Pythagoras's Theorem.*

By M. Edouard Collignon.

The square described on the hypotenuse of a right-angled triangle is equal to the squares described on the other two sides.

About half a hundred proofs of this theorem have been given, but few of them have been "ocular," that is, few have shown how the two smaller squares may be decomposed so as to fit into the largest square. One of the most elegant of the ocular proofs is that of Henry Perigal, and was discovered about 1830. A demonstration of its correctness is not difficult to obtain, but the following demonstration is believed to be new. It depends somewhat on algebra, and presupposes a simple lemma.

Perigal's Construction.

Figure 23.

Triangle ABC is right-angled.
BCED is the square on the hypotenuse, ACKH and ABFG are the squares on the other sides.

Find O the centre of the square ABFG, which may be done by drawing the two diagonals (not shown in the figure), and through it draw two straight lines, one of which is parallel to BC, and the other perpendicular to BC. The square ABFG is then divided into four quadrilaterals equal in every respect.

Through the mid points of the sides of the square BCED draw parallels to AB and AC as in the figure.

The parts numbered 1, 2, 3, 4, 5 will be found to coincide with 1′, 2′, 3′, 4′, 5′.

Lemma. The square described on a diagonal of a square is double of the square described on a side.

* This is the 47th proposition of the 1st book of Euclid's *Elements*, and is called in France *le pont aux ânes*, the *asses' bridge*. This name, in English-speaking countries, is bestowed on the 5th proposition of the 1st book of Euclid's *Elements*.

FIGURE 24.

This is easily seen from an inspection of Figure 24, and from what is known about congruent triangles.

Triangles 1, 3 may be slid (that is, without moving them out of the plane) vertically into coincidence with 1′, 3′; and triangles 2, 4 may be slid horizontally into coincidence with 2′, 4′.

Demonstration of the Theorem.

FIGURE 25.

From the properties of parallelograms

$$MM' = CB \text{ and } NN' = MM'.$$

Denote the sides BC, CA, AB by a, b, c; and calculate the segments AM, AN, ... determined by the vertices of the inscribed square MNM′N′ on the sides of the square ABFG.

From the construction of the figure, we have

$$AM = BN \qquad OI = \frac{c}{2} \qquad ON = OM = \frac{a}{2}.$$

By the lemma, the side of the inscribed square,

$$MN = \frac{a}{2}\sqrt{2} = \frac{a}{\sqrt{2}}.$$

Hence the area of the inscribed square is $\frac{a^2}{2}$ and the area of the four equal triangles AMN, ... is the difference $c^2 - \frac{a^2}{2}$.

Each triangle is therefore the quarter of this difference, that is,

$$\tfrac{1}{4}\left(c^2 - \frac{a^2}{2}\right).$$

But the area of each of the triangles

$$= \tfrac{1}{2}AM \times AN.$$

Hence
$$AM \times AN = AN \times NB = \frac{c^2}{2} - \frac{a^2}{4},$$

and the product of the segments AN, NB is known.

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Now the sum of the segments AN, NB is also known, since it is equal to AB or c.

The two segments are therefore the roots of an equation of the second degree

$$(1) \qquad t^2 - ct + \frac{c^2}{2} - \frac{a^2}{4} = 0;$$

whence, after simplification,

$$(2) \qquad t = \frac{c}{2} \pm \tfrac{1}{2}\sqrt{a^2 - c^2}.$$

The difference of the roots is therefore equal to the radical

$$\sqrt{a^2 - c^2};$$

and if, on the segment AN, we cut off AP equal to BN, this difference is PN.

The mid point of PN is I, and consequently

$$\mathrm{IN} = \tfrac{1}{2}\sqrt{a^2 - c^2}.$$

Since triangle OIN has its sides respectively perpendicular to those of the given triangle BAC, these two triangles are similar; therefore

$$(3) \qquad \frac{\mathrm{IN}}{\mathrm{AC}} = \frac{\mathrm{ON}}{\mathrm{BC}} = \frac{\mathrm{OI}}{\mathrm{BA}} = \tfrac{1}{2}.$$

Hence $\qquad \mathrm{IN} = \dfrac{b}{2}$ and $\mathrm{PN} = b$.

Finally, we have the relation

$$(4) \qquad b = \sqrt{a^2 - c^2},$$

that is, $\qquad b^2 + c^2 = a^2,$

which proves the theorem.

Post-scriptum.

On peut justifier le théorème sur l'un quelconque des triangles AMN qu'on forme pour la démonstration de Henry Perigal.

On connaît les trois côtés de ce triangle: $\mathrm{MN} = \dfrac{a}{\sqrt{2}}$, a désignant l'hypothénuse CB; AM et AN sont les racines d'une équation du second degré

$$t^2 - ct + \frac{c^2}{2} - \frac{a^2}{4} = 0.$$

Sans résoudre l'équation, appelons t' et t'' les racines ; nous avons à former $t'^2 + t''^2$ et vérifier que cette somme reproduit $\frac{a^2}{2}$.

Or, la somme des carrés des racines d'une équation algébrique quelconque

$$x^m + px^{m-1} + qx^{m-2} + \ldots = 0$$

s'exprime par la fonction $p^2 - 2q$, car $-p$ est la somme des racines, et q la somme de leurs produits deux à deux. Ici on a

$$p = -c \qquad q = \frac{c^2}{2} - \frac{a^2}{4}$$

et

$$p^2 - 2q = c^2 - \left(c^2 - \frac{a^2}{2}\right) = \frac{a^2}{2}$$

ce qui justifie le résultat annoncé.

On a Problem in Rigid Dynamics.

By G. M. K. Leggett.