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Affine Actions of $U_q(sl(2))$ on Polynomial Rings

To Len Krop in honor of his retirement.

Jeffrey Bergen

Abstract. We classify the affine actions of $U_q(sl(2))$ on commutative polynomial rings in $m \geq 1$ variables. We show that, up to scalar multiplication, there are two possible actions. In addition, for each action, the subring of invariants is a polynomial ring in either m or $m-1$ variables, depending upon whether q is or is not a root of 1.

Montgomery and Smith [3] examine the actions of the quantum group $U_q(sl(2))$ on a polynomial ring in one variable. A natural direction for generalization is to try to realize $U_q(sl(2))$, or more generally $U_q(sl(n))$, as differential operators on quantum n-space. For $U_q(sl(2))$, this was also done in [3]. In [2], this was done for $U_q(sl(n))$, for any n.

Another natural direction is to try to find all actions of $U_q(sl(2))$ on commutative polynomial rings in $m \geq 1$ variables and to then describe the invariants of these actions. The definition of $U_q(sl(2))$ has evolved slightly since the Montgomery–Smith paper, and we use the newer definition to determine all affine actions of $U_q(sl(2))$ on commutative polynomial rings. We will show, following a change of variables, that these actions are trivial on $m-1$ variables, and the behavior on the last variable is very similar to the situation described in [3]. The main result of this paper will be the following theorem.

Theorem 4 *Consider an affine action of $U_q(sl(2))$ on the commutative polynomial ring $R = k[x_1, \dots, x_m]$ such that $\sigma^2 \neq 1$. Then there exist $y_1, \dots, y_m, \in R$ such that*

(i) *R is the polynomial ring $k[y_1, \dots, y_m]$;*
(ii) *$\sigma(y_i) = y_i$ and $\delta_E(y_i) = \delta_F(y_i) = 0$, for $2 \leq i \leq m$,*

where σ is an automorphism, δ_E is a $(\sigma, 1)$-skew derivation, and δ_F is a $(1, \sigma^{-1})$-skew derivation. Furthermore, the only two possibilities, up to scalar multiplication, for the action of $U_q(sl(2))$ on ${y_1}^n$, for $n \geq 1$, are

(i) $\sigma({y_1}^n) = q^{2n}{y_1}^n$, $\delta_E({y_1}^n) = \frac{q^{2n}-1}{q^2-1}{y_1}^{n+1}$, $\delta_F({y_1}^n) = \frac{q^{-2n}-1}{q^{-2}-1}{y_1}^{n-1}$;
(ii) $\sigma({y_1}^n) = q^{-2n}{y_1}^n$, $\delta_E({y_1}^n) = \frac{q^{-2n}-1}{q^{-2}-1}{y_1}^{n-1}$, $\delta_F({y_1}^n) = \frac{q^{2n}-1}{q^2-1}{y_1}^{n+1}$.

Using Theorem 4, we can easily describe the invariants of these actions.

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Corollary 5 *Consider an affine action of $H = U_q(sl(2))$ on the commutative polynomial ring $R = k[x_1, \dots, x_m]$ with $\sigma^2 \neq 1$.*

(i) *If q is not a root of 1, then the subring of invariants R^H is a commutative polynomial ring in $m-1$ variables.*

(ii) *If q is a root of 1 and t is the smallest positive integer such that $q^{2t} = 1$, then the subring of invariants R^H is a commutative polynomial ring in m variables and R is a free R^H-module of rank t.*

We now introduce the terminology and notation that will be used throughout this paper. Additional background material on $U_q(sl(2))$ and Hopf algebras can be found in [1]. We will let k denote a field and $R = k[x_1, \dots, x_m]$ will be the commutative polynomial ring over k. There will be no assumptions made about the characteristic of k, and we let $0 \neq q \in k$ be such that $q^2 \neq 1$. It will not be important whether or not q is a root of 1 until Corollary 5.

Next, we let $U_q(sl(2))$ be the k-algebra generated by K, K^{-1}, E, F subject to the relations

$$KK^{-1} = K^{-1}K = 1, \quad KE = q^2 EK, \quad KF = q^{-2} FK, \quad EF - FE = \frac{K - K^{-1}}{q - q^{-1}}.$$

Observe that these relations require that $q^2 \neq 1$, and this guarantees that $U_q(sl(2))$ is not commutative.

Since $U_q(sl(2))$ is a Hopf algebra, we also need to examine its comultiplication, counit, and antipode. The comultiplication Δ in $U_q(sl(2))$ is given by

$$\Delta(K) = K \otimes K, \quad \Delta(E) = E \otimes K + 1 \otimes K, \quad \Delta(F) = F \otimes 1 + K^{-1} \otimes F.$$

Observe that $U_q(sl(2))$ is also not cocommutative. In addition, the counit ϵ and antipode S are given by

$$\begin{aligned} &\epsilon(K) = 1, && \epsilon(E) = 0, && \epsilon(F) = 0, \\ &S(K) = K^{-1}, && S(E) = -EK^{-1}, && S(F) = -KF. \end{aligned}$$

If H is a Hopf algebra, then an algebra A is called an H-module algebra if A is a left H-module with the added properties that

$$h(1) = \epsilon(h)1 \quad \text{and} \quad h(ab) = \sum_{(h)} h_{(1)}(a)h_{(2)}(b),$$

for all $a, b \in A$ and $h \in H$, where $\Delta(h) = \sum_{(h)} h_{(1)} \otimes h_{(2)}$ is comultiplication applied to h. When we refer to an *action* of $H = U_q(sl(2))$ on $R = k[x_1, \dots, x_m]$ or say that *H acts* on R, we mean that R is an H-module algebra. If A is a commutative domain, we let $Q(A)$ denote its quotient field.

When $U_q(sl(2))$ acts on R, since $\Delta(K) = K \otimes K$ and K is invertible, K acts as an automorphism. We will let σ denote the automorphism of R induced by K. If g is an automorphism, a k-linear map d is called a $(g, 1)$-*skew derivation* if

$$d(ab) = d(a)g(b) + ad(b),$$

for all $a, b \in R$. Analogously, d is called a $(1, g)$-*skew derivation* if

$$d(ab) = d(a)b + g(a)d(b).$$

Since $\Delta(E) = E \otimes K + 1 \otimes K$ and $\Delta(F) = F \otimes 1 + K^{-1} \otimes F$, E acts as a $(K, 1)$-skew derivation and F acts as a $(1, K^{-1})$-skew derivation. We will let δ_E and δ_F denote the skew derivations of R induced, respectively, by E and F. It is important to note that we will always assume that $\sigma^2 \neq 1$. Observe that if $\sigma^2 \neq 1$ then $\delta_E \neq 0$ and $\delta_F \neq 0$, whereas if $\sigma^2 = 1$, then either $q^4 = 1$ or $\delta_E = \delta_F = 0$.

The invariants of the action of H on R is the subalgebra

$$R^H = \{a \in R \mid h(a) = \epsilon(h)a, \text{ for all } h \in H\}.$$

In our situation, observe that

$$R^H = \{a \in R \mid \sigma(a) = a\} \cap \{a \in R \mid \delta_E(a) = 0\} \cap \{a \in R \mid \delta_F(a) = 0\}.$$

We say that the action of $U_q(sl(2))$ on $k[x_1, \dots, x_m]$ is *affine* if $\sigma(x_i)$ has degree one, for $1 \leq i \leq m$. Certainly when $U_q(sl(2))$ acts on $k[x_1]$, as in [3], the action must be affine. However, in general, actions of $U_q(sl(2))$ on $k[x_1, \dots, x_m]$ need not be affine.

We begin the work needed to prove Theorem 4 with the following lemma.

Lemma 1 *Let $d \neq 0$ be either a $(g, 1)$ or $(1, g)$-skew derivation of a commutative domain A where $g \neq 1$. Then there exists $0 \neq \lambda \in Q(A)$ such that $d = \lambda(g - 1)$.*

Proof First, suppose d is a $(g, 1)$-skew derivation and let $a, b \in A$. Since A is commutative, if $a, b \in A$, we have

$$d(ab) = d(ba) = d(b)g(a) + bd(a) = d(a)b + g(a)d(b).$$

Therefore d is also a $(1, g)$-skew derivation, and it suffices to consider $(1, g)$-skew derivations.

Let $r \in A$; since $g \neq 1$, we can choose $a \in A$ such that $g(a) \neq a$. Since A is commutative, we have

$$d(a)r + g(a)d(r) = d(ar) = d(ra) = d(r)a + g(r)d(a).$$

If we subtract $d(a)r + d(r)a$ from the far left and far right of the previous equation, we obtain

$$(g(a) - a)d(r) = d(a)(g(r) - r).$$

Since $g(a) \neq a$, we can divide both sides of the previous equation by $g(a) - a$, and if we let $\lambda = (g(a) - a)^{-1}d(a)$, we obtain

$$d(r) = \lambda(g(r) - r).$$

Thus, $d = \lambda(g - 1)$. ∎

In light of Lemma 1, when $H = U_q(sl(2))$ acts on $R = k[x_1, \dots, x_m]$, the fixed points of σ are the same as the kernel of both δ_E and δ_F. Therefore, when we examine R^H, it will suffice to study the fixed points of σ.

Lemma 2 *Let A be a commutative domain with an automorphism σ such that $\sigma^2 \neq 1$ and let $0 \neq e, f \in Q(A)$. If $\delta_E = e(\sigma - 1)$ and $\delta_F = f(\sigma^{-1} - 1)$, then $\sigma, \delta_E, \delta_F$ induce an action of $U_q(sl(2))$ on A if and only if*

(i) $\sigma(e) = q^2 e$, $\sigma(f) = q^{-2} f$,

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(ii) $ef = \frac{q^3}{(q^2-1)^2}$,
(iii) $\delta_E(A) \subseteq A$, $\delta_F(A) \subseteq A$.

Proof In order for $\sigma, \delta_E, \delta_F$ to induce an action of $U_q(sl(2))$ on A, they need to satisfy the same relations satisfied, respectively, by K, E, F, in $U_q(sl(2))$. Using the facts that A is commutative, σ is an automorphism, $\delta_E = e(\sigma-1)$, and $\delta_F = f(\sigma^{-1}-1)$, it is easy to see that

$$\sigma(ab) = \sigma(a)\sigma(b), \quad \delta_E(ab) = \delta_E(a)\sigma(b) + a\delta_E(b),$$
$$\delta_F(ab) = \delta_F(a)b + \sigma^{-1}(a)\delta_F(b),$$

for all $a, b \in A$. Thus the actions of $\sigma, \delta_E, \delta_F$ on A are consistent with the comultiplication of K, E, F in $U_q(sl(2))$.

Next, we need to find necessary and sufficient conditions on e, f such that $\sigma, \delta_E, \delta_F$ satisfy the relations

$$\sigma\delta_E = q^2\delta_E\sigma, \quad \sigma\delta_F = q^{-2}\delta_F\sigma, \quad \delta_E\delta_F - \delta_F\delta_E = \frac{\sigma - \sigma^{-1}}{q - q^{-1}}.$$

If $a \in A$, we have

$$\sigma\big(\delta_E(a)\big) = \sigma\big(e\sigma(a) - ea\big) = \sigma(e)\big(\sigma^2(a) - \sigma(a)\big),$$
$$q^2\delta_E\big(\sigma(a)\big) = q^2\big(e\sigma^2(a) - e\sigma(a)\big) = q^2e\big(\sigma^2(a) - \sigma(a)\big)$$

and

$$\sigma\big(\delta_F(a)\big) = \sigma\big(f\sigma^{-1}(a) - fa\big) = \sigma(f)\big(a - \sigma(a)\big),$$
$$q^{-2}\delta_F\big(\sigma(a)\big) = q^{-2}\big(fa - f\sigma(a)\big) = q^{-2}f\big(a - \sigma(a)\big).$$

Since there exists $a \in A$ such that $\sigma(a) \neq a$, the above equations show that $\sigma\delta_E = q^2\delta_E\sigma$ and $\sigma\delta_F = q^{-2}\delta_F\sigma$ if and only if $\sigma(e) = q^2e$ and $\sigma(f) = q^{-2}f$.

We will now compute $\delta_E\delta_F$ and $\delta_F\delta_E$. In light of the previous argument, we may assume that $\sigma(e) = q^2e$ and $\sigma(f) = q^{-2}f$. If $a \in A$, we have

$$\begin{aligned} \delta_E\big(\delta_F(a)\big) &= e\sigma\big(f\sigma^{-1}(a) - fa\big) - e\big(f\sigma^{-1}(a) - fa\big) \\ &= e\sigma(f)a - e\sigma(f)\sigma(a) - ef\sigma^{-1}(a) + efa \\ &= (1+q^{-2})efa - q^{-2}ef\sigma(a) - ef\sigma^{-1}(a) \end{aligned} \tag{1}$$

and

$$\begin{aligned} \delta_F\big(\delta_E(a)\big) &= f\sigma^{-1}\big(e\sigma(a) - ea\big) - f\big(e\sigma(a) - ea\big) \\ &= \sigma^{-1}(e)fa - \sigma^{-1}(e)f\sigma^{-1}(a) - ef\sigma(a) + efa \\ &= (1+q^{-2})efa - ef\sigma(a) - q^{-2}ef\sigma^{-1}(a). \end{aligned} \tag{2}$$

Subtracting equation (2) from equation (1) gives us

$$(\delta_E\delta_F - \delta_F\delta_E)(a) = (1-q^{-2})ef\big(\sigma(a) - \sigma^{-1}(a)\big),$$

therefore

$$\delta_E\delta_F - \delta_F\delta_E = (1-q^{-2})ef(\sigma - \sigma^{-1}).$$

Since there exists $a \in A$ such that $\sigma^2(a) \neq a$, the previous equation shows us that

$$\delta_E\delta_F - \delta_F\delta_E = \frac{\sigma - \sigma^{-1}}{q - q^{-1}}$$

if and only if

$$(1 - q^{-2})ef = \frac{1}{q - q^{-1}}.$$

However, this is clearly equivalent to

$$ef = \Big(\frac{1}{1 - q^{-2}}\Big)\Big(\frac{1}{q - q^{-1}}\Big) = \frac{q^3}{(q^2 - 1)^2}.$$

Finally, since e, f need not belong to A, in order to have an action of $U_q(sl(2))$ on A, we also need to add the conditions that $\delta_E(A) \subseteq A$ and $\delta_F(A) \subseteq A$. ■

The next lemma will exploit the fact that $k[x_1, \dots, x_m]$ is a unique factorization domain.

Lemma 3 *Let $R = k[x_1, \dots, x_m]$, and suppose $y \in R$ has degree one and $0 \neq d \in Q(R)$ such that $dy, \frac{y}{d} \in R$ with $d \notin k$. Then there exists $0 \neq \alpha \in k$ such that either $y = \alpha d$ or $y = \frac{\alpha}{d}$.*

Proof One possibility is that either d or $\frac{1}{d}$ belongs to R, and we first consider the case where $d \in R$. Since R is a unique factorization domain and $d \notin k$, we have $d = p_1 \cdots p_s$, where $s \geq 1$ and each p_i is an irreducible polynomial. However, y has degree one and

$$\frac{y}{d} = \frac{y}{p_1 \cdots p_s} \in R,$$

therefore it must be the case that $s = 1$ and p_1 has degree one. As a result, $d = p_1$ and $y = \alpha d$, for some $0 \neq \alpha \in k$. An identical argument then shows that if $\frac{1}{d} \in R$, then $y = \frac{\alpha}{d}$, for some $0 \neq \alpha \in k$.

In light of the previous argument, it suffices to show that either $d \in R$ or $\frac{1}{d} \in R$. Therefore, by way of contradiction, we will assume that neither d nor $\frac{1}{d}$ belong to R. Since R is a unique factorization domain, we can write

$$d = \frac{p_1 \cdots p_s}{q_1 \cdots q_t},$$

where $s, t \geq 1$ and every p_i, q_j is an irreducible polynomial such that no p_i is a multiple in R of any q_j Recall that

$$dy = \Big(\frac{p_1 \cdots p_s}{q_1 \cdots q_t}\Big) y \quad \text{and} \quad \frac{y}{d} = \frac{q_1 \cdots q_t}{p_1 \cdots p_s} y$$

both belong to R. Since $dy \in R$, we see that $p_1 \cdots p_s y$ is a multiple in R of q_1, hence y is a multiple in R of q_1. Similarly, since $\frac{y}{d} \in R$, we know that $q_1 \cdots q_t y$ is a multiple in R of p_1, hence y is also a multiple in R of p_1. However, y has degree one, therefore there exist $0 \neq \beta, \gamma \in k$ such that $y = \beta q_1$ and $y = \gamma p_1$. This immediately leads to the contradiction that p_1 is a multiple in R of q_1, concluding the proof. ■

Suppose that δ_1 is a $(\sigma, 1)$-skew derivation and δ_2 is a $(1, \sigma^{-1})$-skew derivation such that

$$\sigma\delta_1 = q^2\delta_1\sigma, \quad \sigma\delta_2 = q^{-2}\delta_2\sigma, \quad \delta_1\delta_2 - \delta_2\delta_1 = \alpha(\sigma - \sigma^{-1}), \tag{3}$$

where $0 \neq \alpha \in k$. It is easy to see that, for any $0 \neq \beta \in k$, there exists a unique $0 \neq \beta' \in k$ such that

$$\sigma(\beta\delta_1) = q^2(\beta\delta_1)\sigma, \quad \sigma(\beta'\delta_2) = q^{-2}(\beta'\delta_2)\sigma,$$
$$(\beta\delta_1)(\beta'\delta_2) - (\beta'\delta_2)(\beta\delta_1) = \frac{\sigma - \sigma^{-1}}{q - q^{-1}}.$$

Therefore, for any $\sigma, \delta_1, \delta_2$ satisfying (3) and $0 \neq \beta \in k$, we see that $\sigma, \beta\delta_1, \beta'\delta_2$ represent an action of $U_q(sl(2))$ on $k[x_1, \dots, x_m]$. As a result, finding actions of $U_q(sl(2))$ on $k[x_1, \dots, x_m]$ reduces to finding $\sigma, \delta_1, \delta_2$ satisfying (3) and if $0 \neq \gamma, \gamma' \in K$ then $\sigma, \gamma\delta_1, \gamma'\delta_2$ represents essentially the same action. In this situation, we say that $\sigma, \delta_1, \delta_2$ and $\sigma, \gamma\delta_1, \gamma'\delta_2$ are scalar multiples. Thus, up to scalar multiplication, it suffices to find triples $\sigma, \delta_1, \delta_2$ satisfying (3).

Theorem 4 *Consider an affine action of $U_q(sl(2))$ on the commutative polynomial ring $R = k[x_1, \dots, x_m]$ such that $\sigma^2 \neq 1$. Then there exist $y_1, \dots, y_m, \in R$ such that*

(i) *R is the polynomial ring $k[y_1, \dots, y_m]$;*
(ii) *$\sigma(y_i) = y_i$ and $\delta_E(y_i) = \delta_F(y_i) = 0$, for $2 \leq i \leq m$.*

Furthermore, the only two possibilities, up to scalar multiplication, for the action of $U_q(sl(2))$ on ${y_1}^n$, for $n \geq 1$, are

(i) $\sigma({y_1}^n) = q^{2n}{y_1}^n$, $\delta_E({y_1}^n) = \frac{q^{2n}-1}{q^2-1}{y_1}^{n+1}$, $\delta_F({y_1}^n) = \frac{q^{-2n}-1}{q^{-2}-1}{y_1}^{n-1}$;
(ii) $\sigma({y_1}^n) = q^{-2n}{y_1}^n$, $\delta_E({y_1}^n) = \frac{q^{-2n}-1}{q^{-2}-1}{y_1}^{n-1}$, $\delta_F({y_1}^n) = \frac{q^{2n}-1}{q^2-1}{y_1}^{n+1}$.

Proof Given an action of $U_q(sl(2))$ on $R = k[x_1, \dots, x_m]$, Lemma 1 implies that there exist $0 \neq e, f \in Q(R)$ such that $\delta_E = e(\sigma - 1)$ and $\delta_F = f(\sigma^{-1} - 1)$. Recall that we only need to find δ_E and δ_F up to scalar multiplication. Therefore, given σ, Lemma 2 tells us that it suffices to find $0 \neq e \in Q(R)$ such that $\sigma(e) = q^2 e$ and

$$e\big(\sigma(a) - a\big), \quad \frac{1}{e}\big(\sigma^{-1}(a) - a\big) \in R, \tag{4}$$

for all $a \in R$. Observe, in this situation, we are letting $f = \frac{1}{e}$ and it immediately follows that $\sigma(f) = q^{-2}f$.

Choose $1 \leq i \leq m$ and let $y = \sigma(x_i) - x_i$. If $y \neq 0$, then y has degree one and, from (4), it follows that

$$ey = e(\sigma(x_i) - x_i) \in R \quad \text{and} \quad \frac{1}{e}y = -\frac{1}{e}\big(\sigma^{-1}(\sigma(x_i)) - \sigma(x_i)\big) \in R.$$

By Lemma 3, there exists $0 \neq \alpha \in k$ such that $y = \alpha e$ or $y = \frac{\alpha}{e}$. Thus, at least one of e or $\frac{1}{e}$ belongs to R. However, since σ is not the identity on e, we have $e \notin k$. Therefore, at most one of e or $\frac{1}{e}$ belongs to R.

It follows from the argument above that exactly one of e or $\frac{1}{e}$ belongs to R and we will let e' denote the one that does. As a result, $y = \alpha e'$ and e' has degree one.

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Thus, every nonzero element of the form $\sigma(x_i) - x_i$ is a scalar multiple of e'. If we let $F = \sigma - 1$, then F is a linear map from the vector space $kx_1 + \cdots + kx_m$ to the vector space ke'. Furthermore, since $\sigma \neq 1$, there is some i such that $\sigma(x_i) \neq x_i$. Hence, F is not the zero map; thus the image of F has dimension one and the kernel of F has dimension $m - 1$.

We can let $y_1 = e'$ and then choose a basis $\{y_2, \ldots, y_m\}$ for the kernel of F. Since $\{y_1, y_2, \ldots, y_m\}$ consists of m linearly independent degree one polynomials, R is equal to the polynomial ring $k[y_1, \ldots, y_m]$. In addition, since $F = \sigma - 1$, we immediately see that

$$\sigma(y_i) = y_i \quad \text{and} \quad \delta_E(y_i) = \delta_F(y_i) = 0,$$

for $2 \le i \le m$. At this point, all that remains is to examine the action of $\sigma, \delta_E, \delta_F$ on y_1.

Since $y_1 = e'$, we now have two cases to consider: either $y_1 = e$ or $y_1 = \frac{1}{e}$. If $y_1 = e$, then since $\sigma(e) = q^2 e$, we have

$$\sigma(y_1) = q^2 y_1, \quad \delta_E(y_1) = e(\sigma(y_1) - y_1) = y_1(q^2 y_1 - y_1) = (q^2 - 1){y_1}^2,$$
$$\delta_F(y_1) = \frac{1}{e}(\sigma^{-1}(y_1) - y_1) = \frac{1}{y_1}(q^{-2} y_1 - y_1) = (q^{-2} - 1).$$

However, we are finding δ_E and δ_F up to scalar multiplication. Therefore, without loss of generality, we may assume

$$\sigma(y_1) = q^2 y_1, \quad \delta_E(y_1) = {y_1}^2, \quad \delta_F(y_1) = 1.$$

It now easily follows, by mathematical induction, that if $n \ge 1$, we have

$$\sigma({y_1}^n) = q^{2n} {y_1}^n \quad \delta_E({y_1}^n) = \frac{q^{2n} - 1}{q^2 - 1} {y_1}^{n+1}, \quad \delta_F({y_1}^n) = \frac{q^{-2n} - 1}{q^{-2} - 1} {y_1}^{n-1}.$$

The remaining possibility is that $y_1 = \frac{1}{e}$. Since $\sigma(e) = q^2 e$, we have $\sigma(\frac{1}{e}) = q^{-2} \frac{1}{e}$, therefore

$$\sigma(y_1) = q^{-2} y_1, \quad \delta_E(y_1) = e(\sigma(y_1) - y_1) = \frac{1}{y_1}(q^{-2} y_1 - y_1) = q^{-2} - 1,$$
$$\delta_F(y_1) = \frac{1}{e}(\sigma^{-1}(y_1) - y_1) = y_1(q^2 y_1 - y_1) = (q^2 - 1){y_1}^2.$$

Since we are finding δ_E and δ_F up to scalar multiplication, without loss of generality, we may assume that

$$\sigma(y_1) = q^{-2} y_1, \quad \delta_E(y_1) = 1, \quad \delta_F(y_1) = {y_1}^2.$$

Mathematical induction can now be used to show that, for $n \ge 1$,

$$\sigma({y_1}^n) = q^{-2n} {y_1}^n, \quad \delta_E({y_1}^n) = \frac{q^{-2n} - 1}{q^{-2} - 1} {y_1}^{n-1}, \quad \delta_F({y_1}^n) = \frac{q^{2n} - 1}{q^2 - 1} {y_1}^{n+1}. \qquad \blacksquare$$

We conclude our paper with any easy application of Theorem 4.

Corollary 5 *Consider an affine action of $H = U_q(sl(2))$ on the commutative polynomial ring $R = k[x_1, \ldots, x_m]$ with $\sigma^2 \neq 1$.*

(i) *If q is not a root of 1, then the subring of invariants R^H is a commutative polynomial ring in $m - 1$ variables.*

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(ii) *If q is a root of* 1 *and t is the smallest positive integer such that $q^{2t} = 1$, then the subring of invariants R^H is a commutative polynomial ring in m variables and R is a free R^H-module of rank t.*

Proof Since $\sigma, \delta_E, \delta_F$ all have the same invariants, R^H is equal to the invariants of σ. By Theorem 4, R is the polynomial ring $k[y_1, \dots, y_m]$, $\sigma(y_1) = \alpha y_1$, where $\alpha = q^2$ or $\alpha = q^{-2}$, and $\sigma(y_i) = y_i$, for $2 \le i \le m$. If $r \in R$, we can express r uniquely as $r = \sum_{i=0}^{n} p_i {y_1}^i$, where $n \ge 0$ and each $p_i \in k[y_2, \dots, y_m]$. Applying σ, we have

$$\sigma(r) = \sigma\Big(\sum_{i=0}^{n} p_i {y_1}^i\Big) = \sum_{i=0}^{n} \sigma(p_i)\sigma(y_1)^i = \sum_{i=0}^{n} p_i \alpha^i {y_1}^i. \tag{5}$$

In light of (5), $\sigma(r) = r$ if and only if $\alpha^i p_i = p_i$, for $0 \le i \le n$. If we are in the case where q is not a root of 1, then α is not a root of 1 and we see that $\sigma(r) = r$ if and only if $p_i = 0$, for $i \ge 1$. Thus, $r \in R^H$ if and only if $r = p_0 \in k[y_2, \dots, y_m]$. Therefore, in this case, $R^H = k[y_2, \dots, y_m]$.

On the other hand, if q is a root of 1, let t is the smallest positive integer such that $q^{2t} = 1$. Therefore t is the smallest positive integer such that $\alpha^t = 1$ and it follows from (5) that $\sigma(r) = r$ if and only if $p_i = 0$ whenever i is not a multiple of t. Therefore R^H is the polynomial ring $k[{y_1}^t, y_2, \dots, y_m]$ and, as a R^H-module, we have

$$R = R^H \oplus R^H y_1 \oplus \cdots \oplus R^H {y_1}^{t-1}.$$

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