Canad. Math. Bull.Vol. 34 (3), 1991 pp. 351–356

TWO CONTINUA HAVING A PROPERTY OF J. L. KELLEY

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ABSTRACT. In proving the contractibility of certain hyperspaces J. L. Kelley identified and defined a certain uniformness property which he called Property 3.2. It is known that the classes of locally connected continua, homogeneous continua and hereditarily indecomposable continua have Property 3.2. In this paper we prove that two examples of indecomposable continua developed respectively by the authors have Property 3.2. One is the example of a nonchainable atriodic tree-like continuum with positive span which was defined by the first author, and the other is a nonchainable, noncircle-like continuum which has the cone=hyperspace property which was defined by the second author. Each of the examples is an inverse limit of an inverse system having a single bonding map.

1. **Introduction.** In [1], the first author constructed a nonchainable, atriodic tree-like continuum. By an *unfolding technique* applied to that continuum, the second author constructed a nonchainable, noncircle-like continuum having the cone=hyperspace property [5]. Throughout this paper we will refer to these continua as Y and X, respectively. Each of these continua is described by an inverse limit of an inverse system having a single bonding map.

In a private conversation with the first author, Sam B. Nadler, Jr. raised the question of whether the continuum Y has Property 3.2, an uniformness property defined by J. L. Kelley in [2]. In his book *Hyperspaces of Sets,* Nadler refers to this property as property κ [3, 16.10]. The purpose of this paper is to show that each of these continua has Property 3.2, answering in the affirmative the question of Nadler. We first prove that X has Property 3.2, and then we prove that Y is a confluent image of X. R. W. Wardle [6, Theorem 4.3] proved that confluent mappings preserve Property 3.2.

By a *continuum* we mean a compact, connected metric space, and by a *mapping* we mean a continuous function. If X is a continuum, the hyperspace of subcontinua $C(X)$ consists of all nonempty subcontinua of X with the Hausdorff metric $H(A,B) = \inf\{\epsilon > 0 : A \subseteq N_\epsilon(B) \text{ and } B \subseteq N_\epsilon(A)\}$, where $N_\epsilon(A)$ denotes the union of all ϵ-balls about points in A. A continuum X has *Property 3.2* provided that if $\epsilon > 0$, there exists $\delta > 0$ such that if $a, b \in X, d(a,b) < \delta$ and $a \in A \in C(X)$ then there exists $B, b \in B \in C(X)$, such that $H(A,B) < \epsilon$. In [6], Wardle defined Property 3.2 at a point, say a, by letting δ depend on a and ϵ. Thus X has Property 3.2 if and only if it has Property 3.2 at each point.

A mapping f of a continuum X onto a continuum Y is said to be *confluent* provided that for each subcontinuum K of Y, each component of $f^{-1}(K)$ is thrown by f onto K.

Received by the editors September 7, 1989.
AMS subject classification: 54B20, 54F15.
Key words and phrases: indecomposable continuum, hyperspaces, property κ.

The continuum T is a *triod* provided that there is a subcontinuum K such that $T-K$ has at least three components. A continuum which contains no triods is called *atriodic*.

If $X_1, X_2, X_3, \ldots$ is a sequence of spaces and $f_1, f_2, f_3, \ldots$ is a sequence of mappings such that, for each positive integer $i, f_i\colon X_{i+1} \longrightarrow X_i$, then by the *inverse limit* of the inverse system $\{X_i, f_i\}$ is meant the subset of the product of the spaces $X_1, X_2, X_3, \ldots$ to which the point $(x_1, x_2, x_3, \ldots)$ belongs if and only if $f_i(x_{i+1}) = x_i$ for each positive integer i. Suppose f is a mapping of a space X into X and, for each positive integer $i, X_i = X$ and $f_i = f$. We denote by $\varprojlim\{X, f\}$ the inverse limit of the inverse system $\{X_i, f_i\}$. The projection of the inverse limit to the i^{th} factor space will be denoted by π_i.

For completeness and notational convenience we include abbreviated descriptions of the continua defined in [1] and [5].

2. **The continua X and Y and the mapping Φ.** Let T denote the simple triod $\{(r,\theta) : 0 \leq r \leq 1 \text{ and } \theta = 0, \theta = \pi/2 \text{ or } \theta = \pi\}$ in polar coordinates. We denote by J the junction point $(0,0) = (0,\pi/2) = (0,\pi)$, by A the point $(1,\pi/2)$, by B the point $(1,\pi)$ and by C the point $(1,0)$. If $0 \leq r \leq 1$, we let rA be the point $(r,\pi/2)$, while rB denotes (r,π) and rC denotes $(r,0)$.

Let d_1 be the metric on T which measures the length of the shortest arc in T between each pair of points. Let $f\colon T \longrightarrow T$ be the unique piecewise linear mapping such that $f(J) = B, f(C) = C, f(\frac{1}{3}B) = J, f(\frac{1}{2}B) = \frac{1}{2}A, f(\frac{2}{3}B) = J, f(B) = C, f(\frac{1}{4}A) = J, f(\frac{1}{2}A) = A$, $f(\frac{3}{4}A) = J$ and $f(A) = C$, and such that f is linear on $[J,C]$, $[J,\frac{1}{3}B]$, $[\frac{1}{3}B,\frac{1}{2}B]$, $[\frac{1}{2}B,\frac{2}{3}B]$, $[\frac{2}{3}B,B]$, $[J,\frac{1}{4}A]$, $[\frac{1}{4}A,\frac{1}{2}A]$, $[\frac{1}{2}A,\frac{3}{4}A]$ and $[\frac{3}{4}A,A]$. Figure 1 is a schematic representation of the *graph* of f.

For each positive integer n, put $T_n = T$ and $f_n = f$. Let $Y = \varprojlim\{T, f\}$, and let ρ_1 denote the metric on Y determined by d_1. In [1], the first author proved that Y is an atriodic, tree-like continuum having positive span.

Let (T, d_1) and $f\colon T \longrightarrow T$ be as defined above. Let S denote the decomposition space obtained by identifying the disjoint union $T \times \{1,2\}$ at the pairs of points having first projections A, B, C, J, and $\frac{1}{2}A$. We consider $S = \bigcup_{i=1}^{8} Z_i$, where the arcs Z_i are as labelled in Figure 2 below. We denote by A, B, C, J and $\frac{1}{2}A$ the identified points in $T \times \{1,2\}$.

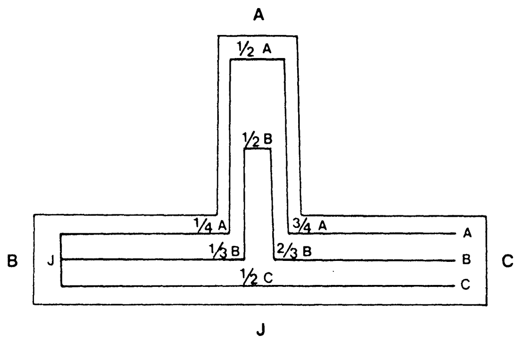

FIGURE 1

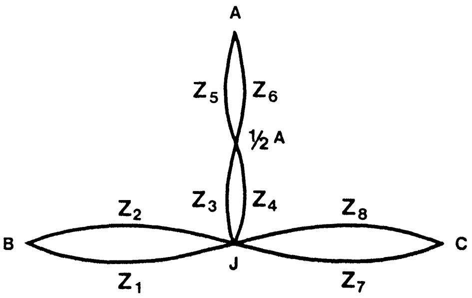

FIGURE 2

Let d be the metric on S which measures the length of the shortest arc in S, with distances in Z_i, $1 \leq i \leq 8$, coinciding with d_1 in T. Define $g: S \longrightarrow S$ as the piecewise linear map induced by the map $f: T \longrightarrow T$ by *unfolding* f at each of A, B, C, J, and $\frac{1}{2}A$. Figure 3 provides the schematics of the *graphs* of $g|Z_i$ for each integer $1 \leq i \leq 8$. By superimposing the eight *graphs* in Figure 3, we obtain the *graph* of g given in Figure 4.

For each positive integer n, put $S_n = S$ and $g_n = g$. Let $X = \varprojlim\{S, g\}$ and let ρ denote the metric on X determined by d. The diameter of S is 2, and thus the diameter of X does not exceed 2. The continuum X is a nonchainable, noncircle-like continuum having the cone=hyperspace property [5].

In [1] or [5] it is shown that every nondegenerate, proper subcontinuum of Y or of X, respectively, is an arc. Furthermore, if α is a proper subcontinuum of Y or X, there exists a positive integer N such that $m \geq N$ implies $\pi_m(\alpha)$ is a subset of $T - \{A, B, J, \frac{1}{2}A\}$ or $S - \{A, B, J, \frac{1}{2}A\}$, respectively, and $f|\pi_m(\alpha)$ or $g|\pi_m(\alpha)$, respectively, is linear.

Let $k: S \longrightarrow T$ be the mapping which collapses S onto T in the natural manner. From the definition of g, we have that $kg = fk$, and thus, inductively, $kg^n = f^n k$ for each positive integer n. For each $(x_1, x_2, x_3, \ldots)$ in X, let $\Phi((x_1, x_2, x_3, \ldots)) = \big(k(x_1), k(x_2), k(x_3), \ldots\big)$. Then $f\big(k(x_{n+1})\big) = k\big(g(x_{n+1})\big) = k(x_n)$, and thus $\Phi((x_1, x_2, x_3, \ldots)) \in Y$. It follows immediately that Φ is a mapping of X onto Y.

3. **The continuum X has Property 3.2.** Let M be a continuum and x be a point of M. That M has Property 3.2 at x is equivalent to the following: for each $\alpha \in C(M), x \in \alpha$, and each sequence $y_1, y_2, y_3, \ldots$ of points in M converging to x, there exists a sequence $\beta_1, \beta_2, \beta_3, \ldots$ with $y_n \in \beta_n \in C(M)$ for each integer n, such that $\beta_1, \beta_2, \beta_3 \ldots$ converges to α.

THEOREM 3.1. *The continuum X has Property 3.2.*

PROOF. Let $x = (x_1, x_2, x_3, \ldots)$ be a point of X, and $y^1, y^2, y^3, \ldots$ be a sequence of points of X converging to x, where $y^i = (y^i_1, y^i_2, y^i_3, \ldots)$ for each positive integer i. Let $\alpha \in C(X)$ with $x \in \alpha$. In [4], J. Segal proved that the hyperspace operation commutes with inverse limits. Considering the nature of the homeomorphism from $\varprojlim\{C(S), \hat{g}\}$

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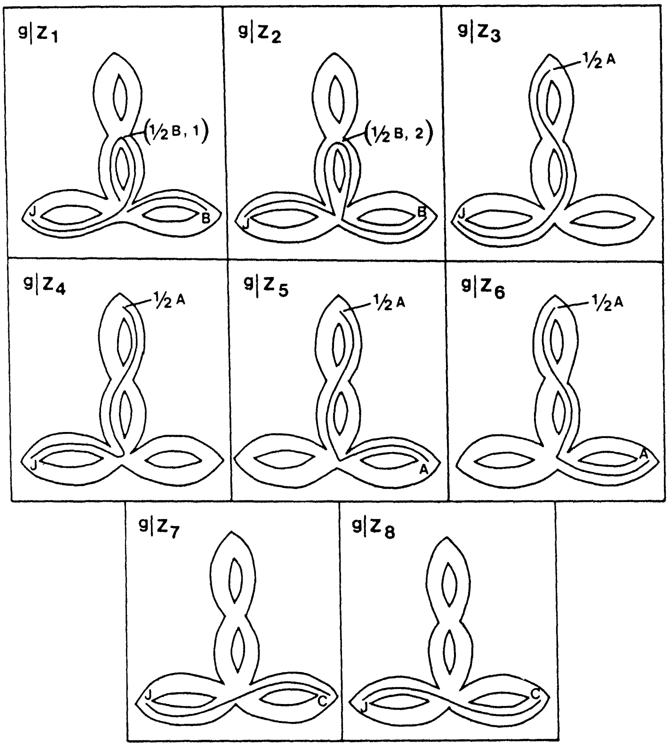

FIGURE 3

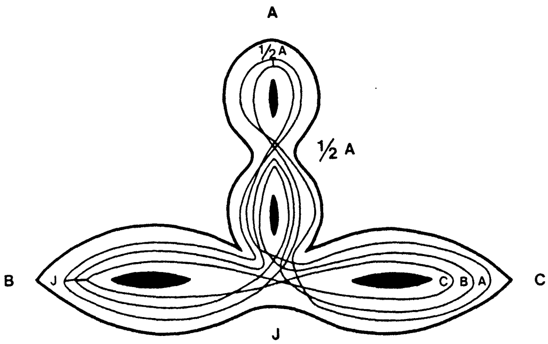

FIGURE 4

onto $C(X)$ as given in [3, 1.169], where $\hat{g}\colon C(S) \longrightarrow C(S)$ is defined by $\hat{g}(M) = \{g(m) : m \in M\}$ for each $M \in C(S)$, we consider $\varprojlim\{C(S), \hat{g}\}$ as identical with $C(X)$. Thus for $\alpha \in C(X)$, we denote $\pi_n[\alpha]$ by α_n and write $\alpha = (\alpha_1, \alpha_2, \alpha_3, \ldots)$.

Let N_0 be a positive integer such that $m \geq N_0$ implies $\alpha_m \subseteq S - \{A, B, J, \frac{1}{2}A\}$ and

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$g|\alpha_m$ is linear. For each $\epsilon > 0$, since the projection mappings $\pi_n: X \longrightarrow S_n$ are $\frac{1}{2^{n-1}}$-mappings, there exists a positive integer N and a positive sequence $\delta_N, \delta_{N+1}, \delta_{N+2}, \ldots$ such that $m \geq N$ and $p = (p_1, p_2, p_3, \ldots)$ and $q = (q_1, q_2, q_3, \ldots)$ in X with $d(p_m, q_m) < \delta_m$ implies $\rho(p, q) < \epsilon$. Thus there exists an increasing sequence of positive integers $N_1, N_2, N_3, \ldots$, with $N_1 \geq N_0$, and a decreasing positive sequence $\delta_{N_1}, \delta_{N_1+1}, \delta_{N_1+2}, \ldots$ such that

1) if i is a positive integer, $N_i \leq m < N_{i+1}$ and $p = (p_1, p_2, p_3, \ldots)$ and $q = (q_1, q_2, q_3, \ldots)$ belong to X with $d(p_m, q_m) < \delta_m$, then $\rho(p, q) < \frac{1}{i}$,
2) if $u, v \in S$ with $d(u, v) < \delta_{m+1}$, then $d\big(g(u), g(v)\big) < \delta_m$, and
3) $N_{\delta_m}(\alpha_m)$ is a subset of $S - \{A, B, J, \frac{1}{2}A\}$.

There exists an increasing sequence $K_1, K_2, K_3, \ldots$ such that if i is a positive integer and $m \geq K_i$, then $y^m_{N_i} \in N_{\delta_{N_i}}(x_{N_i})$. For $m < K_1$, let $\beta^m = X$. For each positive integer i and each m such that $K_i \leq m < K_{i+1}$, let β^m be the arc in $C(X)$ containing y^m and such that $\beta^m_{N_i}$ is the arc in S_{N_i} irreducible about $\{y^m_{N_i}\} \cup \alpha_{N_i}$. That β^m is uniquely determined follows because $\beta^m_{N_i} \subseteq N_{\delta_{N_i}}(\alpha_{N_i}) \subseteq S - \{A, B, J, \frac{1}{2}A\}, y^m \in \beta^m, g$ is linear on each of the components of $g^{-n}(N_{\delta_{N_i}}(\alpha_{N_i}))$ for each positive integer n, and g throws each component of $g^{-n}\big(N_{\delta_{N_i}}(\alpha_{N_i})\big)$ onto a component of $g^{-n+1}\big(N_{\delta_{N_i}}(\alpha_{N_i})\big)$ for each positive integer n.

To see that the sequence $\beta^1, \beta^2, \beta^3, \ldots$ in $C(X)$, with $y^i \in \beta^i$ for each positive integer i, converges to α, let t be a positive integer. Then N_t is a positive integer and $\delta_{N_t} > 0$ such that if $p = (p_1, p_2, p_3, \ldots)$ and $q = (q_1, q_2, q_3, \ldots)$ belong to X with $d(p_{N_t}, q_{N_t}) < \delta_{N_t}$, then $\rho(p, q) < \frac{1}{t}$. Furthermore, K_t is a positive integer such that if $m \geq K_t$, then $d(y^m_{N_t}, x_{N_t}) < \delta_{N_t}$. But $m \geq K_t$ implies there exists a positive integer $r, r \geq t$, such that $K_r \leq m < K_{r+1}$, and thus $N_{\delta_{N_r}}(\alpha_{N_r}) \supseteq \beta^m_{N_r}$ and $\alpha_{N_r} \subseteq \beta^m_{N_r}$. Therefore $N_{\delta_{N_t}}(\alpha_{N_t}) \supseteq \beta^m_{N_t}$ and $\alpha_{N_t} \subseteq \beta^m_{N_t}$, and thus $H(\alpha, \beta^m) < \frac{1}{t}$. This establishes that X has Property 3.2.

4. **The continuum Y has Property 3.2.** As previously mentioned, the confluent image of a continuum having Property 3.2 also has Property 3.2 [6]. Thus we prove

THEOREM 4.1. *The mapping $\Phi: X \longrightarrow Y$ is confluent.*

PROOF. The mapping $k: S \longrightarrow T$ collapses S onto T in such a way that $k^{-1}(t) = t$ for $t \in \{A, B, C, J, \frac{1}{2}A\}$, and $k^{-1}(t) = \{(t, 1), (t, 2)\}$ otherwise. Let $y = (y_1, y_2, y_3, \ldots)$ be a point of Y and consider $\Phi^{-1}(y)$. If $y = (C, C, C, \ldots) \in Y$, then $\Phi^{-1}(y) = (C, C, C, \ldots) \in X$. Suppose that $y_i \neq C$ for some positive integer i. Then there exists an integer N such that $m \geq N$ implies $y_m \notin \{A, B, C, J, \frac{1}{2}A\}$. Thus $k^{-1}(y_m) = \{(y_m, 1), (y_m, 2)\}$. The mapping $g: S \longrightarrow S$ is defined so that if $w \in T$ and $f(w) \notin \{A, B, C, J, \frac{1}{2}A\}$, then $g((w, 1)) \neq g((w, 2))$ but $kg((w, 1)) = kg((w, 2)) = f(w)$. So for $m > N, g((y_m, 1)) \neq g((y_m, 2))$. It follows immediately that $\Phi^{-1}(y)$ is the two element set $\{u, v\}$ such that for $m \geq N, \pi_m(u) = (y_m, i_m)$ and $\pi_m(v) = (y_m, j_m)$ where $i_m \neq j_m$ and $i_m, j_m \in \{1, 2\}$. Thus $\Phi: X \longrightarrow Y$ is a 2-to-1 mapping except at the point $(C, C, C, \ldots)$.

Let $\alpha \in C(Y), \alpha = (\alpha_1, \alpha_2, \ldots)$ in $\varprojlim\{C(T), \hat{f}\}$. If $\alpha \neq Y$, there exists a positive integer N_1 such that $m > N_1$ implies that $\alpha_m \subseteq T - \{A, B, J, \frac{1}{2}A\}$. We consider two cases.

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CASE 1. If $(C, C, C, \ldots) \in \alpha$, then $\alpha_m = [w_m, C] \subseteq (J, C]$ and $k^{-1}(\alpha_m)$ is the arc in S_m containing C and having endpoints $(w_m, 1)$ and $(w_m, 2)$. Thus $\Phi^{-1}(\alpha)$ is an arc in X.

CASE 2. There exists $N_2 \geq N_1$ such that $m \geq N_2$ implies that $\alpha_m \subseteq T - \{A, B, C, J, \frac{1}{2}A\}$. Then $m > N_2$ implies $k^{-1}(\alpha_m) = \{\alpha_m \times \{1\}, \alpha_m \times \{2\}\}$, and $g(\alpha_m \times \{1\}) = \alpha_{m-1} \times \{i\}$ and $g(\alpha_m \times \{2\}) = \alpha_{m-1} \times \{j\}$, where $i, j \in \{1, 2\}$ and $i \neq j$. Thus $\Phi^{-1}(\alpha)$ consists of two arcs in X each of which is thrown by Φ onto α. Hence Φ is confluent, and so Y has Property 3.2.

REFERENCES

1. W. T. Ingram, *An atriodic tree-like continuum with positive span*, Fund. Math. **77**(1972), 99–107.
2. J. L. Kelley, *Hyperspaces of a continuum*, Trans. Amer. Math. Soc. **52**(1942), 22–36.
3. Sam B. Nadler, Jr., *Hyperspaces of Sets*. Marcel Dekker, Inc., New York, 1978.
4. J. Segal, *Hyperspaces of the inverse limit space*, Proc. Amer. Math. Soc. **10**(1959), 706–709.
5. D. D. Sherling, *Concerning the cone=hyperspace property*, Can. J. Math. **35**(1983), 1030–1048.
6. R. W. Wardle, *On a property of J. L. Kelley*, Houston J. Math. **3**(1977), 291–299.

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