Michigan,
Dearborn, Michigan