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SOME REMARKS ON SET-VALUED DYNAMICAL SYSTEMS

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Abstract

It is shown that under some conditions a collection of continuous mappings gives rise to a set-valued dynamical system. Using this it is further shown that under some other conditions the system $\dot{x}(t) \in F(x(t))$ is equivalent to a set-valued dynamical system.

1. Introduction

In mathematical economics a number of phenomena involving time can be modelled as $\dot{x}(t) = f(x(t))$, $x(t) \in C$, where f has discontinuities on the boundary of C. In some circumstances this system is equivalent to $\dot{x}(t) \in F(x(t))$, where F is an upper semicontinuous compact-convex valued correspondence (see, for instance, Champsaur, Drèze and Henry [2]).

Other phenomena can be described by $\dot{x}(t) = f(x(t), u(t))$, where $x(\,.\,)$ is a state function and $u(\,.\,)$ a control function. Here f is a continuous mapping.

Another way of modelling dynamic economic phenomena is by means of a so-called set-valued dynamical system, abbreviated as SVDS. This is done for instance by Cherene in his monograph [3].

In all these cases we are interested in the behaviour of trajectories; hence the fundamental object of study should be that of a trajectory.

In this paper we will show that, given a particular set of continuous functions, there is a SVDS with as trajectories just these continuous functions. Further, we will show that, under some conditions, $\dot{x}(t) \in F(x(t))$ is equivalent to a SVDS and that, under some other conditions, $\dot{x}(t) = f(x(t), u(t))$ is equivalent to $\dot{x}(t) \in F(x(t))$.

2. Set-valued dynamical systems

In the sequel X will stand for a complete metric space with metric δ_x and CX will denote the set of all non-empty compacta of X. The letter T will stand for the set $[0, \infty)$.

DEFINITION 1. *The mapping G: $X \times T \to CX$ is called a SVDS if and only if:*
(1) $G(x, 0) = x$ *for all* $x \in X$,
(2) $G(G(x, r), t) = G(x, r + t)$ *for all* $x \in X$ *and for all* $r, t \in T$, *and*
(3) G *is upper semi-continuous in* x *for every* $t \in T$ *and continuous in* t *for every fixed* $x \in X$.

Closely related definitions can be found in Cherene [3], Roxin [9] and Kloeden [6].

DEFINITION 2. *A trajectory of a SVDS with name G starting at x is a mapping* $x(\,.\,)$: $T \to X$ *such that* $x(0) = x$ *and* $x(r + t) \in G(x(r), t)$ *for all* $r, t \in T$.

Repeating almost *ad verbatim* the proofs given by Roxin [9] and Kloeden [6] it can easily be shown that:

THEOREM 1. *Every trajectory is continuous. Further, let* $\bar{x} \in G(x, t)$; *then there is a trajectory* $x(\,.\,)$ *such that* $x(0) = x$ *and* $x(t) = \bar{x}$.

THEOREM 2. (*Barbashin's theorem.*) *Let t be an arbitrary number of T and* $\{x_i(\,.\,)\}$ *a collection of trajectories such that* $x_i(0) \to x_0$. *Then there is a subsequence* $\{x_{i_j}(0)\}$ *and a trajectory* $x_0(\,.\,)$ *such that* $x_{i_j}(\,.\,)$ *converges uniformly on* $[0, t]$ *to* $x_0(\,.\,)$.

Now we consider the problem of constructing a SVDS given a set S of continuous mappings from T to X. Let S satisfy the following properties:
(a) For all $x \in X$, there exists $x(\,.\,) \in S$ with $x(0) = x$.
(b) For all $\hat{t} \in T$ and for all $x(\,.\,) \in S$, there exists $\tilde{x}(\,.\,) \in S$ such that $\tilde{x}(t) = x(t + \hat{t})$ for all $t \in T$.
(c) For all $\bar{x}(\,.\,) \in S$ and for all $x(\,.\,) \in S$ with $\bar{x}(\hat{t}) = x(0)$ for some $\hat{t} \in T$, there exists $\tilde{x}(\,.\,) \in S$ with $\tilde{x}(t) = \bar{x}(t)$ for $t \in [0, \hat{t}]$ and $\tilde{x}(t) = x(t - \hat{t})$ for all $t \geqslant \hat{t}$.
(d) For all $\{x_i(\,.\,)\} \subset S$ with $x_i(1) = x_{i+1}(0)$, $i = 1, 2, \ldots,$ there exists $x(\,.\,) \in S$ with $x(t) = x_i(t)$ for $i - 1 \leqslant t \leqslant i$.

(e) For all $t \in T$ and for all $\{x_i(\,.\,)\} \subset S$ with $x_i(0) \to x_0$, there exists $\{x_{i_j}(\,.\,)\} \subset \{x_i(\,.\,)\}$ and $x_0(\,.\,) \in S$ such that $x_{i_j}(\,.\,) \to x_0(\,.\,)$ uniformly on $[0, t]$.

THEOREM 3. *Let S satisfy* (a) *to* (e); *then G: $X \times T \to CX$ defined by $G(x, t) = \cup \{x(t) | x(\,.\,) \in S,\ x(0) = x\}$ is a SVDS. Further, the trajectories of G are precisely the elements of S.*

PROOF. One trivially has that $G(x, 0) = x$ for all $x \in X$. Further, (b) and (c) imply that $G(G(x, r), t) = G(x, r + t)$ for all $x \in X$ and for all $r, t \in T$. Using well-known characterizations of upper and lower semi-continuity (see, for instance, Hildenbrand [5]), property (3) leads to G being compact valued and upper semi-continuous in x for every finite $t \in T$. Now take a fixed $X \in CX$ and define $\eta(t) = G(X, t) = \cup_{x \in X} G(x, t)$. Then (e) also implies the continuity of $\eta(\,.\,)$; hence we are done with the first part of the theorem.

Now take a trajectory $y(\,.\,)$ of G defined by S. Because of (c) and (d), it suffices to prove the existence of a mapping $x(\,.\,) \in S$ such that $y(t) = x(t)$ for $t \in [0, 1]$. But (c) and the definition of G imply that, for all $q \in \{1, 2, \ldots\}$, there exists $x_q(\,.\,) \in S$ with $y(p/(2^q)) = x_q(p/(2^q))$ for $p = 1, 2, \ldots, 2^q$. Applying (e) to $\{x_q(\,.\,)\}$ leads to the existence of an element $x(\,.\,) \in S$ such that $y(t) = x(t)$ for all dyadic numbers in $[0, 1]$. The mappings $y(\,.\,)$ and $x(\,.\,)$ being continuous leads to $y(t) = x(t)$ for $t \in [0, 1]$ and we are done with the proof.

The theorem above stresses the importance of the notion of trajectory: the properties showing how to patch together trajectories, (b) to (d), the property of uniform convergence, (e), and the fact that the starting points of the trajectories form a complete metric space, (a), completely determine a SVDS. Further, Theorem 3 can be of use in proving that a particular system in a SVDS.

3. The differential system $\dot{x}(t) \in F(x(t))$

In this section of the paper, F will denote an upper semi-continuous correspondence from R^p to the set of all non-empty convex compact subsets of R^p such that, for some $\alpha > 0$,

$$\sup_{w \in F(z)} |w| \leqslant \alpha(1 + |z|).$$

DEFINITION 3. *The mapping $z(\,.\,)$ is called a solution of $\dot{x}(t) \in F(x(t))$ if and only if $\dot{z}(t) \in F(z(t))$ almost everywhere on $[0, \infty]$ and, for all $t > 0$, $z(\,.\,)$ restricted to $[0, t]$ is absolutely continuous.*

Let $S_t(z^0)$ denote the restrictions to $[0, t]$ of all solutions $z(\,.\,)$ of $\dot{x}(t) \in F(x(t))$ with $z(0) = z^0$. Then:

THEOREM 4. (*Castaing and Valadier* [1].) *For all $t \in [0, \infty)$ and for all $z^0 \in R^p$, $S_t(z^0)$ is non-empty and compact in $C_u([0, t]; R^p)$, the space of continuous functions from $[0, t] \to R^p$ endowed with the uniform convergence topology. Further, for every $t \in [0, \infty)$, the mapping $z^0 \to S_t(z^0)$ is upper semi-continuous.*

By means of this result it is easy to prove that:

THEOREM 5. *Let S denote all the solutions of $\dot{x}(t) \in F(x(t))$; then G defined by S is a SVDS.*

We would like to remark that Theorem 5 was first proved by Roxin [10, Theorem 5.1] under the stronger assumption of continuity of F.

In our opinion it is conceptually elegant to start with set-valued dynamical systems and to consider $\dot{x}(t) \in F(x(t))$ to be a special case of it since, for instance, a lot of stability results can be phrased and proved in terms of SVDS's. To give an example, we discuss a result taken from Champsaur, Drèze and Henry [2]. These authors define an *equilibrium point* of $\dot{x}(t) \in F(x(t))$ to be a point $\bar{z}$ such that $0 \in F(\bar{z})$. Let G be the set-valued dynamical system associated with $\dot{x}(t) \in F(x(t))$; then the definition of equilibrium point can be rephrased as follows:

DEFINITION 4. *A point $\bar{z}$ is an equilibrium point if there is a trajectory $z(\,.\,)$ of G such that $z(t) = \bar{z}$ for all $t \in [0, \infty)$.*

Defining the notions *limit point*, *quasi-stability* and *Lyapunov-function* as is done in [2], we have the following result:

THEOREM 6. *If there is a Lyapunov function for G then G is quasi-stable.*

The proof being analogous to that of Theorem 6.1 of [2], we omit it.

Categorizing $\dot{x}(t) \in F(x(t))$ and $\dot{x}(t) = f(x(t), u(t))$, see below, as set-valued dynamical systems is, however, not only useful when studying Lyapunov-stability. For instance, the notion of *funnel*, extensively investigated for ordinary differential equations without uniqueness and so on, has been studied in the framework of set-valued dynamical systems by Kloeden [8].

In general one can say, following Kloeden [7], that set-valued dynamical systems "enable concepts and different modes of behaviour to be investigated in some generality without their inherent features being obscured by circumstantial details pertaining to a particular function or analytical representation".

4. The differential system $\dot{x}(t) = f(x(t), u(t))$

Let us say that a system of differential equations is equivalent to $\dot{x}(t) \in F(x(t))$, where F is as in Section 2, if the solutions of the first system are precisely those of $\dot{x}(t) \in F(x(t))$. Champsaur, Drèze and Henry [2] show that, under certain circumstances, the system $\dot{x}(t) = f(x(t))$ for $x(t) \in C$ is equivalent to $\dot{x}(t) \in F(x(t))$. In this section, however, we will prove that, under certain conditions, $\dot{x}(t) = f(x(t), u(t))$ for $u(t) \in U$ is equivalent to $\dot{x}(t) \in F(x(t))$. Here f will be a continuous mapping from $R^p \times R^n \to R^p$ such that $|f(x, u)| \leqslant \alpha(1 + |x|)$ for all $x, u \in U$ and for some $\alpha > 0$.

DEFINITION 5. *The mapping $z(\,.\,)$ is called a solution of $\dot{x}(t) = f(x(t), u(t))$ for $u(t) \in U$ if there is a mapping $u(\,.\,)$: $[0, \infty) \to U$ such that, for every $t > 0$, $u(\,.\,)$ is Lebesgue-measureable on $[0, t]$ such that $\dot{z}(t) = f(z(t), u(t))$ almost everywhere on $[0, \infty)$. Further, $z(\,.\,)$ has to be absolutely continuous on $[0, t]$ for every $t \in (0, \infty)$.*

We will prove the following:

THEOREM 7. *When U is compact and $f(x, U) := F(x)$ is convex for all $x \in R^p$, then $\dot{x}(t) = f(x(t), u(t))$ for $u(t) \in U$ is equivalent to $\dot{x}(t) \in F(x(t))$.*

PROOF. It is easy to see that F is as in Section 2. Hence there remains to be proved that a solution of $\dot{x}(t) \in F(x(t))$ is a solution of $\dot{x}(t) = f(x(t), u(t))$ for $u(t) \in U$. Let $z(\,.\,)$ be such a solution. Take $\bar{t} \in (0, \infty)$; then we know that $z(\,.\,)$ is absolutely continuous on $[0, \bar{t}]$ and

$$\dot{z}(t) = f(z(t), u_t) \quad \text{almost everywhere for } t \in [0, \bar{t}\,] \quad \text{and} \quad u_t \in U.$$

As $\dot{z}(\,.\,)$ is measurable on $[0, \bar{t}]$, there is a sequence of compact subsets $\{\Delta_i\} \subset [0, \bar{t}]$ such that $\Delta_1 \subset \Delta_2 \subset \Delta_3 \subset \ldots$ and $[0, \bar{t}] - (\Delta_1 \cup \Delta_2 \cup \ldots)$ has measure zero and, further, the restriction of $\dot{z}(\,.\,)$ to Δ_i is continuous (Lusin's theorem). Without loss of generality, we may assume that the measure of Δ_1 is greater than zero. Now define $D_i = \{(t, u) | t \in \Delta_t,\ \dot{z}(t) = f(z(t), u) \text{ for } u \in U\}$ and $D = D_1 \cup D_2 \cup \ldots.$

Since the measure of Δ_i is greater than zero, we immediately have that $D_t \neq \emptyset$. Further, D_i is trivially bounded. Now take a sequence $\{(t_j, u_j)\} \subset D_i$

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such that $t_j \to \hat{t}$ and $u_j \to \hat{u}$. Then $\dot{z}(t_j) \to \dot{z}(\hat{t})$ and $z(t_j) \to z(\hat{t})$. As f is continuous, we have that $\dot{x}(\hat{t}) = f(x(\hat{t}), \hat{u})$; hence $(\hat{t}, \hat{u}) \in D_i$ and therefore D_i is compact. Defining $\Delta = \{t | (t, u) \in D \text{ for some } u \in U\}$, we have that $\Delta \subset \Delta_1 \cup \Delta_2 \cup \ldots$ and, further, that the measure of Δ is equal to the measure of $\Delta_1 \cup \Delta_2 \cup \ldots$. Now the application of a selection lemma (Fleming and Rishel [4, page 199, Lemma B]), implies the existence of a measurable function $u(\,.\,)$ on $[0, \bar{t}]$ such that $(t, u(t)) \in D$ for almost all $t \in \Delta$; hence $\dot{z}(t) = f(z(t), u(t))$ almost everywhere on $[0, \bar{t}]$.

The proof of the foregoing theorem is a slight alteration of a technique in Fleming and Rishel [4]. Further, we would like to remark that implicit in Theorem 7 is the existence of solutions to $\dot{x}(t) = f(x(t), u(t))$ when $f(x, U)$ is convex, U is compact and $|f(x, u)| \leqslant \alpha(1 + |x|)$ for all $x, u \in U$.

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