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RADIATING DEMIANSKI-TYPE METRICS AND THE EINSTEIN-MAXWELL FIELDS

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Abstract

A Demianski-type metric is investigated in connection with the Einstein-Maxwell fields. Using complex vectorial formalism, some exact solutions of Einstein-Maxwell field equations for source-free electromagnetic fields plus pure radiation fields are obtained. The radiating Demianski solution, the Debney-Kerr-Schild solution and the Brill solution are derived as particular cases.

1. Introduction

Two physically important vacuum solutions of the Einstein equations representing the gravitational fields of rotating bodies are well-known in the literature. They are the Kerr [11] solution and the NUT [12] solution. The corresponding charged versions of these solutions are also available now ([5], [8]).

Vaidya Patel and Bhatt [15] have obtained the non-static generalisation of the above two vacuum solutions. They have used the field equations $R_{\alpha\beta} = \sigma\xi_\alpha\xi_\beta$, $\xi_\alpha\xi^\alpha = 0$, and obtained the radiating Kerr and the radiating NUT solutions. Bhatt and Patel [2] have extended these non-static solutions to a situation where there is a source-free electromagnetic field.

Here it should be noted that Bonnor and Vaidya [3], [4] and Patel and Misra [14] have also obtained some Kerr-Schild type solutions of the Einstein-Maxwell field equations, corresponding to source-free electromagnetic field plus pure radiation. Demianski [9] has obtained a new vacuum solution of the Einstein field equations depending on four arbitrary constants, by the method of a complex

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co-ordinates transformation. This solution includes the Kerr solution and NUT solution as particular cases. Patel [13] has obtained a radiating version of the Demianski solution corresponding to the field equations $R_{\alpha\beta} = \sigma\xi_\alpha\xi_\beta;\ \xi_\alpha\xi^\alpha = 0$. The present paper is an extension of the argument of the paper by Patel [13] to a situation where there is a source-free electromagnetic field. We shall solve the Einstein-Maxwell field equations

$$R_{\alpha\beta} = -[E_{\alpha\beta} + \sigma\xi_\alpha\xi_\beta], \qquad \xi_\alpha\xi^\alpha = 0, \tag{1}$$

$$E_{\alpha\beta} = -F_{\alpha\beta}F^\delta_\beta + \frac{g_{\alpha\beta}}{4}F_{\tau\delta}F^{\tau\delta},$$

$$F^{\alpha\beta}{}_{;\beta} = 0, \tag{2}$$

where $F_{\alpha\beta}$ and σ are respectively the electromagnetic field tensor and the density of flowing radiation. The semicolon indicates covariant differentiation.

The line of attack will be the complex vectorial formalism developed by Cahne, Debever and Defrise [6]. The detailed account of this formalism has been given by Israel [10] and we shall use his notation. This formalism is briefly discussed in the previous paper [1], and so is not repeated here. In this complex vectorial formalism the field equations (1) and (2) become

$$R = 0, \qquad E_{p\bar{q}} = -2F_p\overline{F}_q - \sigma\delta_p^2\bar{\delta}_q^2, \tag{3}$$

$$dF^+ = 0. \tag{4}$$

Here R is a scalar curvature, $E_{p\bar{q}}$ is the Hermitian tensor corresponding to the trace-free part of the Ricci tensor and F^+ is the self-dual part of the electromagnetic field tensor F_{ab}. Thus

$$F^+ = F_pZ^p \tag{5}$$

where $\{Z^p\}$ is the basis for the 3-complex space of self-dual bivectors. We use the conventions of our previous paper [1].

2. Metric and the Maxwell equations

Consider the metric in the form

$$ds^2 = 2(du+gGd\beta)(dr+hGd\beta) - 2L(du+gGd\beta)^2 - M^2\{(dy^2/G^2)+G^2d\beta^2\} \tag{6}$$

where L and M are functions of coordinates u, y and r, and h, G and g are functions of y only. For the metric (6), the expressions for components $E_{p\bar{q}}$ and the scalar curvature R are given in [1].

Since $E_{1\bar{2}} = 0$, it follows from (3) that either (i) $F_1 = 0$ or (ii) $F_2 = 0$ or (iii) $F_1 = F_2 = 0$. The last case is not possible because from (3) we find that F_3 also vanishes. In this paper we discuss the case $F_1 = 0$. Therefore we can take

$$F^+ = \phi Z^2 + \psi Z^3 \tag{7}$$

where $F_2 = \phi$ and $F_3 = \psi$ are functions of co-ordinates u, y and r.

Since $F_{\bar{1}} = 0$, it follows from the field equation (3) that $E_{1\bar{1}} = 0$ and $E_{1\bar{3}} = 0$. These two relations involve only one unknown function M. They can be solved to get

$$M^2 = F^2(X^2 + Y^2), \quad F^2 = fY, \quad 2f = (gG)_y \tag{8}$$

with

$$X_u = (-Y + z)_\theta, \quad X_\theta = Y_u, \quad X_r = -1, \quad Y_r = 0. \tag{9}$$

Here and in what follows θ and z are defined by the differential relations

$$g dy = G d\theta, \qquad h dy = G dz \tag{10}$$

and the suffixes denote partial derivatives.

Using F^+ given by (7) and M^2 given by (8) and (9), the Maxwell equations (4) give us the following relations for ϕ and ψ:

$$M^2\psi_r + \psi[MM_r - i(gG)_y] = 0, \tag{11}$$

$$hi\psi_r + gi\psi_u - G\psi_y = 0, \tag{12}$$

$$\sqrt{2}M[\phi_r + (M_r/M)\phi] - G\psi_y = 0, \tag{13}$$

$$\begin{aligned} &hi\phi_r + gi\phi_u - G\phi_y + M(\psi_u + L_r)/\sqrt{2} + (\phi/M)[giM_u - (MG)_y - hiM_r] \\ &+ \sqrt{2}\psi[M_u + LM_r + i(Gh)_y/M^2 - iL(gG)_y/2M^2] = 0. \end{aligned} \tag{14}$$

The equations (8) and (11) give us the solution for ψ as

$$\psi = K(X - iY)^{-2} \tag{15}$$

where K is a complex function of u and y only. Substituting the value of ψ in (12) we find

$$K_{1u} = K_{2\theta}, \; K_{1\theta} = -K_{2u}, \qquad K = K_1 + iK_2 \tag{16}$$

where K_1 and K_2 are real functions of co-ordinates u and y. Using the value of ψ given by (15) and (16) in (11), we find the form of the function ϕ as

$$\phi = \frac{gi}{\sqrt{2}M}\left(\frac{K}{X - iY}\right)_u + \frac{hi}{\sqrt{2}M} \cdot \frac{K}{(X - iY)^2}. \tag{17}$$

Substituting the value of ϕ, ψ and M in (14) we find that

$$K = \ell Y \tag{18}$$

where ℓ is a complex function of y only. From (15) and (17) we can obtain the electromagnetic field tensor F_{ij}.

3. Remaining Einstein-Maxwell equations and their solutions

In this case we shall solve the remaining equations of (3) for a comparatively simple case in which $f = Y$. The fact that $f = Y$ implies that Y is a function

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of y only. From the equation (9) we have

$$X = au - r, \qquad Y = z - a\theta + b, \tag{19}$$

where a and b are constants of integration. We shall use the variable α instead of y, defined by the equation $G(y) = \sin\alpha$. From the results (16) and (18) it is easy to see that $K = $ constant.

Now the equations which have to be satisfied in this case are

$$R = 0, \qquad E_{3\bar{3}} = -2F_3\overline{F}_3, \qquad E_{2\bar{3}} = -2F_2\overline{F}_3. \tag{20}$$

Using the expressions for R, $E_{3\bar{3}}$ and $E_{2\bar{3}}$ given by [1] in (20), we obtain the following form of the function $2L$:

$$2L = 2a - 1 + (2E^*Y + 2F^*Y)/(X^2 + Y^2) \tag{21}$$

where

$$E^* + K\overline{K}/(4Y) = -k\theta + n, \qquad F^* = m + ku \tag{22}$$

and

$$N^* = (a-1)Y - k\theta + n \tag{23}$$

Here k, n and m are constants of integration and N^* stands for $(h\sin\alpha)y/2$. The functions g and h can be determined from $Y = z - a\theta + b$ and the result (23).

From (8), (19), (23) and the fact that $f = Y$ and $G = \sin\alpha$, one can obtain the following differential equations for the functions Y and N^*:

$$(1-p^2)Y_{pp} - 2pY_p = 2N^* - 2aY, \tag{24}$$

$$(1-p^2)N^*_{pp} - 2pN^*_p = -2kY + (a-1)(2N^* - 2a) \tag{25}$$

where $p = \cos\alpha$.

If we set $q > 0$ ($q = \sqrt{1-4k}$) then it can be easily seen that the equations (24) and (25) are equivalent to

$$(1-p^2)Z'_{pp} - 2pZ'_p + n(n+1)Z' = 0 \tag{26}$$

$$(1-p^2)Z_{pp} - 2pZ_p + \ell(\ell+1)Z = 0 \tag{27}$$

with

$$1 - q = n(n+1), \qquad 1 + q = \ell(\ell+1),$$
$$Z' = N^* + (q - 2a + 1)Y/2, \qquad Z = N^* - (q + 2a - 1)Y/2. \tag{28}$$

The solutions for (26) and (27) are, from [7]:

$$Z' = a_1P_n(p) + a_2Q_n(p), \tag{29}$$

$$Z = b_1P_\ell(p) + b_2Q_\ell(p), \tag{30}$$

where a_1, a_2, b_1 and b_2 are constants and $P_n(p)$ and $Q_n(p)$ are Legendre functions of the first and second kind respectively. The series expressions of $P_n(p)$ and $Q_n(p)$ are

$$P_n(p) = p - (n-1)(n+2)/(3!)p^3 + (n-1)(n+2)(n-3)(n-4)/(5!)p^5 - \cdots, \quad (31)$$

$$Q_n(p) = 1 - n(n+1)/(2!)p^2 + n(n+1)(n-2)(n-3)/(4!)p^4 - \cdots. \quad (32)$$

The series for $P_n(p)$ and $Q_n(p)$ are convergent for $q^2 < 1$ and are continuous in the interval $(-1, 1)$. Knowing Z and Z' from (29) and (30), the result (28) gives us

$$qY = Z - Z', \quad 2qN^* = (q - 1 - 2a)Z + (2a - 1 + q)Z'. \quad (33)$$

Then the unknown functions g and h are obtained as

$$g \sin\alpha = -\int 2Y \sin\alpha d\alpha, \qquad h \sin\alpha = -\int 2N^* \sin\alpha\, d\alpha. \quad (34)$$

These integrals can be evaluated in the interval $(-1, 1)$. The density of the flowing radiation is given by

$$\sigma = -2k/(X^2 + Y^2). \quad (35)$$

Now we are ready with the forms of all unknown functions L, M, g, h and G. The metric (6) then can be written as

$$ds^2 = 2\left[du - \left(\int 2Y \sin\alpha\, d\alpha\right) d\beta\right]\left[dr - \left(\int 2N^* \sin\alpha\, d\alpha\right) d\beta\right] - (X^2 + Y^2)(d\alpha^2 + \sin^2\alpha\, d\beta^2) - 2L\left[du\left(\int 2Y \sin\alpha\, d\alpha\right) d\beta\right] \quad (36)$$

with

$$X = au - r, \qquad E^* = -K\overline{K}/(4Y) + N^* + (a-1)Y,$$
$$2L = 2a - 1 + [2E^*Y + 2X(-ku + m)]/(X^2 + Y^2).$$

The functions Y and N^* are given by (33).

The metric (36) is the charged version of the radiating Demianski-type metric. However if the electromagnetic field is switched off, we are back to the radiating Demianski-type metric given by Patel [13] (taking $a_1 = 0$). The relation $q > 0$ is equivalent to $k > 1/4$. If we choose $k \leq 0$, it is clear from (35) that the radiation density is positive.

To obtain the charged version of the Demianski metric, let us choose $a_1 = 0$ in (29), and if we choose $k = 0$ then $q = 1$ and consequently $n = 0$ and $l = 1$ in

(28). In this case we get the electromagnetic field only, and we have

$$X = au - r, \qquad -Y = b_1 \cos\alpha - a_2 + b_2 \cos\alpha \log\left(\tan\frac{\alpha}{2}\right),$$
$$g \sin\alpha = b_1 \sin^2\alpha - 2a_2 \cos\alpha + 2b_2 \cos\alpha - b_2 \sin^2\alpha \log(\tan\alpha);$$
$$h \sin\alpha = 2aa_2 \sin\alpha + (a-1) g \sin\alpha. \tag{37}$$

Using the results (37), the metric (36) can be written as

$$\begin{aligned} ds^2 &= 2(du + g\sin\alpha\, d\beta)[(2aa_2 \sin\alpha + g\sin\alpha)d\beta - d\bar{r}] \\ &\quad - (\bar{r}^2 + y^2)(d\alpha^2 + \sin^2\alpha\, d\beta^2) \\ &\quad + [1 - (2m\bar{r} + 2YE^*)/(\bar{r}^2 + Y^2)](du + g\sin\alpha\, d\beta)^2 \end{aligned} \tag{38}$$

with $X = au - r = \bar{r}$; $E^* = K\overline{K}/4Y$.

The Brill [5] metric can be obtained from (38) by taking $b_1 = b_2 = 0$. Then the final form of the metric (38) is

$$\begin{aligned} ds^2 &= 2(du + g\sin\alpha\, d\beta)[(2aa_2 \sin\alpha - g\sin\alpha)d\beta - d\bar{r}] \\ &\quad + [1 - (2m\bar{r} - 2a_2E^*)/(\bar{r}^2 + a_2^2)](du + g\sin\alpha\, d\beta)^2 \\ &\quad - (\bar{r}^2 + a_2^2)(d\alpha^2 + \sin^2\alpha\, d\beta^2) \end{aligned} \tag{39}$$

with

$$X = au - r = \bar{r}, \quad g\sin\alpha = -2a_2\cos\alpha, \quad E^* = n - K\overline{K}/(4a_2).$$

To obtain the Debney, Kerr and Schild [8] metric we put $b_2 = a_2 = 0$ in (38); then (38) can be written as

$$\begin{aligned} ds^2 &= 2(du + g\sin\alpha\, d\beta)(d\bar{r} + g\sin\alpha\, d\beta) \\ &\quad + [1 - (2m\bar{r} - 2b_1\cos\alpha E^*)/(\bar{r}^2 + b_1^2\cos^2\alpha)](du + g\sin\alpha\, d\beta)^2 \\ &\quad - (\bar{r}^2 + b_1^2\cos^2\alpha)(d\alpha^2 + \sin^2\alpha d\beta^2) \end{aligned} \tag{40}$$

with

$$X = au - r, \qquad g\sin\alpha = b_1\sin^2\alpha, \qquad E^* = n - K\overline{K}/4b_1\cos\alpha.$$

The metric (40) is the Debney-Kerr-Schild metric.

The case $f \neq Y$ and the other mathematical details regarding the metric (6) will be reported elsewhere.

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