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NUCLEAR AND INTEGRAL POLYNOMIALS

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Abstract

Let E be a Banach space whose dual E^* has the approximation property, and let m be an index. We show that E^* has the Radon-Nikodým property if and only if every m-homogeneous integral polynomial from E into any Banach space is nuclear. We also obtain factorization and composition results for nuclear polynomials.

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Many authors have studied nuclear and integral polynomials between Banach spaces (see, for example, [2–5, 7]). In the present paper, we continue this study obtaining a characterization of the Radon-Nikodým property in terms of these classes of polynomials, as well as factorization and composition results for these and related classes.

First, we extend to the polynomial setting the following well-known result due to Grothendieck:

THEOREM 1 ([11, Theorem VIII.4.6]). *Let E be a Banach space such that E^* has the approximation property. Then E^* has the Radon-Nikodým property if and only if every integral operator on E is nuclear. In this case, the integral and the nuclear norms coincide.*

We also give results about the composition of integral polynomials with weakly compact operators and of weakly compact polynomials with integral operators. We

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characterize the polynomials that factorize through a nuclear operator into a Hilbert space.

We show that, as in the linear case, the nuclear polynomials factorize through diagonal polynomials from ℓ_∞ into ℓ_1 and also from c_0 into ℓ_1. Using this result, we show that a polynomial P is nuclear if and only if it may be written in the form $P = Q \circ T$ where T is a compact operator and Q is a Pietsch integral polynomial.

Finally, we show that not every nuclear polynomial is 1-dominated and obtain a sufficient condition for this to happen.

Throughout, E, F, G, X, Y and Z denote Banach spaces, E^* is the dual of E, and B_E stands for its closed unit ball. By $\mathbb{N}$ we represent the set of all natural numbers and by $\mathbb{K}$ the scalar field (real $\mathbb{R}$ or complex $\mathbb{C}$). The notation $E \equiv F$ means that E and F are isometrically isomorphic. The definition of the Radon-Nikodým property may be found in [11, Definition III.1.3]. Recall that E^* has the Radon-Nikodým property if and only if E is an Asplund space. By an *operator* from E into F we always mean a bounded linear mapping. We use $\mathscr{L}(E, F)$ for the space of all operators from E into F.

Given $m \in \mathbb{N}$, we denote by $\mathscr{P}(^mE, F)$ the space of all m-homogeneous (continuous) polynomials from E into F endowed with the supremum norm given by

$$\|P\| = \sup\{\|P(x)\| : x \in B_E\} \quad \text{for all } P \in \mathscr{P}(^mE, F).$$

Recall that with each $P \in \mathscr{P}(^mE, F)$ we can associate a unique symmetric m-linear (continuous) mapping $\widehat{P} : E \times \overset{(m)}{\cdots} \times E \to F$ so that

$$P(x) = \widehat{P}\left(x, \overset{(m)}{\ldots}, x\right) \quad (x \in E).$$

For the general theory of polynomials on Banach spaces, we refer to [12] and [16].

Given $1 \le r < \infty$, a polynomial $P \in \mathscr{P}(^mE, F)$ is *r-dominated* [15] if there exists a constant $k > 0$ such that, for all $n \in \mathbb{N}$ and $(x_i)_{i=1}^n \subset E$, we have

$$\left(\sum_{i=1}^n \|P(x_i)\|^{r/m}\right)^{m/r} \le k \sup_{x^* \in B_{E^*}} \left(\sum_{i=1}^n |x^*(x_i)|^r\right)^{m/r}.$$

Note that, for $m = 1$, we obtain the class of *(absolutely) r-summing operators.* If $T \in \mathscr{L}(E, F)$ is r-summing, the least of the constants k that satisfy the above inequality for $m = 1$ is denoted by $\pi_r(T)$.

An m-linear mapping $T : E \times \overset{(m)}{\cdots} \times E \to F$ is *nuclear* [2] if there are bounded sequences $(x_{ji}^*)_{i=1}^\infty \subset E^*$ $(1 \le j \le m)$ and $(y_i)_{i=1}^\infty \subset F$ with

$$\sum_{i=1}^\infty \|x_{1i}^*\| \cdots \|x_{mi}^*\| \|y_i\| < \infty$$

such that

$$T(x_1, \dots, x_m) = \sum_{i=1}^{\infty} x_{1i}^*(x_1) \cdots x_{mi}^*(x_m) y_i \quad (x_j \in E,\ 1 \le j \le m).$$

The *nuclear norm* of T is

$$\|T\|_{\mathrm{N}} := \inf \sum_{i=1}^{\infty} \|x_{1i}^*\| \cdots \|x_{mi}^*\| \|y_i\|,$$

where the infimum is taken over all sequences satisfying the definition.

A polynomial $P \in \mathscr{P}(^mE, F)$ is *nuclear* [2] if it can be written in the form

$$P(x) = \sum_{i=1}^{\infty} x_i^*(x)^m y_i \quad (x \in E), \tag{1}$$

where $(x_i^*) \subset E^*$ and $(y_i) \subset F$ are bounded sequences such that

$$\sum_{i=1}^{\infty} \|x_i^*\|^m \|y_i\| < \infty. \tag{2}$$

We denote by $\mathscr{P}_{\mathrm{N}}(^mE, F)$ the space of all m-homogeneous nuclear polynomials from E into F endowed with the nuclear norm

$$\|P\|_{\mathrm{N}} := \inf \sum_{i=1}^{\infty} \|x_i^*\|^m \|y_i\|,$$

where the infimum is taken over all sequences $(x_i^*) \subset E^*$ and $(y_i) \subset F$ which satisfy (1) and (2). We denote by $\mathscr{N}(E, F)$ the space of all nuclear operators from E into F.

The following definition of integral m-linear mapping was given in [7] and extends the one given in [17] for multilinear functionals.

An m-linear mapping $T : E \times \overset{(m)}{\cdots} \times E \to F$ is (*Grothendieck*) *integral* if there exists a constant $C \ge 0$ such that, for every $n \in \mathbb{N}$ and all families $(x_{ji})_{i=1}^n \subset E$ $(1 \le j \le m)$ and $(f_i^*)_{i=1}^n \subset F^*$, we have

$$\left| \sum_{i=1}^{n} \langle T(x_{1i}, \dots, x_{mi}), f_i^* \rangle \right| \le C \sup_{\substack{x_j^* \in B_{E^*} \\ 1 \le j \le m}} \left\| \sum_{i=1}^{n} x_1^*(x_{1i}) \cdots x_m^*(x_{mi}) f_i^* \right\|_{F^*}.$$

For $m = 1$, we obtain the *integral operators* [11, Definition VIII.2.6]. The integral norm $\|T\|_{\mathrm{I}}$ is the infimum of all constants C that satisfy the definition.

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In [22], T is said to be integral if there exists a regular F^{**}-valued countably additive, Borel measure $\mathscr{G}$, of bounded variation, on the product $B_{E^*} \times \overset{(m)}{\cdots} \times B_{E^*}$, endowed with the weak-star topology, such that

$$(3) \qquad T(x_1, \ldots, x_m) = \int_{B_{E^*} \times \overset{(m)}{\cdots} \times B_{E^*}} x_1^*(x_1) \cdots x_m^*(x_m)\, d\mathscr{G}(x_1^*, \ldots, x_m^*)$$

for all $x_j \in E$ $(1 \leq j \leq m)$. The integral norm of T is the infimum of the variation of $\mathscr{G}$, taken over all measures $\mathscr{G}$ as above.

From [7] and [22] it is easy to see that both notions of integral m-linear mapping are equivalent and that the two definitions of integral norm coincide.

We say that a polynomial $P \in \mathscr{P}(^mE, F)$ is *(Grothendieck) integral* if there exists a constant $C \geq 0$ such that, for every $n \in \mathbb{N}$ and all families $(x_i)_{i=1}^n \subset E$ and $(f_i^*)_{i=1}^n \subset F^*$, we have

$$\left| \sum_{i=1}^n \langle P(x_i), f_i^* \rangle \right| \leq C \sup_{x^* \in B_{E^*}} \left\| \sum_{i=1}^n [x^*(x_i)]^m f_i^* \right\|_{F^*} .$$

The symbol $\mathscr{P}_\mathrm{I}(^mE, F)$ denotes the space of all m-homogeneous integral polynomials from E into F, endowed with the integral norm $\|P\|_\mathrm{I} := \inf C$, where the infimum is taken over all constants C that satisfy the definition. By $\mathscr{I}(E, F)$ we denote the space of all integral operators from E into F.

An m-linear mapping $T : E \times \overset{(m)}{\cdots} \times E \to F$ is *Pietsch integral* [2] if it can be written in the form (3), where $\mathscr{G}$ is F-valued. The Pietsch integral norm $\|T\|_\mathrm{PI}$ of T is the infimum of the variation of the measures $\mathscr{G}$.

A polynomial $P \in \mathscr{P}(^mE, F)$ is *Pietsch integral* [2] if it can be written in the form

$$P(x) = \int_{B_{E^*}} [x^*(x)]^m \, d\mathscr{G}(x^*) \quad (x \in E)$$

where $\mathscr{G}$ is an F-valued regular countable additive Borel measure, of bounded variation, defined on $(B_{E^*}, \text{weak-}*)$. The Pietsch integral norm of P is $\|P\|_\mathrm{PI} := \inf |\mathscr{G}|(B_{E^*})$, where $|\mathscr{G}|$ is the variation of $\mathscr{G}$, and the infimum is taken over all measures satisfying the definition.

In the literature, the concept 'integral polynomial' has been used sometimes for what we call Pietsch integral polynomials and sometimes (as we do) for the (Grothendieck) integral polynomials.

Every nuclear polynomial is Pietsch integral, and every Pietsch integral polynomial is integral. Moreover, if P is nuclear, we have $\|P\|_\mathrm{I} \leq \|P\|_\mathrm{PI} \leq \|P\|_\mathrm{N}$.

We use the notation $\otimes^m E := E \otimes \overset{(m)}{\cdots} \otimes E$ for the m-fold tensor product of E, $\otimes_\epsilon^m E := E \otimes_\epsilon \overset{(m)}{\cdots} \otimes_\epsilon E$ for the m-fold injective tensor product of E, and $\otimes_\pi^m E$ for the m-fold projective tensor product of E (see [9] for the theory of tensor products). By

$\otimes_s^m E := E \otimes_s \overset{(m)}{\cdots} \otimes_s E$ we denote the m-fold symmetric tensor product of E, that is, the set of all elements $u \in \otimes^m E$ of the form

$$u = \sum_{j=1}^{n} \lambda_j x_j \otimes \overset{(m)}{\cdots} \otimes x_j \quad (n \in \mathbb{N}, \lambda_j \in \mathbb{K}, x_j \in E, 1 \leq j \leq n).$$

By $\otimes_{\epsilon,s}^m E$ we denote the closure of $\otimes_s^m E$ in $\otimes_\epsilon^m E$. Analogously, $\otimes_{\pi,s}^m E$ is the closure of $\otimes_s^m E$ in $\otimes_\pi^m E$. For symmetric tensor products, we refer to [13].

If $P \in \mathcal{P}({}^m E, F)$, we define its linearization $\overline{P} : \otimes_s^m E \to F$ by

$$\overline{P}\left(\sum_{i=1}^{n} \lambda_i x_i \otimes \overset{(m)}{\cdots} \otimes x_i\right) = \sum_{i=1}^{n} \lambda_i P(x_i)$$

for all $\lambda_i \in \mathbb{K}, x_i \in E$ $(1 \leq i \leq n)$.

The following lemma will be needed.

LEMMA 2 ([9, Theorem 16.6]). *Suppose that E^* has the Radon-Nikodým property and the approximation property. Then $(E \otimes_\epsilon F)^* \equiv E^* \otimes_\pi F^*$ for every Banach space F.*

We can now prove the following

THEOREM 3. *Suppose that E^* has the approximation property. Then the following assertions are equivalent*:

(a) *E^* has the Radon-Nikodým property.*

(b) *For every $m \in \mathbb{N}$ and every Banach space F, we have $\mathcal{P}_N({}^m E, F) = \mathcal{P}_I({}^m E, F)$.*

(c) *There is $m \in \mathbb{N}$ such that, for every Banach space F, we have $\mathcal{P}_N({}^m E, F) = \mathcal{P}_I({}^m E, F)$.*

Moreover, if these conditions are satisfied, we have

$$\|P\|_I \leq \|P\|_N \leq \frac{m^m}{m!} \|P\|_I$$

for every $P \in \mathcal{P}_I({}^m E, F)$.

PROOF. (a) $\Rightarrow$ (b). Let $P \in \mathcal{P}_I({}^m E, F)$. Then the associated m-linear mapping $\widehat{P}$ is integral [7], and its linearization $\widetilde{\widehat{P}} : \otimes_\epsilon^m E \to F$ is well defined and integral [22]. Since E^* has the Radon-Nikodým property, by [19, Theorem 1.9] and induction, the space $(\otimes_\epsilon^m E)^*$ has also the Radon-Nikodým property. By Lemma 2 and induction, we have $(\otimes_\epsilon^m E)^* \equiv \otimes_\pi^m E^*$.

Since E^* has the approximation property, $\otimes_\pi^m E^*$ has also the approximation property [9, Exercise 5.4]. By Theorem 1, $\widetilde{\widehat{P}} : \otimes_\epsilon^m E \to F$ is nuclear. Clearly, the

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restriction of $\overline{\overline{P}}$ to $\otimes^m_{\epsilon,s}E$ coincides with $\overline{P}$, which is also nuclear and hence Pietsch integral. By [22], P is Pietsch integral. Since E is Asplund, P is nuclear (see [2, Proposition 1] or [5, Theorem 1.4]).

(b) $\Rightarrow$ (c) is obvious.

(c) $\Rightarrow$ (a). It is proved in [6] that the equality $\mathscr{P}_{\mathrm{N}}({}^mE, F) = \mathscr{P}_{\mathrm{I}}({}^mE, F)$ for some m implies that $\mathscr{N}(E, F) = \mathscr{I}(E, F)$. Since this is true for all F, applying Theorem 1, we have that E^* has the Radon-Nikodým property.

Assume now that the three equivalent assertions hold. Let $P \in \mathscr{P}_{\mathrm{I}}({}^mE, F)$. We know that $\|P\|_{\mathrm{I}} \leq \|P\|_{\mathrm{N}}$. By Theorem 1, $\|\overline{\overline{P}}\|_{\mathrm{I}} = \|\overline{\overline{P}}\|_{\mathrm{PI}} = \|\overline{\overline{P}}\|_{\mathrm{N}}$. Hence,

$$
\begin{aligned}
\|P\|_{\mathrm{N}} &\leq \frac{m^m}{m!}\|\widehat{P}\|_{\mathrm{N}} && \text{(by [2])}\\
&= \frac{m^m}{m!}\|\widehat{P}\|_{\mathrm{PI}} && \text{(by [1, Theorem 2.3])}\\
&= \frac{m^m}{m!}\|\overline{\overline{P}}\|_{\mathrm{PI}} && \text{(by [22])}\\
&= \frac{m^m}{m!}\|\overline{\overline{P}}\|_{\mathrm{I}} && \\
&= \frac{m^m}{m!}\|\widehat{P}\|_{\mathrm{I}} && \text{(by [22])}\\
&\leq \frac{m^m}{m!}\|P\|_{\mathrm{I}},
\end{aligned}
$$

and the proof is finished. □

We now consider the extension to the polynomial setting of the following result [11, Theorem VIII.4.12]:

THEOREM 4. *Consider the operators $T \in \mathscr{L}(E, F)$ and $S \in \mathscr{L}(F, G)$. Then:*

(a) *If T is integral and S is weakly compact, then $S \circ T$ is nuclear.*

(b) *If T is weakly compact and S is integral then $S \circ T$ is nuclear into G^{**}.*

Most of the possible extensions to polynomials fail. However, we obtain:

PROPOSITION 5. *Let $P \in \mathscr{P}({}^mE, F)$, $S \in \mathscr{L}(F, Y)$, and $T \in \mathscr{L}(X, E)$. Then*

(a) *If P is integral and S is weakly compact, then $S \circ P$ is Pietsch integral and its linearization $\overline{S \circ P} : \otimes^m_{\epsilon,s}E \to Y$ is nuclear.*

(b) *If T is weakly compact and P is integral, then $P \circ T$ is nuclear into F^{**}, and $\overline{P \circ T} : \otimes^m_{\epsilon,s}X \to F^{**}$ is nuclear.*

(c) *If T is integral and P is weakly compact then $P \circ T$ is Pietsch integral.*

PROOF. (a) Since P is integral, its linearization $\overline{P} : \otimes^m_{\epsilon,s} E \to F$ is well-defined and integral [7]. By Theorem 4, $\overline{S \circ P} = S \circ \overline{P}$ is nuclear and, hence, Pietsch integral. By [22], $S \circ P$ is Pietsch integral.

In general, $S \circ P$ is not nuclear. For instance, the polynomial $P : C[0,1] \to \mathbb{C}$, given by $P(f) = \int_0^1 f(t)^2\,dt$, is integral. However, if $S : \mathbb{C} \to \mathbb{C}$ is the identity on $\mathbb{C}$, then $P = S \circ P$ is not nuclear [1, Remark 2.4].

(b) There are a reflexive space G, and operators $A \in \mathcal{L}(X, G)$ and $B \in \mathcal{L}(G, E)$ such that $T = B \circ A$ [11, Corollary VIII.4.9]. Consider the operator

$$\otimes^m B := B \otimes \overset{(m)}{\cdots} \otimes B : \otimes^m_{\epsilon,s} G \to \otimes^m_{\epsilon,s} E.$$

Then $\overline{P} \circ (\otimes^m B) = \overline{P \circ B}$ is integral, so $P \circ B$ is integral, hence it is Pietsch integral as a polynomial with values in F^{**}. Since G is Asplund, $P \circ B$ is nuclear from G into F^{**} [5, Theorem 1.4]. Easily, $P \circ T = P \circ B \circ A$ is nuclear with values in F^{**}.

The operator $\overline{P} \circ (\otimes^m B) = \overline{P \circ B} : \otimes^m_{\epsilon,s} G \to F^{**}$ is Pietsch integral. Since G^* has the Radon-Nikodým property, so does $\left(\otimes^m_{\epsilon,s} G\right)^*$ [19, Theorem 1.9]. Then $\overline{P \circ B}$ is nuclear into F^{**}. Therefore, $\overline{P \circ T} = \overline{P} \circ (\otimes^m T) = \overline{P} \circ (\otimes^m B) \circ (\otimes^m A)$ is nuclear into F^{**}.

(c) Since P is weakly compact, there are a reflexive space G, a polynomial $Q \in \mathcal{P}(^mE, G)$ and an operator $B \in \mathcal{L}(G, F)$ such that $P = B \circ Q$ [20, Theorem 3.7]. Since T is integral, $Q \circ T$ is an integral polynomial [7]. As in (a), $B \circ Q \circ T = P \circ T$ is Pietsch integral.

We do not know if $P \circ T$ is nuclear. □

Our next goal is to show that a polynomial P is nuclear if and only if it may be written in the form $P = Q \circ T$ where Q is a Pietsch integral polynomial and T is a compact operator. To this end, we first show that every nuclear polynomial factorizes through a diagonal polynomial from ℓ_∞ into ℓ_1, and from c_0 into ℓ_1. This extends the result in the linear case, and might be well known but we have only found a mention to a part of it in [21, page 114]. For completeness, we sketch the proof.

PROPOSITION 6. *Let $P \in \mathcal{P}(^mE, F)$. The following assertions are equivalent:*

(a) *P is nuclear.*

(b) *There are operators $u \in \mathcal{L}(E, \ell_\infty)$ and $v \in \mathcal{L}(\ell_1, F)$ and a polynomial $M_\lambda \in \mathcal{P}(^m\ell_\infty, \ell_1)$ of the form $M_\lambda(z) = (\lambda_n z_n^m)_{n=1}^\infty$, where $\lambda = (\lambda_n) \in \ell_1$ and $z = (z_n) \in \ell_\infty$, such that the following diagram commutes*

$$\begin{array}{ccc} E & \xrightarrow{P} & F \\ {\scriptstyle u}\downarrow & & \uparrow{\scriptstyle v} \\ \ell_\infty & \xrightarrow[M_\lambda]{} & \ell_1 \end{array}$$

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(c) *There are compact operators $u \in \mathcal{L}(E, c_0)$ and $v \in \mathcal{L}(\ell_1, F)$, and a polynomial $M'_\lambda \in \mathcal{P}(^m c_0, \ell_1)$ of the form $M'_\lambda(y) = (\lambda_n y_n^m)_{n=1}^\infty$, where $\lambda = (\lambda_n) \in \ell_1$ and $y = (y_n) \in c_0$, such that the following diagram commutes*

$$\begin{tikzcd} E \arrow[r, "P"] \arrow[d, "u"'] & F \\ c_0 \arrow[r, "M'_\lambda"'] & \ell_1 \arrow[u, "v"'] \end{tikzcd}$$

PROOF. (a) $\Rightarrow$ (b). If P is nuclear, there are bounded sequences $(x_n^*) \subset E^*$ and $(y_n) \subset F$ such that formulas (1) and (2) hold. Define u, M_λ and v by

$$\begin{aligned} u(x) &= \left(\frac{x_n^*(x)}{\|x_n^*\|}\right)_{n=1}^\infty && (x \in E) \\ M_\lambda(z) &= \left(\|x_n^*\|^m \|y_n\| z_n^m\right)_{n=1}^\infty && (z = (z_n) \in \ell_\infty) \\ v(e_n) &= \frac{y_n}{\|y_n\|}, \end{aligned}$$

where (e_n) is the unit vector basis of ℓ_1.

(b) $\Rightarrow$ (c). Given $\lambda = (\lambda_n) \in \ell_1$, we can find $\alpha = (\alpha_n) \in c_0$, with $\alpha_n > 0$, and $\tau = (\tau_n) \in \ell_1$ such that $\lambda_n = \alpha_n \tau_n$ for all n [18, 3, Exercise 12]. Define

(i) the operator $b \in \mathcal{L}(\ell_\infty, c_0)$ by $b(z) = (\alpha_n^{1/2m} z_n)_{n=1}^\infty$ for $z = (z_n) \in \ell_\infty$,
(ii) the operator $a \in \mathcal{L}(\ell_1, \ell_1)$ by $a(w) = (\alpha_n^{1/2} w_n)_{n=1}^\infty$ for $w = (w_n) \in \ell_1$, and
(iii) the polynomial $M \in \mathcal{P}(^m c_0, \ell_1)$ by $M(y) = (\tau_n y_n^m)_{n=1}^\infty$ for $y = (y_n) \in c_0$.

Easily, a and b are compact, and $M_\lambda = a \circ M \circ b$.

(c) $\Rightarrow$ (a). Since

$$M'_\lambda(y) = \left(\lambda_n y_n^m\right)_{n=1}^\infty = \sum_{n=1}^\infty \lambda_n y_n^m e_n = \sum_{n=1}^\infty \lambda_n [e_n(y)]^m e_n$$

for all $y = (y_n) \in c_0$, it follows that M'_λ is nuclear. It is easy to prove that $P = v \circ M'_\lambda \circ u$ is nuclear. □

THEOREM 7. *Given $P \in \mathcal{P}(^m E, F)$, we have that P is nuclear if and only if there are a Banach space G, a compact operator $T \in \mathcal{L}(E, G)$ and a Pietsch integral polynomial $Q \in \mathcal{P}(^m G, F)$ such that $P = Q \circ T$.*

PROOF. If P is nuclear, consider the factorization of Proposition 6,(c), and take $G = c_0$, $T = u$, and $Q = v \circ M'_\lambda$. Conversely, if $P = Q \circ T$ as in the statement, we can find a reflexive space Z and operators $A \in \mathcal{L}(E, Z)$ and $B \in \mathcal{L}(Z, G)$ such that $T = B \circ A$ [11, page 260]. Then $Q \circ B$ is Pietsch integral [8]. Since Z is Asplund, $Q \circ B$ is nuclear [5, Theorem 1.4]. Easily, $Q \circ T = Q \circ B \circ A$ is nuclear. □

We now characterize the polynomials that factorize through a nuclear operator into a Hilbert space. This extends [10, Theorem 5.31] to the polynomial setting.

PROPOSITION 8. *Let* $P \in \mathscr{P}(^m E, F)$. *Then the following assertions are equivalent*:

(a) *There are a Banach space* G, *a* 2-*summing operator* $T \in \mathscr{L}(E, G)$, *and a* 2-*dominated polynomial* $Q_1 \in \mathscr{P}(^m G, F)$ *such that* $P = Q_1 \circ T$.
(b) *There are a Hilbert space* H, *an operator* $S \in \mathscr{N}(E, H)$ *and a polynomial* $Q \in \mathscr{P}(^m H, F)$ *such that* $P = Q \circ S$.

PROOF. (a) $\Rightarrow$ (b). Since Q_1 is 2-dominated, there are a Banach space Z, a 2-summing operator $B \in \mathscr{L}(G, Z)$, and a polynomial $R \in \mathscr{P}(^m Z, F)$ such that $Q_1 = R \circ B$ [21]. Since $B \circ T$ is the composition of two 2-summing operators, there are a Hilbert space H, an operator $S \in \mathscr{N}(E, H)$, and an operator $U \in \mathscr{L}(H, Z)$ such that $B \circ T = U \circ S$ [10, Theorem 5.31]. Therefore, (b) follows with $Q = R \circ U$.

(b) $\Rightarrow$ (a). Since S is nuclear, there are operators $u \in \mathscr{L}(E, c_0)$, $M \in \mathscr{N}(c_0, \ell_1)$, and $v \in \mathscr{L}(\ell_1, H)$ such that $S = v \circ M \circ u$ (Proposition 6). Then, $M \circ u$ is nuclear and therefore 2-summing. The operator $v \in \mathscr{L}(\ell_1, H)$ is 2-summing [10, Theorem 3.4], so the polynomial $Q \circ v$ is 2-dominated [21]. We have proved (a) with $G = \ell_1$, $T = M \circ u$, and $Q_1 = Q \circ v$. □

COROLLARY 9. *If* $T \in \mathscr{L}(E, G)$ *is* 2-*summing and* $Q_1 \in \mathscr{P}(^m G, F)$ *is* 2-*dominated, then* $Q_1 \circ T$ *is nuclear.*

PROOF. By Proposition 8, there are a Hilbert space H, an operator $S \in \mathscr{N}(E, H)$ and a polynomial $Q \in \mathscr{P}(^m H, F)$ such that $Q_1 \circ T = Q \circ S$. By [14, 3.1.9], the composition of a nuclear operator with a polynomial is nuclear, so $Q_1 \circ T$ is nuclear. □

REMARK 10. Not every nuclear polynomial satisfies the assertions of Proposition 8. Indeed, if $P \in \mathscr{P}_{\mathrm{N}}(^m E, F)$ satisfies Proposition 8, then we may write $P = Q \circ S$ with S a nuclear (hence, 1-summing) operator. So, P is 1-dominated [21]. Theorem 11 gives many examples of nuclear polynomials which are not 1-dominated and hence they cannot factorize through a nuclear operator.

If $P \in \mathscr{P}(^m E, F)$ is 2-dominated and $T \in \mathscr{L}(F, G)$ is 2-summing, the composition $T \circ P$ is not necessarily nuclear. Indeed, let $i : \ell_1 \to \ell_2$ be the natural inclusion, and let $R : \ell_2 \to \mathbb{K}$ be the polynomial given by $R(x) = \sum_{n=1}^{\infty} x_n^2$. Since i is 1-summing, it is 2-summing, and so $P := R \circ i$ is 2-dominated [21]. If $T : \mathbb{K} \to \mathbb{K}$ is the identity on $\mathbb{K}$, which is obviously 2-summing, we have that $P = T \circ P$ is not nuclear [4, Proposition 2.3].

We now investigate conditions for a nuclear polynomial to be 1-dominated. We first obtain a characterization of the 1-dominated diagonal polynomials from ℓ_∞ into ℓ_1.

THEOREM 11. *Let $M_\lambda \in \mathcal{P}(^m\ell_\infty, \ell_1)$ be given by $M_\lambda(x) = (\lambda_n x_n^m)_{n=1}^\infty$ for all $x = (x_n) \in \ell_\infty$, where $\lambda = (\lambda_n) \in \ell_1$. Then M_λ is 1-dominated if and only if $\lambda \in \ell_{1/m}$.*

PROOF. Suppose that $\lambda \in \ell_{1/m}$. If the field is complex, let $T \in \mathcal{L}(\ell_\infty, \ell_1)$ be given by $T(x) = \left(|\lambda_n|^{1/m} e^{i\theta_n/m} x_n\right)_{n=1}^\infty$ for all $x = (x_n) \in \ell_\infty$, where $\lambda_n = |\lambda_n| e^{i\theta_n}$. Define $P \in \mathcal{P}(^m\ell_m, \ell_1)$ by $P(x) = (x_n^m)_{n=1}^\infty$ for all $x = (x_n) \in \ell_m$, and let $i : \ell_1 \to \ell_m$ be the natural inclusion. Since i is 1-summing, $P \circ i \circ T \in \mathcal{P}(^m\ell_\infty, \ell_1)$ is 1-dominated [21]. Now,

$$\begin{aligned} P \circ i \circ T(x) &= P \circ i \left(\left(|\lambda_n|^{1/m} e^{i\theta_n/m} x_n\right)_{n=1}^\infty\right) \\ &= \left(|\lambda_n| e^{i\theta_n} x_n^m\right)_{n=1}^\infty = \left(\lambda_n x_n^m\right)_{n=1}^\infty = M_\lambda(x). \end{aligned}$$

So M_λ is 1-dominated.

If the field is real, we write $M_\lambda = M_\mu + M_\nu$ with $\mu = (\mu_n)$ and $\nu = (\nu_n)$, where $\mu_n \geq 0$ and $\nu_n \leq 0$ for all n. Then, by the above argument, M_μ and M_ν are 1-dominated and so is M_λ.

Conversely, suppose that M_λ is 1-dominated. Then there are a space F, a 1-summing operator $T \in \mathcal{L}(\ell_\infty, F)$ and a polynomial $Q \in \mathcal{P}(^mF, \ell_1)$ such that $M_\lambda = Q \circ T$ [21]. Then, since T is 1-summing, we have

$$\begin{aligned} \sum_{n=1}^r |\lambda_n|^{1/m} &= \sum_{n=1}^r \|M_\lambda(e_n)\|^{1/m} = \sum_{n=1}^r \|Q \circ T(e_n)\|^{1/m} \\ &\leq \|Q\|^{1/m} \sum_{n=1}^r \|T(e_n)\| \\ &\leq \|Q\|^{1/m} \pi_1(T) \sup \left\{ \sum_{n=1}^r |x^*(e_n)| : x^* \in B_{\ell_\infty^*} \right\} \\ &= \|Q\|^{1/m} \pi_1(T) \sup \left\{ \sum_{n=1}^r |y^*(e_n)| : y^* \in B_{\ell_1} \right\} \\ &\leq \|Q\|^{1/m} \pi_1(T) \end{aligned}$$

for all $r \in \mathbb{N}$. Therefore, $\sum_{n=1}^\infty |\lambda_n|^{1/m}$ is convergent. □

This theorem shows that, unlike the linear case, a nuclear polynomial is not necessarily 1-dominated.

Finally, we obtain a sufficient condition for a nuclear polynomial to be 1-dominated.

COROLLARY 12. *Let $P \in \mathscr{P}_N({}^mE, F)$, so it satisfies* (1) *and* (2). *Suppose*

$$\sum_{n=1}^{\infty} \|x_n^*\| \, \|y_n\|^{1/m} < \infty.$$

Then P is 1-dominated.

PROOF. Since P is nuclear, by Proposition 6, it admits a factorization through a diagonal polynomial $M_\lambda \in \mathscr{P}({}^m\ell_\infty, \ell_1)$, where

$$\lambda_n = \|x_n^*\|^m \|y_n\| \quad (n \in \mathbb{N}).$$

By Theorem 11, M_λ is 1-dominated. By [15, Theorem 9], P is 1-dominated. □

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