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ON DOUBLY TRANSITIVE PERMUTATION GROUPS OF DEGREE PRIME SQUARED PLUS ONE

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Abstract

A doubly transitive permutation group of degree $p^2 + 1$, p a prime, is proved to be doubly primitive for $p \neq 2$. We also show that if such a group is not triply transitive then either it is a normal extension of $PSL(2, p^2)$ or the stabilizer of a point is a rank 3 group.

We will show that the groups described in the title are doubly primitive for $p > 2$ and sometimes they are even triply transitive.

THEOREM A. *Let G be a doubly transitive permutation group of degree $p^2 + 1$, p a prime. Then either G is doubly primitive or $p = 2$ and G is the Frobenius group of order 20.*

THEOREM B. *Let G be a doubly transitive permutation group of degree $p^2 + 1$, p an odd prime. Assume that G_α contains two distinct Sylow p-subgroups. Then either a) G is triply transitive, or b) The stabilizer of a point is primitive rank 3 group of degree p^2 and subdegrees 1, $2(p-1)$, $(p-1)^2$. Moreover, the stabilizer of a point is isomorphic to a subgroup of $S_p \int S_2$, the wreath product of the symmetric groups of degrees p and 2.*

Groups described in b) are discussed in Higman (1970).

COROLLARY: *Let G be a doubly transitive permutation group of degree $p^2 + 1$, p a prime. Then one of the following is true:*

(*a*) *G is 3-transitive,*

(*b*) *$PSL(2, p^2) \subseteq G \subseteq P\Gamma L(2, p^2)$ in its natural representation,*

(*c*) *G is a Frobenius group of order 20 and $p = 2$,*

(*d*) *The stabilizer of a point is primitive rank 3 group of degree p^2 and subdegrees 1, $2(p-1)$, $(p-1)^2$. Moreover, the stabilizer of a point is isomorphic to a subgroup of $S_p \int S_2$, the wreath product of the symmetric groups of degrees p and 2.*

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NOTATIONS. We use notations of Wielandt (1964) for permutation groups and notations of Ryser (1963) for the parameters of a block design. If G acts on Ω and $T \subseteq G$ we define $F(T) = \{x \in \Omega \mid xt = x \text{ for all } t \in T\}$.

We start with the following lemma:

LEMMA. *Let G be a doubly transitive permutation group of degree $p^2 + 1$ on a set Ω. Here p is a prime. Then*:

a) *If $|G| \equiv 0(p^3)$ then G contains A_{p^2+1}.*

b) *There is no nontrivial block design with $\lambda = 1$ on Ω.*

c) *If G is sharply doubly transitive then $p = 2$ and $|G| = 20$.*

PROOF. Part a) is a result of Tsuzuku (1968), and part b) follows from the incidence equations of a block design and the Fisher inequality (see Ryser (1963)). In c) G contains a regular normal subgroup and if c) is not true, $p^2 + 1 = 2^x$ for some integer x. This is impossible since $p^2 + 1 \equiv 2(4)$.

PROOF OF THEOREM A. Assume that G is not doubly primitive. Let Ω be the set on which G acts and let $\alpha \in \Omega$. It follows that G_α has a complete system of inprimitivity sets on $\Omega - \{\alpha\}$. Let $\Lambda_0 = \{\Delta_1, \Delta_2, \cdots, \Delta_p\}$ be such a system and let $\Lambda = \Lambda_0 - \{\Delta_1\}$. Let $\beta \in \Delta_1$. We have that $|\Lambda_0| = p$. Let P be a Sylow p-subgroup of G contained in G_α. By the lemma, $|P| = p^2$. Let K be the kernel of the action of G_α on Λ_0 and let H be the stabilizer of Δ_1 in G_α in its action on Λ_0. Let A be the kernel of H on Δ_1. By the lemma we have that either $G_{\alpha\beta} \neq 1$ or we are done. Hence we can assume that $G_{\alpha\beta} \neq 1$. It follows that $H_\beta = G_{\alpha\beta} \neq 1$. Clearly H is transitive on Δ_1 and G_α is transitive on Λ_0. We can also assume that $p > 2$.

Since G_α/K is transitive permutation group of degree p, $|G_\alpha/K|_p = p$ so that $|K|_p = p$. Let P_0 be a Sylow p-subgroup of K. We can assume that $P_0 \subseteq P$. Since $|H : G_{\alpha\beta}| = p$ we get that $|A|_p = |G_{\alpha\beta}|_p = 1$. We use Wielandt (1964), 11.6, 11.7, without referring to them. First we prove that $A = 1$. Suppose $A \neq 1$. The lemma and lemma 1.1 of Praeger (submitted) implies that A fixes a point in some $\Delta_i \neq \Delta_1$. Thus A fixes at least two blocks of Λ_0 setwise. However H is either transitive or semiregular on Λ, and since its normal subgroup A fixes a block in Λ we get $A \subseteq K$. Since $|A|_p = 1$, $A \lhd K$ and K is transitive on each Δ_i we conclude that A is trivial on each Δ_i so that $A = 1$. This contradicts $A \neq 1$.

We now break the proof into two cases.

CASE 1. We assume that G_α/K is nonsolvable. It follows that G_α is doubly transitive on Λ. Since $H = G_{\alpha\beta}K$ we have that $G_{\alpha\beta}$ is transitive on Λ. The lemma and lemma 2 of Atkinson (1972/73) imply that $\Delta_1 - \{\beta\}$ is not $G_{\{\alpha,\beta\}}$-invariant. It follows that there exists a $G_{\alpha\beta}$-orbit Γ_0, on $\Delta_1 - \{\beta\}$ such

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that $\Gamma_0 g \not\subset \Delta_1$ for $g \in G_{\{\alpha,\beta\}} - G_{\alpha\beta}$. Set $\Gamma = \Gamma_0 g$ and $\Sigma = \{\Delta \in \Lambda \mid \Delta \cap \Gamma \neq \emptyset\}$. Since Σ is a $G_{\alpha\beta}$-orbit on Λ we get that $|\Sigma| = p - 1$ so that $\Gamma_0 = \Delta - \{\beta\}$ and $|\Delta \cap \Gamma| = 1$ for every $\Delta \in \Lambda$. If $K_\beta \neq 1$ then K_β fixes the point in $\Delta \cap \Gamma$ for all $\Delta \in \Lambda$ so that $|F(K_\beta)| > 2$. This contradicts our lemma because of B1 of O'Nan (1972).

Therefore $K_\beta = 1$. Since K is transitive on Δ_1 we have that $K = P_0$. Since $H \simeq H^{\Delta_1}$ and $K \lhd H$, H is metacyclic of order dividing $p(p-1)$, so that H/K is cyclic of order dividing $p - 1$. This contradicts the assumption that G_α/K is nonsolvable. Thus we have:

Case 2. We assume that G_α/K is solvable. In this case G_α/K is a Frobenius group so that $G_{\alpha\beta}/K_\beta$ is semiregular on Λ. Let $t = |G_{\alpha\beta} : K_\beta|$. Then $t \mid p - 1$. Since $P_0 \subseteq K \subseteq H$ and $|A|_p = 1$ we get that K is transitive on Δ_1 and therefore on each Δ_i.

Assume that K is not faithful on some $\Delta \in \Lambda_0$ and let M be the kernel of K on Δ. Since K is transitive on Δ, $|K : M|_p = p$ so that $|M|_p = 1$. Hence M cannot be transitive on any $\Delta_i, 1 \leqq i \leqq p$. Since $M \lhd K$ and $|\Delta_i| = p$ we get that M fixes all points of Ω. Since this is impossible, K is faithful on each Δ_i.

If $K_\beta = 1$ then $|G_{\alpha\beta}| = t$ and $|H| = tp$. Then H is solvable so that $G_{\alpha\beta}$ is semiregular on both $\Delta_1 - \{\beta\}$ and Λ. Hence $G_{\alpha\beta\gamma} = 1$ for $\gamma \in \Omega - \{\alpha, \beta\}$ and G is a Zassenhaus group of order $t(p^2 + 1)p^2$ where $t \mid p - 1$. Since $p \neq 2$, this contradicts Feit (1960). Therefore $K_\beta \neq 1$.

By B 1 of O'Nan (1972), $F(K_\beta) = \{\alpha, \beta\}$. It follows that K_β fixes no point of Δ_2 so that K has at least two classes of subgroups of index p. This implies that K is nonsolvable and consequently K is doubly transitive on each Δ_i. By Theorem D of O'Nan (1975) we get that G is a normal extension of $PSL(n, q)$ for some $n \geqq 3$. This contradicts our lemma part b). Therefore the assumption that G is not doubly primitive is false and the theorem is proved.

Proof of Theorem B. Assume that G is not triply transitive. By Tsuzuku (1968), we can assume that $|G|_p = p^2$. Let Ω be the set on which G acts and let $\alpha, \beta \in \Omega, \alpha \neq \beta$. By assumption G_α contains two distinct Sylow p-subgroups. By Theorem A, G_α is primitive on $\Omega - \{\alpha\}$ and by assumption it is not doubly transitive. By Wielandt (1969), there is a subgroup N, of index 2 in G_α such that $N = X \times Y, X, Y$ intransitive on $\Omega - \{\alpha\}$. Since G_α is primitive, N is transitive so that X and Y have, each, p orbits of size p on $\Omega - \{\alpha\}$. Let P be a Sylow p-subgroup of G contained in N; then we can write $P = P_1 \times P_2, P_1 \subseteq X, P_2 \subseteq Y, |P_1| = |P_2| = p$. If X is not faithful on one of its orbits the kernel on this orbit must be transitive on some other orbit or else the kernel would fix Ω. This implies that $|X|_p \geqq p^2$ which is impossible. Thus X is faithful on its orbits. The same is true for Y.

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Let $\Lambda = \{\Lambda_i \mid 1 \leqq i \leqq p\}$ be the set of X-orbits on $\Omega - \{\alpha\}$ and let $\Gamma = \{\Gamma_i \mid 1 \leqq i \leqq p\}$ be the set of Y-orbits on $\Omega - \{\alpha\}$. Suppose X is solvable. Then $P_1 \lhd N$. Let $t \in G_\alpha - N$. Then $(P_1)^t \lhd N$ and if $(P_1)^t \neq P_1$ then $(P_1)^t P_1$ is a normal Sylow p-subgroup of N and therefore of G_α, contradicting the fact that G_α contains at least two Sylow p-subgroups. Thus $(P_1)^t = P_1$ and $P_1 \lhd G_\alpha$, contradicting the primitivity of G_α on $\Omega - \{\alpha\}$. We conclude that X is nonsolvable and therefore doubly transitive on each of its orbits. The same is true for Y.

Since Λ is a complete system of imprimitivity sets for the action of N on $\Omega - \{\alpha\}, N$ is transitive on Λ and therefore Y is transitive on Λ. If Y has a kernel, $V \neq 1$, on Λ then $|Y : V|_p = p$ and since $V \lhd Y$, V is either transitive or trivial on Γ_1. Since Y is faithful on Γ_1, V is transitive on it so that $|V|_p = p$. This implies that $p^2 \| |Y|$ which is impossible. Hence Y is faithful on Λ and since it is unsolvable, Y is doubly transitive on Λ. Certainly we can assume that $\Lambda_1 = \beta^X$ and $\Gamma_1 = \beta^Y$. Put $W = \{y \in Y \mid \Lambda_1 y = \Lambda_1\}$. Then $Y_\beta \subseteq W$ and since $|Y : Y_\beta| = |Y : W| = p$ we get that $W = Y_\beta$.

Hence Y_β is transitive on $\Lambda - \{\Lambda_1\}$. Since X is transitive on Λ_1 and $[X, Y_\beta] = 1$ we obtain that Y_β fixes Λ_1 pointwise. Thus $F(Y_\beta) = \Lambda_1 \cup \{\alpha\}$. By symmetry X_β is transitive on $\Gamma - \{\Gamma_1\}$ and $F(X_\beta) = \Gamma_1 \cup \{\alpha\}$. Now $p^2 = |N : N_\beta| = |X : X_\beta| |Y : Y_\beta|$ implies that $N_\beta = X_\beta \times Y_\beta$. The previous paragraphs imply that $\Gamma_1 - \{\beta\}$, $\Lambda_1 - \{\beta\}$ and $(\bigcup_{i=1} \Lambda_i) - \Gamma_1 - \Lambda_1$ are the N_β-orbits on $\Omega - \{\alpha, \beta\}$. Their sizes are $p-1$, $p-1$, $(p-1)^2$ respectively. Also, Γ_1, contains one point from each Λ_i.

Since $|G_{\alpha\beta} : N_\beta| = 2$ we can choose $t \in G_{\alpha\beta} - N_\beta$. We have that $G_{\alpha\beta} = N_\beta \langle t \rangle$ and $G_\alpha = N \langle t \rangle$ because $t^2 \in N_\beta$. Suppose that t fixes both $\Gamma_1 - \{\beta\}$ and $\Lambda_1 - \{\beta\}$ as sets. Then $(X_\beta)^t$ acts on each of these sets and $(X_\beta)^t$ fixes $\Gamma_1 - \{\beta\}$ pointwise. Thus $(X_\beta)^t \cap Y_\beta = 1$ as Y is faithful on Γ_1. Since $(X_\beta)^t \subseteq N_\beta$ and $(X_\beta)^t$ fixes Γ_1 pointwise we have that $(X_\beta)^t$ acts trivially on Λ. Then $(X_\beta)^t$ is contained in the kernel of the action of N_β on Λ, namely X_β. Hence $(X_\beta)^t = X_\beta$.

Let $g \in G_\alpha$ and put $g = t^i h, h \in N$ for some integer i. Then since $X \lhd N, (X_\beta)^g \cap G_{\alpha\beta} = (X_\beta)^h \cap G_{\alpha\beta} \subseteq X \cap G_{\alpha\beta} = X_\beta$. Thus X_β is a strongly closed subgroup of $G_{\alpha\beta}$ in G_α. We now apply our lemma and B of O'Nan (1972) to get a contradiction.

Therefore t does not fix $\Gamma_1 - \{\beta\}$ and $\Lambda_1 - \{\beta\}$ and since t normalizes N_β, it must interchange these sets. We conclude that G_α is a rank 3 group on $\Omega - \{\alpha\}$ and the sizes of the $G_{\alpha\beta}$-orbits are 1, $2(p-1)$, $(p-1)^2$. Using Higman (1970) we are done.

We remark that the proof of Theorem B is also a proof for the following extension of Wielandt (1969):

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THEOREM C. *Let G be a primitive but not doubly transitive permutation group of degree p^2. Assume that G_α contains two distinct Sylow p-subgroups. Then G is either rank 3 or rank 4 permutation group with sub-degrees* $1, 2(p-1), (p-1)^2$ or $1, (p-1), (p-1), (p-1)^2$.

In fact the rank 4 case does not occur because of Proposition 0.1 of Iwasaki (1973) that states that we are in case I and proposition 1.1 of Iwasaki (1973).

PROOF OF THE COROLLARY. By Theorems A and B and the lemma we can assume that $p \neq 2$, G_α contains a unique Sylow p-subgroup P and $|P| = p^2$. Now $P \lhd G_\alpha$ and since G_α is primitive, P is regular on $\Omega - \{\alpha\}$. By a result of Hering, Kantor and Seitz (1972) we get that G has a normal subgroup M such that $G \subsetneq \mathrm{Aut}(M)$, where M is either $PSL(2, p^2)$ or sharply 2-transitive, (because the degree is $p^2 + 1$). If M is sharply 2-transitive, so is G and $|G| = |M| = 20$ and $p = 2$. This proves the corollary.

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