GENERALIZATION OF HÖLDER'S THEOREM TO ORDERED MODULES

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Hölder's theorem on archimedean groups states:

An ordered (abelian) group G is order isomorphic to an ordered subgroup of the ordered group R of real numbers if and only if it is archimedean.

We comprehend this theorem in the following setting: G is a Z-module and R is the completion with respect to the open interval topology of the ordered field Q; Q itself is the ordered quotient field of the ordered domain Z.

Rephrasing the situation, we raise the following question: We start with a fully ordered domain A, let K be its ordered quotient field. We endow K with the open interval topology and consider $\hat{K}$, the topological completion of K. Is it possible to impose a compatible order structure on $\hat{K}$ and if this can be done, when can we say that an ordered A-module M is order isomorphic to an ordered A-submodule of $\hat{K}$? In Theorem 3.1, we obtain a set of necessary and sufficient conditions for this isomorphism to hold.

In the case of an ordered abelian group G, there is another condition which is equivalent to being archimedean. G is archimedean if and only if it is o-simple, i.e., it has no non-trivial convex subgroup. We show by means of two examples (4.3, 4.4) that this situation does not generalize in the same way to o-simple modules. There exist fully ordered o-simple A-modules which are not order isomorphic to ordered A-submodules of $\hat{K}$. However, when A is archimedean, all these concepts coincide (Theorem 4.2).

1. The completion field associated with a fully ordered domain. Let A be a fully ordered domain and K its ordered quotient field. Let K be given *the open interval topology*, that is, a fundamental system of neighbourhoods of the origin 0, is taken to be the family of open intervals $(-a, a)$ with $a > 0$ in K. It is natural to consider the topological completion of K, when it exists. In what follows, our results will hold good for any fully ordered field. We have the following well-known proposition.

PROPOSITION 1.1. *With the open interval topology, the additive group $(K, +)$ of a fully ordered field K is a Hausdorff topological group.*

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PROPOSITION 1.2. *With the open interval topology, every fully ordered field is a Hausdorff topological field.*

Proof. First we prove that multiplication is continuous. Let $x, y \in K$ and let a given neighbourhood of xy contain the open interval $xy + (-a, a)$ with $a > 0$. We first prove that there exist $b, c > 0$ in K such that $|x|b < \frac{1}{3}a$, $|y|c < \frac{1}{3}a$, and $bc < \frac{1}{3}a$. If $x = 0$, choose $b > 0$ arbitrarily. If $x \neq 0$, $|x|a/4|x| = \frac{1}{4}a < \frac{1}{3}a$. Thus, $b = a/4|x|$ will suffice. Similarly, there exists c > 0 such that $|y|c < \frac{1}{3}a$. Suppose that $bc \geqq \frac{1}{3}a$; then by the same argument, there exists $d > 0$ such that $bd < \frac{1}{3}a$. Now $d < c$, if not, $c \leqq d$ and $bc \leqq bd < \frac{1}{3}a$, a contradiction. For such an element d, $|y|d < |y|c < \frac{1}{3}a$ and $bd < \frac{1}{3}a$.

We claim that

$$\{x + (-c, c)\}\{y + (-b, b)\} \subseteq xy + (-a, a).$$

In fact, if $B_1 \in (-c, c)$ and $B_2 \in (-b, b)$, then

$$\begin{aligned} |B_1y + xB_2 + B_1B_2| &\leqq |B_1y| + |xB_2| + |B_1B_2| \\ &< c|y| + |x|b + cb \\ &< \tfrac{1}{3}a + \tfrac{1}{3}a + \tfrac{1}{3}a \\ &= a. \end{aligned}$$

Hence, $B_1y + xB_2 + B_1B_2 \in (-a, a)$ so that $(x + B_1)(y + B_2) \in xy + (-a, a)$, proving our claim.

Let ϕ: $K^\cdot \to K^\cdot$ be the inverse map $x \to x^{-1}$. A neighbourhood of x^{-1} may be taken to be (t, t'), where either $0 < t < x^{-1} < t'$ or $t < x^{-1} < t' < 0$. In either case, $(1/t', 1/t)$ is a neighbourhood of x and $\phi((1/t', 1/t)) \subseteq (t, t')$. Thus ϕ is continuous. By Proposition 1.1, K is a Hausdorff topological field.

Let $\hat{K}$ be the uniform completion of the topological field K. As K is Hausdorff, it is homeomorphic to a dense subspace of $\hat{K}$. Thus we may assume that K is embedded in $\hat{K}$. We wish to show that $\hat{K}$ is a Hausdorff topological field. For this we observe that the completion of a field, though a ring, need not, in general, be a field. The following result from (**3**) furnishes a criteria to test whether the completion $\hat{K}$ is a topological field or not.

PROPOSITION 1.3. *Let K be a Hausdorff topological field and let $\hat{K}$ be the topological completion of K. For $\hat{K}$ to be a topological field, it is necessary and sufficient that:*

(∗) *under the inverse map $x \to x^{-1}$ of $K^\cdot$ onto $K^\cdot$ the image of every Cauchy filter in the additive uniform structure of K, for which* 0 *is not a cluster point, is again a Cauchy filter for this additive structure.*

Remark 1.4. If 0 is not a cluster point of a Cauchy filter $\mathfrak{F}$ (in the uniform structure of K), there exists $F \in \mathfrak{F}$ which is disjoint from an open interval $(-a, a)$ of 0. Thus, by intersecting each member of a basis $\mathfrak{B}$ of $\mathfrak{F}$ with F, we obtain a basis $\mathfrak{B}'$ of $\mathfrak{F}$. The image of $\mathfrak{B}'$ under the inverse map makes sense.

The filter $\mathfrak{F}'$ generated by this image of $\mathfrak{B}'$ is called the image of $\mathfrak{F}$ under the inverse map.

PROPOSITION 1.5. *If K is a fully ordered field, then the condition* $(*)$ *is satisfied for the open interval topology of K. Thus, the topological completion $\hat{K}$ of K with respect to the open interval topology is a Hausdorff topological field.*

Proof. That $\hat{K}$ is Hausdorff would follow from the fact that K is Hausdorff. Now, let $\mathfrak{F}$ be a Cauchy filter in the additive uniform structure of K such that 0 is not a cluster point. Let $F \in \mathfrak{F}$, $(-a, a)$, $\mathfrak{B}' \subseteq \mathfrak{F}$, and $\mathfrak{F}'$ be as in the remark above. Let $(-c, c)$ be an arbitrary neighbourhood of the origin with $c > 0$. Choose $b > 0$ with $0 < b < a^2c$ (for e.g., $b = \frac{1}{2}a^2c$). Since $\mathfrak{F}$ is a Cauchy filter, there exists $G \in \mathfrak{B}'$ so that $|-x + g| < b$ for all $x, g \in G$. Furthermore, $|g| > a$ for every $g \in G$. Hence

$$\left| -\frac{1}{x} + \frac{1}{g} \right| = \left| \frac{-g + x}{xg} \right| < \frac{b}{|x|\,|g|} < \frac{b}{a^2} < c.$$

Now $1/x$ and $1/g$ are elements of $(G)^{-1} \in \mathfrak{F}'$. Thus, $\mathfrak{F}'$ contains "small sets". Therefore, $\mathfrak{F}'$ is a Cauchy filter.

COROLLARY 1.6. *The multiplicative structure over $\hat{K}^{\cdot}$ is a structure of a complete space.*

Proof. K is a (commutative) topological field and the additive uniform structure of $\hat{K}$ is that of a complete, Hausdorff space. The corollary is now a consequence of a result of (**3**, p. 85).

Definition 1.7. Let A be a fully ordered integral domain. Then the field $\hat{K}$ obtained above is called *the completion field associated with A.*

2. The extension of the order to the completion field $\hat{K}$. Throughout the remainder of this paper, A will denote a fully ordered domain, K its ordered quotient field, and $\hat{K}$ the completion field associated with A.

We now extend the order of K to $\hat{K}$ endowing $\hat{K}$ with the structure of a fully ordered field. We recall that $\hat{K}$ is the space of minimal Cauchy filters of K. If $\mathfrak{F}$ is one such minimal Cauchy filter, call $\mathfrak{F}$ positive if and only if

(α) there exists $F \in \mathfrak{F}$ consisting entirely of positive elements of K.

We now have the following theorem.

THEOREM 2.1. *With the above definition, $\hat{K}$ is a fully ordered field and the open interval topology of $\hat{K}$ is equivalent to the topology of the completion.*

Proof. That $\hat{K}$ is an ordered field is shown in three steps: (1) $\hat{K}$ is an ordered abelian group; (2) $\hat{K}$ is fully ordered; (3) $\hat{K}$ is an ordered ring. In (2), one uses the fact that if the minimal filter $\mathfrak{F}$ is not the neighbourhood system at the origin, then 0 is not a cluster point of $\mathfrak{F}$. The easy proofs are left to the reader.

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For the second conclusion, we recall that in the topology of the completion, a basis at a point $\mathfrak{F}$ consists of the family $\{\mathfrak{F}_a\colon a > 0 \text{ in } K\}$, where $\mathfrak{F}_a$ is the set of filters sharing with $\mathfrak{F}$ a small set of order a. Now, given $\mathfrak{F}$, there exists $(a, b) \in \mathfrak{F}$ with $a < \mathfrak{F} < b$ and (a, b) can be made as small as we please. This proves that a basis in $\hat{K}$ is given by the open intervals (a, b) in $\hat{K}$. On the other hand, given an open interval in $\hat{K}$, there is an infinite number of elements of K belonging to this interval. A small enough sub-interval of K gives rise to a set $\mathfrak{F}_x$; thus, the open interval topology of $\hat{K}$ is weaker than the completion topology and this completes the proof.

Definition 2.2. The field $\hat{K}$, endowed with the structure of a fully ordered field as in Theorem 2.1, is called *the ordered completion field associated with the fully ordered domain A.*

3. A structure theorem for fully ordered modules. We are now ready to generalize Hölder's theorem on archimedean groups. We have the following theorem.

THEOREM 3.1. *Let A be a fully ordered domain and $\hat{K}$ the ordered completion field associated with A. A non-zero, fully ordered A-module M is order isomorphic to an ordered A-submodule of $\hat{K}$ if and only if the following two conditions are satisfied*:

(i) *M is a torsion-free A-module*;

(ii) *There exists an element $m_0 \neq 0$ in M such that whenever $m, n \in M$ with $m < n$, there exist a, b in A with $b > 0$ such that $bm \leqq am_0 < bn$.*

Proof. We will prove the sufficiency first. Fix an element m_0 satisfying condition (ii). We may assume that $m_0 > 0$. Given $m \in M$, let $n \in M$ be an arbitrary element such that $-m < n$; then there exists $-a, b \in A$, $b > 0$ such that $b(-m) \leqq (-a)m_0 < bn$; hence $am_0 \leqq bm$. Similarly, if $n \in M$ is such that $m < n$, there exists $a', b' \in A$, $b' > 0$, such that $b'm \leqq a'm_0 < b'n$. Thus there exists

(**) a quadruple $(a, b, a', b') \in A$ with $b, b' > 0$ so that $am_0 \leqq bm$, and $b'm \leqq a'm_0$.

Consider all possible quadruples (a, b, a', b') satisfying the condition (**). We claim that in the ordered quotient field K of A, $a/b \leqq a'/b'$ for such a quadruple (a, b, a', b'). If not, $a/b > a'/b'$. Thus, $ba' < b'a$. From $am_0 \leqq bm$ and $b'm \leqq a'm_0$, we have that $bb'm \leqq ba'm_0 < b'am_0 \leqq b'bm$, a contradiction. The inequality in the middle is strict since M is a torsion-free A-module.

In K, the interval $[a/b, a'/b'] = \{x \in K, a/b \leqq x \leqq a'/b'\}$ is closed in the topology of K. Let

$$\mathfrak{B} = \left\{\left[\frac{a}{b}, \frac{a'}{b'}\right], \text{ where } (a, b, a', b') \text{ is a quadruple in } A \text{ satisfying } (**)\right\}.$$

Our first claim is: $\mathfrak{B}$ is a base of a filter $\mathfrak{F}$ in K. Clearly, $\mathfrak{B}$ is not empty and the empty set does not belong to $\mathfrak{B}$, as a closed interval contains at least one point. Let

$$\left[\frac{a}{b}, \frac{a'}{b'}\right], \left[\frac{a_1}{b_1}, \frac{a_1'}{b_1'}\right] \in \mathfrak{B}.$$

Then $a/b \leqq a_1'/b_1'$; if not,

$$(1) \qquad \frac{a}{b} > \frac{a_1'}{b_1'} \quad \text{and} \quad a_1'b < ab_1'.$$

From $am_0 \leqq bm$ and $b_1'm \leqq a_1'm_0$ we have that $ab_1'm_0 \leqq b_1'bm \leqq ba_1'm_0$. But from (1), $a_1'bm_0 < ab_1'm_0$ (strict inequality, since M is torsion-free), a contradiction. Similarly, $a_1/b_1 \leqq a'/b'$. Thus, the intersection of the two closed intervals is the closed interval $[c, d]$, where

$$c = \max\left\{\frac{a}{b}, \frac{a_1}{b_1}\right\} \quad \text{and} \quad y = \min\left\{\frac{a'}{b'}, \frac{a_1'}{b_1'}\right\}$$

and, evidently, the quadruple which determines c and d satisfies (**).

Secondly, we claim that $\mathfrak{B}$ contains arbitrarily small sets in the uniform structure of K. Let $a/b > 0$ be given in K and assume that $a, b > 0$. Hence, $bm < bm + \frac{1}{2}am_0$. By condition (ii) of the hypothesis, there exist λ, μ in A, $\lambda > 0$, such that

$$(2) \qquad \mu bm \leqq \lambda m_0 < \mu(bm + \tfrac{1}{2}am_0).$$

Similarly, there exist λ', μ' in A, $\mu' > 0$, with

$$(3) \qquad \mu'(bm - \tfrac{1}{2}am_0) \leqq \lambda' m_0 < \mu' bm.$$

Rewriting (2) and (3) we have that

$$(2') \qquad \mu bm \leqq \lambda m_0 < \mu bm + \mu(\tfrac{1}{2}a)m_0,$$

$$(3') \qquad \mu' bm - \mu'(\tfrac{1}{2}a)m_0 \leqq \lambda' m_0 < \mu' bm.$$

Put $s' = \mu b$, $r' = \lambda$, $r = \lambda'$, and $s = \mu' b$. We have that $s, s' > 0$ and the quadruple (r, s, r', s') satisfies the condition (**).

$$\frac{r'}{s'} - \frac{r}{s} = \frac{\lambda}{\mu b} - \frac{\lambda'}{\mu' b} = \frac{1}{b}\left(\frac{\lambda}{\mu} - \frac{\lambda'}{\mu'}\right).$$

But

$$(2'') \qquad \lambda m_0 - \mu bm < \mu(\tfrac{1}{2}a)m_0,$$

$$(3'') \qquad \mu' bm - \lambda' m_0 \leqq \mu'(\tfrac{1}{2}a)m_0.$$

Multiply (2″) by μ', (3″) by μ and adding we obtain $(\lambda\mu' - \lambda'\mu)m_0 < 2\mu\mu'(\frac{1}{2}a)m_0$. This implies that $(\lambda\mu' - \lambda'\mu) < \mu\mu' a$. Therefore,

$$\frac{1}{b}\left(\frac{\lambda}{\mu} - \frac{\lambda'}{\mu'}\right) < \frac{a}{b}.$$

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Thus,

$$0 < \frac{r'}{s'} - \frac{r}{s} < \frac{a}{b}.$$

Now, for any $x, y \in [r/s, r'/s']$ we have that $|x - y| < r'/s' - r/s < a/b$. Thus, the closed interval $[r/s, r'/s']$ belongs to $\mathfrak{B}$ and is a small set of order $V_{a/b}$ in K, where $V_{a/b}$ is an entourage determined by $(-a/b, a/b)$. Therefore, $\mathfrak{B}'$, the filter generated by $\mathfrak{B}$, is a Cauchy filter. Let $\mathfrak{F}$ be the unique minimal Cauchy filter contained in $\mathfrak{B}'$.

Define θ: $M \rightarrow \hat{K}$ as follows:

For an arbitrary $m \in M$, define $\theta(m) = \mathfrak{F}$ as above. Thus θ is a well-defined mapping.

It is a rather simple conclusion that θ is not only a group, but even a module homomorphism preserving order relation.

To show that θ is one-to-one, we shall show that $\theta(m) = \mathfrak{F} > 0$ in K implies that $m > 0$. Since $\mathfrak{F} > 0$, there exists $F \in \mathfrak{F}$ consisting entirely of positive elements. Since the filter generated by $\mathfrak{B}$ associated with m is finer than $\mathfrak{F}$, there exists a closed interval $[r/s, r'/s']$ consisting entirely of positive elements. From $s > 0$, $r > 0$, $rm_0 \leqq sm$, we have that $sm > 0$ and $m > 0$. Thus, θ is an o-isomorphism of M into K.

Proof of necessity. The following short proof is due to the referee.

Because of the embedding, $(\mathfrak{F}_1/\mathfrak{F}) < (\mathfrak{F}_2/\mathfrak{F})$ in $\hat{K}$. Between any two elements of $\hat{K}$, there is an element of K. Let

$$(\mathfrak{F}_1/\mathfrak{F}) < a/b < (\mathfrak{F}_2/\mathfrak{F}) \qquad (a, b \in A, b > 0).$$

Then $b\mathfrak{F}_1 < a\mathfrak{F} < b\mathfrak{F}_2$, proving the result.

COROLLARY 3.2. *Let A be a fully ordered integral domain and K, $\hat{K}$ as described above. Let M be an A-module which is order isomorphic to an A-submodule of $\hat{K}$. For every triple (m_0, m_1, m_2) of elements of M with $m_0 > 0$ and $m_1 < m_2$, there exist $r, s \in A$ with $s > 0$ so that $sm_1 \leqq rm_0 < sm_2$ in M. Even strict inequality holds.*

Proof. This follows immediately from the proof of the necessity.

4. o-simple modules. A fully ordered abelian group G is order isomorphic to an ordered subgroup of the ordered field R of real numbers if and only if it has no non-trivial convex subgroup. We generalize this theorem to fully ordered modules over archimedean rings and show by means of two examples that our result will not be true when the underlying ring is non-archimedean.

Definition 4.1. Let A be an ordered ring and M an ordered left A-module. M is called o-*simple* if it has no non-trivial convex A-submodule. The ring A is said to be o-*simple* if it has no non-trivial convex one-sided ideals.

THEOREM 4.2. *Let A be a fully ordered archimedean ring and M a fully ordered A-module. Then the following conditions are equivalent:*

(1) *M is archimedean* (*as an ordered abelian group*);

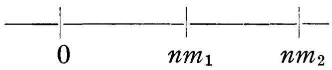
(2) *M is an* o-*simple A-module;*

(3) *M is order isomorphic to an ordered A-submodule of the ordered field of real numbers.*

Proof. We prove (1) $\Leftrightarrow$ (2) and (1) $\Leftrightarrow$ (3).

It is obvious that (1) implies (2). It is easy to see that (2) implies (1) since, if $0 < m_1 < m_2$ in M, then there exists $r \in A_+$ such that $rm_1 \geqq m_2$. But A is archimedean. Hence, there exists a positive integer n such that $n \geqq r$. Now $nm_1 \geqq rm_1 \geqq m_2$. Thus, M is archimedean.

To prove (1) $\Rightarrow$ (3). A is archimedean. Thus, $A \subseteq R$ and the ordered completion field associated with A is R itself. Using Corollary 3.2, it is enough to prove that for every triple (m_0, m_1, m_2) of M with $m_0 > 0$ and $0 < m_1 < m_2$, there exist positive integers n, n' so that $nm_1 \leqq n'm_0 < nm_2$. Since M is archimedean, there exists an integer $n > 0$ so that $n(m_2 - m_1) > m_0$ in M.



Since every non-empty set of positive integers has a smallest element, there exists a non-negative integer r so that $rm_0 < nm_1$ but $(r + 1)m_0 \geqq nm_1$. Now $(r + 1)m_0 < nm_2$; if not, $nm_2 \leqq (r + 1)m_0$, and from $rm_0 < nm_1 < nm_1 + m_0 < nm_2 \leqq (r + 1)m_0$, we have that $m_0 = (r + 1)m_0 - rm_0 > n(m_2 - m_1)$, a contradiction. Thus, if $n' = r + 1$, we have that $nm_1 \leqq n'm_0 < nm_2$.

(3) $\Rightarrow$ (1). Let $0 < m$. Let $m_1 \in M$ be such that $0 < m_1 < 2m_1$. Thus, by Corollary 3.2, there exist $r, s \in A$ with $rm_1 \leqq sm < 2rm_1$. We may assume that r and s are positive. Since A is archimedean, there exists $r' \in A_+$ with $rr' \geqq 1$. Now $m_1 \leqq rr'm_1 \leqq r'sm$. But there exists a positive integer n so that $n \geqq r's$. Thus, $nm \geqq m_1$ in M, and M is archimedean.

The following two examples show that not every o-simple A-module is order isomorphic to an ordered submodule of $\hat{K}$, when A is non-archimedean.

Example 4.3. Let $A = Z[X]$ be the lexically ordered polynomial ring over (the ordered ring) Z. A is an o-simple ring. Consider the A-module $M = Z[X, Y]$. Order M as follows. Let $f(X, Y)$ be a non-zero element of M of total degree $n \geqq 0$. We collect all the terms of $f(X, Y)$ of total degree n and write $f_n \equiv a_n X^n + a_{n-1}X^{n-1}Y + \ldots + a_i X^i Y^{n-i} + \ldots + a_0 Y^n$, where $a_i \in Z$. We point out that in this expression for f_n, we write the powers of X in descending order. We say that $f(X, Y)$ is positive if the first non-vanishing coefficient a_i in f_n is a positive integer. It is easy to check that M is a fully ordered A-module and that M is o-simple, but M fails to satisfy condition (ii) of Theorem 3.1. We have that $Y^2 < XY$ in M and $X > 0$ but it is impossible to find elements r, s in A such that $rY^2 \leqq sX < rXY$, for if $\deg r = n$, from the inequality

$rX^2 \leqq sX$, we obtain $\deg s \geqq n + 1$ since sX is a polynomial of degree greater than or equal to $n + 2$. But then $rXY < sX$.

Example 4.4. We take the same ordered ring A and the module M. We order M in a similar manner, but writing f_n in descending powers of Y. It is clear that M is an ordered A-module with an order different from that of Example 4.3.

Again, M is o-simple, $X > 0$, and $0 < XY < Y^2$, but it is impossible to find elements r, s in A such that $rXY \leqq sX < rY^2$. Indeed, for every r, s in A_+, $sX < rXY$.

It will be interesting to study o-simple modules and o-simple rings. Here we present a useful way of producing o-simple modules.

THEOREM 4.5. *Let A be a fully ordered domain and K the ordered quotient field of A. Let $\hat{K}$ be the ordered completion field associated with A. Then the following are equivalent:*

(1) *A is an* o-*simple ring;*

(2) *Every A-submodule of $\hat{K}$ with the induced order is an* o-*simple A-module;*

(3) *$\hat{K}$ is an* o-*simple A-module;*

(4) *Every A-submodule of K with the induced order is an* o-*simple A-module;*

(5) *K is an* o-*simple A-module.*

Proof. (1) $\Rightarrow$ (2). Let $M \subseteq \hat{K}$ be an ordered A-module. Let N be a non-zero convex A-submodule of M. Let $m_0 > 0$ in M and let $n > 0$ in N. Since $n < 2n$, by Corollary 3.2, there exist r, $s \in A_+$ such that $sn \leqq rm_0 < s(2n)$. Since N is convex in M, this means that $rm_0 \in N$. But A is an o-simple ring. Hence there exists $r' \in A$ with $1 \leqq rr'$ in A. Thus $0 < m_0 \leqq rr'm_0$ and $m_0 \in N$. Therefore, $M_+ \subseteq N$, which implies that $M \subseteq N$. Thus, $N = M$ and M is o-simple.

It is obvious that (2) implies (3). We shall prove that (3) implies (1). Suppose that A is not an o-simple ring. Let I be a non-trivial convex ideal of A. Let $\hat{I}$ be the convex A-submodule of $\hat{K}$ generated by I in $\hat{K}$. $\hat{I} = \{x \in \hat{K}:$ there exists $\lambda, \mu \in I$ with $\lambda \leqq x \leqq \mu\}$. Since $1 \notin I$, it is clear that $1 \notin \hat{I}$. Then $\hat{I} \neq \hat{K}$, contradicting the fact that $\hat{K}$ is an o-simple A-module.

It is obvious that (2) $\Rightarrow$ (4) $\Rightarrow$ (5). As above, it is easy to prove that (5) $\Rightarrow$ (1). This establishes the equivalence of the five conditions.

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