Canad. Math. Bull. Vol. 62 (3), 2019 pp. 663–669
http://dx.doi.org/10.4153/S0008439518000036

Krivine's Function Calculus and Bochner Integration

V. G. Troitsky and M. S. Türer

Abstract. We prove that Krivine's Function Calculus is compatible with integration. Let (Ω, Σ, μ) be a finite measure space, X a Banach lattice, $\boldsymbol{x} \in X^n$, and $f : \mathbb{R}^n \times \Omega \to \mathbb{R}$ a function such that $f(\cdot, \omega)$ is continuous and positively homogeneous for every $\omega \in \Omega$, and $f(\boldsymbol{s}, \cdot)$ is integrable for every $\boldsymbol{s} \in \mathbb{R}^n$. Put $F(\boldsymbol{s}) = \int f(\boldsymbol{s}, \omega)\, d\mu(\omega)$ and define $F(\boldsymbol{x})$ and $f(\boldsymbol{x}, \omega)$ via Krivine's Function Calculus. We prove that under certain natural assumptions $F(\boldsymbol{x}) = \int f(\boldsymbol{x}, \omega)\, d\mu(\omega)$, where the right hand side is a Bochner integral.

1 Motivation

In [Kal12], the author defines a real-valued function of two real or complex variables via $F(s,t) = \int_0^{2\pi} \lvert s + e^{i\theta} t\rvert \, d\theta$. This is a positively homogeneous continuous function. Therefore, given two vectors u and v in a Banach lattice X, one may apply Krivine's Function Calculus to F and consider $F(u,v)$ as an element of X. The author then claims that

$$F(u,v) = \int_0^{2\pi} \lvert u + e^{i\theta} v\rvert \, d\theta, \tag{1}$$

where the right hand side here is understood as a Bochner integral; this is used later in [Kal12] to conclude that $\lVert F(u,v)\rVert \leqslant \int_0^{2\pi} \lVert u + e^{i\theta} v\rVert \, d\theta$ because Bochner integrals have this property: $\lVert \int f \rVert \leqslant \int \lVert f \rVert$. A similar exposition is also found in [DGTJ84, p. 146]. Unfortunately, neither [Kal12] nor [DGTJ84] includes a proof of (1). In this note, we prove a general theorem which implies (1) as a special case.

2 Preliminaries

We start by reviewing the construction of Krivine's Function Calculus on Banach lattices; see [LT79, Theorem 1.d.1] for details. For Banach lattice terminology, we refer the reader to [AA02, AB06].

Fix $n \in \mathbb{N}$. A function $F\colon \mathbb{R}^n \to \mathbb{R}$ is said to be *positively homogeneous* if

$$F(\lambda t_1, \dots, \lambda t_n) = \lambda F(t_1, \dots, t_n) \quad \text{for all } t_1, \dots, t_n \in \mathbb{R} \text{ and } \lambda \geqslant 0.$$

Received by the editors December 26, 2017; revised April 9, 2018.
Published online on Cambridge Core April 22, 2019.
The first author was supported by an NSERC grant.
AMS subject classification: **46B42,** 46A40.
Keywords: Banach lattice, function calculus, Bochner integral.

Let H_n be the set of all continuous positively homogeneous functions from $\mathbb{R}^n$ to $\mathbb{R}$. Let S^n_∞ be the unit sphere of ℓ^n_∞, that is,

$$S^n_\infty = \{(t_1,\dots,t_n) \in \mathbb{R}^n \;:\; \max_{i=1,\dots,n} |t_i| = 1\}.$$

It can be easily verified that the restriction map $F \mapsto F_{|S^n_\infty}$ is a lattice isomorphism from H_n onto $C(S^n_\infty)$. Hence, we can identify H_n with $C(S^n_\infty)$. For each $i = 1,\dots,n$, the i-th coordinate projection $\pi_i\colon \mathbb{R}^n \to \mathbb{R}$ clearly belongs to H_n.

Let X be a (real) Banach lattice and $\boldsymbol{x} = (x_1,\dots,x_n) \in X^n$. Let $e \in X_+$ be such that $x_1,\dots,x_n$ belong to I_e, the principal order ideal of e. For example, one could take $e = |x_1| \vee \cdots \vee |x_n|$. By Kakutani's representation theorem, the ideal I_e equipped with the norm

$$\|x\|_e = \inf\{\lambda > 0 \;:\; |x| \leqslant \lambda e\}$$

is lattice-isometric to $C(K)$ for some compact Hausdorff K. Let $F \in H_n$. Interpreting $x_1,\dots,x_n$ as elements of $C(K)$, we can define $F(x_1,\dots,x_n)$ in $C(K)$ as a composition. We may view it as an element of I_e and, therefore, of X; we also denote it by $\widetilde{F}$ or $\Phi(F)$. It may be shown that, as an element of X, it does not depend on the particular choice of e. This results in a (unique) lattice homomorphism $\Phi\colon H_n \to X$ such that $\Phi(\pi_i) = x_i$. The map Φ will be referred to as *Krivine's function calculus*. This construction allows one to define expressions like $(\sum_{i=1}^n |x_i|^p)^{\frac{1}{p}}$ for $0 < p < \infty$ in every Banach lattice X; this expression is understood as $\Phi(F)$ where $F(t_1,\dots,t_n) = (\sum_{i=1}^n |t_i|^p)^{\frac{1}{p}}$. Furthermore,

$$\big\| F(\boldsymbol{x}) \big\| \leqslant \|F\|_{C(S^n_\infty)} \cdot \Big\| \bigvee_{i=1}^n |x_i| \Big\|. \tag{2}$$

Let L_n be the sublattice of H_n or, equivalently, of $C(S^n_\infty)$, generated by the coordinate projections π_i as $i = 1,\dots,n$. It follows from the Stone–Weierstrass Theorem that L_n is dense in $C(S^n_\infty)$. It follows from $\Phi(\pi_i) = x_i$ that $\Phi(L_n)$ is the sublattice generated by $x_1,\dots,x_n$ in X, hence Range Φ is contained in the closed sublattice of X generated by $x_1,\dots,x_n$. It follows from, e.g., Exercise 8 on [AB06, p. 204] that this sublattice is separable.

Let (Ω,Σ,μ) be a finite measure space and X a Banach space. A function $f\colon\Omega \to X$ is *measurable* if there is a sequence (f_n) of simple functions from Ω to X such that $\lim_n \|f_n(\omega) - f(\omega)\| = 0$ almost everywhere. If, in addition, $\int \|f_n(\omega) - f(\omega)\|\, d\mu(\omega) \to 0$ then f is *Bochner integrable* with $\int_A f\,d\mu = \lim_n \int_A f_n\,d\mu$ for every measurable set A. In the following theorem, we collect a few standard facts about Bochner integral for future reference; we refer the reader to [DU77, Chapter II] for proofs and further details.

Theorem 2.1 *Let $f\colon\Omega \to X$.*

(i) *If f is the almost everywhere limit of a sequence of measurable functions then f is measurable.*

(ii) *If f is separable-valued and there is a norming set $\Gamma \subseteq X^*$ such that x^*f is measurable for every $x^* \in \Gamma$ then f is measurable.*

(iii) *A measurable function f is Bochner integrable iff $\|f\|$ is integrable.*

(iv) *If $f(\omega) = u(\omega)x$ for some fixed $x \in X$ and $u \in L_1(\mu)$ and for all ω then f is measurable and Bochner integrable.*

(v) *If f is Bochner integrable and $T: X \to Y$ is a bounded operator from X to a Banach space Y then $T(\int f\,d\mu) = \int Tf\,d\mu$.*

3 Main theorem

Throughout the rest of the paper, we assume that (Ω, Σ, μ) is a finite measure space, $n \in \mathbb{N}$, and $f: \mathbb{R}^n \times \Omega \to \mathbb{R}$ is such that $f(\cdot, \omega)$ is in H_n for every $\omega \in \Omega$ and $f(\boldsymbol{s}, \cdot)$ is integrable for every $\boldsymbol{s} \in \mathbb{R}^n$. For every $\boldsymbol{s} \in \mathbb{R}^n$, put $F(\boldsymbol{s}) = \int f(\boldsymbol{s}, \omega)\,d\mu(\omega)$. It is clear that F is positively homogeneous.

Suppose, in addition, that F is continuous. Let X be a Banach lattice, $\boldsymbol{x} \in X^n$, and $\Phi: H_n \to X$ the corresponding function calculus. Since $F \in H_n$, $\widetilde{F} = F(\boldsymbol{x}) = \Phi(F)$ is defined as an element of X. On the other hand, for every ω, the function $\boldsymbol{s} \in \mathbb{R}^n \mapsto f(\boldsymbol{s}, \omega)$ is in H_n, hence we may apply Φ to it. We denote the resulting vector by $\tilde{f}(\omega)$ or $f(\boldsymbol{x}, \omega)$. This produces a function $\omega \in \Omega \mapsto f(\boldsymbol{x}, \omega) \in X$.

Theorem 3.1 *Suppose that F is continuous and the function $M(\omega) := \|f(\cdot, \omega)\|_{C(S^n_\infty)}$ is integrable. Then $f(\boldsymbol{x}, \omega)$ is Bochner integrable as a function of ω and $F(\boldsymbol{x}) = \int f(\boldsymbol{x}, \omega)\,d\mu(\omega)$, where the right hand side is a Bochner integral.*

Proof *Special case.* Suppose that $X = C(K)$ for some compact Hausdorff K. By uniqueness of function calculus, Krivine's function calculus Φ agrees with "point-wise" function calculus. In particular,

$$\widetilde{F}(k) = F\big(x_1(k), \dots, x_n(k)\big) \quad \text{and} \quad \big(\tilde{f}(\omega)\big)(k) = f\big(x_1(k), \dots, x_n(k), \omega\big)$$

for all $k \in K$ and $\omega \in \Omega$. We view $\tilde{f}$ as a function from Ω to $C(K)$.

We are going to show that $\tilde{f}$ is Bochner integrable. It follows from $\tilde{f}(\omega) \in \operatorname{Range}\Phi$ that $\tilde{f}$ a separable-valued function. For every $k \in K$, consider the point-evaluation functional $\varphi_k \in C(K)^*$ given by $\varphi_k(x) = x(k)$. Then

$$\varphi_k\big(\tilde{f}(\omega)\big) = \big(\tilde{f}(\omega)\big)(k) = f\big(x_1(k), \dots, x_n(k), \omega\big)$$

for every $k \in K$. By assumption, this function is integrable; in particular, it is measurable. Since the set $\{\varphi_k : k \in K\}$ is norming in $C(K)^*$, Theorem 2.1(ii) yields that $\tilde{f}$ is measurable.

Clearly, $\big|\big(\tilde{f}(\omega)\big)(k)\big| \leqslant M(\omega)$ for every $k \in K$ and $\omega \in \Omega$, so that $\|\tilde{f}(\omega)\|_{C(K)} \leqslant M(\omega)$ for every ω. It follows that $\int \|\tilde{f}(\omega)\|_{C(K)}\,d\mu(\omega)$ exists and, therefore, $\tilde{f}$ is Bochner integrable by Theorem 2.1(iii).

Put $h := \int \tilde{f}(\omega)\,d\mu(\omega)$, where the right-hand side is a Bochner integral. Applying Theorem 2.1(v), we get

$$\begin{aligned} h(k) = \varphi_k(h) = \int \varphi_k\big(\tilde{f}(\omega)\big)\,d\mu(\omega) &= \int f\big(x_1(k), \dots, x_n(k), \omega\big)\,d\mu(\omega) \\ &= F\big(x_1(k), \dots, x_n(k)\big) = \widetilde{F}(k) \end{aligned}$$

for every $k \in K$. It follows that $\int \tilde{f}(\omega)\,d\omega = \widetilde{F}$.

General case. Let $e = |x_1| \vee \cdots \vee |x_n|$. Then $(I_e, \|\cdot\|_e)$ is lattice-isometric to $C(K)$ for some compact Hausdorff K. Note also that $|x| \leqslant \|x\|_e e$ for every $x \in I_e$; this yields $\|x\| \leqslant \|x\|_e \|e\|$, hence the inclusion map $T\colon (I_e, \|\cdot\|_e) \to X$ is bounded. Identifying I_e with $C(K)$, we may view T as a bounded lattice embedding from $C(K)$ into X.

By the construction on Krivine's Function Calculus, Φ actually acts into I_e, *i.e.*, $\Phi = T\Phi_0$, where Φ_0 is the $C(K)$-valued function calculus. By the special case, we know that $\int \tilde{f}(\omega)\, d\mu(\omega) = \widetilde{F}$ in $C(K)$. Applying T, we obtain the same identity in X by Theorem 2.1(v). ∎

Finally, we analyze whether any of the assumptions may be removed. Clearly, one cannot remove the assumption that F is continuous; otherwise, $\widetilde{F}$ would make no sense. The following example shows that, in general, F need not be continuous.

Example 3.2 Let $n = 2$, let μ be a measure on $\mathbb{N}$ given by $\mu(\{k\}) = 2^{-k}$. For each k, we define $f_k = f(\cdot, k)$ as follows. Note that it suffices to define f_k on S_∞^2. Let I_k be the straight line segment connecting $(1,0)$ and $(1, 2^{-k+1})$. Define f_k so that it vanishes on $S_\infty^2 \setminus I_k$, $f_k(1,0) = f_k(1, 2^{-k+1}) = 0$, $f_k(1, 2^{-k}) = 2^k$, and is linear on each half of I_k. Then $f_k \in H_2$ and $F(\boldsymbol{s})$ is defined for every $\boldsymbol{s} \in \mathbb{R}^2$. It follows from $F(\boldsymbol{s}) = \sum_{k=1}^{\infty} 2^{-k} f_k(\boldsymbol{s})$ that $F(1,0) = 0$ and $F(1, 2^{-k}) \geqslant 2^{-k} f_k(1, 2^{-k}) = 1$, hence F is discontinuous at $(1,0)$.

The assumption that M is integrable cannot be removed as well. Indeed, consider the special case when $X = C(S_\infty^n)$ and $x_i = \pi_i$ as $i = 1, \ldots, n$. In this case, Φ is the identity map and $\tilde{f}(\omega) = f(\cdot, \omega)$. It follows from Theorem 2.1(iii) that $\tilde{f}$ is Bochner integrable iff $\|\tilde{f}\|$ is integrable iff M is integrable.

Finally, the assumption that $f(\cdot, \omega)$ is in H_n for every ω may clearly be relaxed to "for almost every ω".

4 Direct proof

In the previous section, we presented a proof of Theorem 3.1 using representation theory. In this section, we present a direct proof. However, we impose an additional assumption: we assume that $f(\cdot, \omega)$ is continuous on S_∞^n uniformly on ω, that is,

(3) for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\big|f(\boldsymbol{s}, \omega) - f(\boldsymbol{t}, \omega)\big| < \varepsilon$ for all $\boldsymbol{s}, \boldsymbol{t} \in S_\infty^n$ and all $\omega \in \Omega$ provided that $\|\boldsymbol{s} - \boldsymbol{t}\|_\infty < \delta$.

In Theorem 3.1, we assumed that F was continuous and M was integrable. Now these two conditions are satisfied automatically. In order to see that F is continuous, fix $\varepsilon > 0$; let δ be as in (3), then

$$\big|F(\boldsymbol{s}) - F(\boldsymbol{t})\big| \leqslant \int \big|f(\boldsymbol{s}, \omega) - f(\boldsymbol{t}, \omega)\big|\, d\mu(\omega) < \varepsilon\mu(\Omega) \tag{4}$$

whenever $\boldsymbol{s}, \boldsymbol{t} \in S_\infty^n$ with $\|\boldsymbol{s} - \boldsymbol{t}\|_\infty < \delta$. The proof of integrability of M will be included in the proof of the theorem.

Theorem 4.1 *Suppose that $f(\cdot, \omega)$ is continuous on S_∞^n uniformly on ω. Then $f(\boldsymbol{x}, \omega)$ is Bochner integrable as a function of ω and $F(\boldsymbol{x}) = \int f(\boldsymbol{x}, \omega)\, d\mu(\omega)$.*

https://doi.org/10.4153/S0008439518000036 Published online by Cambridge University Press

Proof Without loss of generality, by scaling μ and $\boldsymbol{x}$, we may assume that μ is a probability measure and $\|\bigvee_{i=1}^n |x_i|\| = 1$; this will simplify computations. In particular, (2) becomes $\|H(\boldsymbol{x})\| \leqslant \|H\|_{C(S_\infty^n)}$ for every $H \in C(S_\infty^n)$. Note also that $\boldsymbol{x}$ in the theorem is a "fake" variable as $\boldsymbol{x}$ is fixed. It may be more accurate to write $\widetilde{F}$ and $\tilde{f}(\omega)$ instead of $F(\boldsymbol{x})$ and $f(\boldsymbol{x}, \omega)$, respectively. Hence, we need to prove that $\tilde{f}$ as a function from Ω to X is Bochner integrable and its Bochner integral is $\widetilde{F}$.

Fix $\varepsilon > 0$. Let δ be as in (3). It follows from (4) that

$$(5) \qquad \left|F(\boldsymbol{s}) - F(\boldsymbol{t})\right| < \varepsilon \quad \text{whenever } \boldsymbol{s}, \boldsymbol{t} \in S_\infty^n \text{ with } \|\boldsymbol{s} - \boldsymbol{t}\|_\infty < \delta.$$

Each of the $2n$ faces of S_∞^n is a translate of the $(n-1)$-dimensional unit cube B_∞^{n-1}. Partition each of these faces into $(n-1)$-dimensional cubes of diameter less than δ, where the diameter is computed with respect to the $\|\cdot\|_\infty$-metric. Partition each of these cubes into simplices. Therefore, there exists a partition of the entire S_∞^n into finitely many simplices of diameter less than δ. Denote the vertices of these simplices by $\boldsymbol{s}_1, \dots, \boldsymbol{s}_{\boldsymbol{m}}$. Thus, we have produced a triangularization of S_∞^n with nodes $\boldsymbol{s}_1, \dots, \boldsymbol{s}_{\boldsymbol{m}}$.

Let $\boldsymbol{a} \in \mathbb{R}^m$. Define a function $L \colon S_\infty^n \to \mathbb{R}$ by setting $L(\boldsymbol{s}_j) = a_j$ as $j = 1, \dots, m$ and then extending it to each of the simplices linearly; this can be done because every point in a simplex can be written in a unique way as a convex combination of the vertices of the simplex. We write $L = T\boldsymbol{a}$. This gives rise to a linear operator $T \colon \mathbb{R}^m \to C(S_\infty^n)$. For each $j = 1, \dots, m$, let e_j be the j-th unit vector in $\mathbb{R}^m$; put $d_j = Te_j$. Clearly,

$$(6) \qquad T\boldsymbol{a} = \sum_{j=1}^m a_j d_j \quad \text{for every } \boldsymbol{a} \in \mathbb{R}^m.$$

Let $H \in C(S_\infty^n)$. Let $L = T\boldsymbol{a}$ where $a_j = H(\boldsymbol{s}_j)$. Then L agrees with H at $\boldsymbol{s}_1, \dots, \boldsymbol{s}_m$. We write $L = SH$; this defines a linear operator $S \colon C(S_\infty^n) \to C(S_\infty^n)$. Clearly, this is a linear contraction.

Suppose that $H \in C(S_\infty^n)$ is such that $\left|H(\boldsymbol{s}) - H(\boldsymbol{t})\right| < \varepsilon$ whenever $\|\boldsymbol{s} - \boldsymbol{t}\|_\infty < \delta$. Let $L = SH$. We claim that $\left\|L - H\right\|_{C(S_\infty^n)} < \varepsilon$. Indeed, fix $\boldsymbol{s} \in S_\infty^n$. Let $\boldsymbol{s}_{j_1}, \dots, \boldsymbol{s}_{j_n}$ be the vertices of a simplex in the triangularization of S_∞^n that contains $\boldsymbol{s}$. Then $\boldsymbol{s}$ can be written as a convex combination $\boldsymbol{s} = \sum_{k=1}^n \lambda_k \boldsymbol{s}_{j_k}$. Note that $\|\boldsymbol{s} - \boldsymbol{s}_{j_k}\|_\infty < \delta$ for all $j = 1, \dots, n$. It follows that

$$\left|L(\boldsymbol{s}) - H(\boldsymbol{s})\right| = \Big|\sum_{k=1}^n \lambda_k L(\boldsymbol{s}_{j_k}) - \sum_{k=1}^n \lambda_k H(\boldsymbol{s})\Big| \leqslant \sum_{j=1}^n \lambda_k \left|H(\boldsymbol{s}_{j_k}) - H(\boldsymbol{s})\right| < \varepsilon.$$

This proves the claim.

Let $G = SF$. Then (5) and the preceding observation yield $\|G - F\|_{C(S_\infty^n)} < \varepsilon$, so

$$(7) \qquad \left\|G(\boldsymbol{x}) - F(\boldsymbol{x})\right\| < \varepsilon.$$

Similarly, for every $\omega \in \Omega$, apply S to $f(\cdot, \omega)$ and denote the resulting function $g(\cdot, \omega)$. In particular, $g(\boldsymbol{s}_j, \omega) = f(\boldsymbol{s}_j, \omega)$ for every $\omega \in \Omega$ and every $j = 1, \dots, m$. It follows also that

$$\left\|f(\cdot, \omega) - g(\cdot, \omega)\right\|_{C(S_\infty^n)} < \varepsilon$$

for every ω, and therefore

$$\|\tilde{f}(\omega)-\tilde{g}(\omega)\| = \|f(\boldsymbol{x},\omega)-g(\boldsymbol{x},\omega)\| < \varepsilon, \tag{8}$$

where $\tilde{g}(\omega) = g(\boldsymbol{x},\omega)$ is the image under Φ of the function $\boldsymbol{s}\in S^n_\infty \mapsto g(\boldsymbol{s},\omega)$. Note that

$$G(\boldsymbol{s}_j) = F(\boldsymbol{s}_j) = \int f(\boldsymbol{s}_j,\omega)\,d\mu(\omega) = \int g(\boldsymbol{s}_j,\omega)\,d\mu(\omega) \tag{9}$$

for every $j=1,\dots,m$. Since $G = SF = T\boldsymbol{a}$ where $a_j = F(\boldsymbol{s}_j) = G(\boldsymbol{s}_j)$ as $j=1,\dots,m$, it follows from (6) that

$$G = \sum_{j=1}^m G(\boldsymbol{s}_j)d_j. \tag{10}$$

Similarly, for every $\omega\in\Omega$, we have

$$g(\cdot,\omega) = \sum_{j=1}^m g(\boldsymbol{s}_j,\omega)d_j. \tag{11}$$

Applying Φ to (10) and (11), we obtain $\widetilde{G} = G(\boldsymbol{x}) = \sum_{j=1} G(\boldsymbol{s}_j)d_j(\boldsymbol{x})$ and

$$\tilde{g}(\omega) = g(\boldsymbol{x},\omega) = \sum_{j=1}^m g(\boldsymbol{s}_j,\omega)d_j(\boldsymbol{x}) = \sum_{j=1}^m f(\boldsymbol{s}_j,\omega)d_j(\boldsymbol{x}).$$

Together with Theorem 2.1(iv), this yields that $\tilde{g}$ is measurable and Bochner integrable. It now follows from (9) and (10) that

$$\begin{aligned} G(\boldsymbol{x}) &= \sum_{j=1} G(\boldsymbol{s}_j)d_j(\boldsymbol{x}) = \sum_{j=1}^m\Big(\int g(\boldsymbol{s}_j,\omega)d\mu(\omega)\Big)\,d_j(\boldsymbol{x}) \\ &= \int\Big(\sum_{j=1}^m g(\boldsymbol{s}_j,\omega)d_j(\boldsymbol{x})\Big)\,d\mu(\omega) = \int g(\boldsymbol{x},\omega)\,d\mu(\omega). \end{aligned} \tag{12}$$

We will show next that $\tilde{f}$ is Bochner integrable. It follows from (8) and the fact that ε is arbitrary that $\tilde{f}$ can be approximated almost everywhere (actually, everywhere) by measurable functions; hence $\tilde{f}$ is measurable by Theorem 2.1(i). Next, we claim that there exists $\lambda\in\mathbb{R}_+$ such that $|f(\boldsymbol{s},\omega)-f(\mathbf{1},\omega)|\leqslant\lambda$ for all $\boldsymbol{s}\in S^n_\infty$ and all $\omega\in\Omega$. Here $\mathbf{1}=(1,\dots,1)$. Indeed, let $\boldsymbol{s}\in S^n_\infty$ and $\omega\in\Omega$. Find $j_1,\dots,j_l$ such that $\boldsymbol{s}_{j_1}=\mathbf{1}$, $\boldsymbol{s}_{j_k}$ and $\boldsymbol{s}_{j_{k+1}}$ belong to the same simplex as $k=1,\dots,l-1$, and $\boldsymbol{s}_{j_l}$ is a vertex of a simplex containing $\boldsymbol{s}$. It follows that

$$|f(\boldsymbol{s},\omega)-f(\mathbf{1},\omega)| \leqslant |f(\boldsymbol{s},\omega)-f(\boldsymbol{s}_{j_l},\omega)| + \sum_{k=1}^{l-1}|f(\boldsymbol{s}_{j_{k+1}},\omega)-f(\boldsymbol{s}_{j_k},\omega)| \leqslant l\varepsilon\leqslant m\varepsilon.$$

This proves the claim, with $\lambda = m\varepsilon$. It follows that

$$\|\tilde{f}(\omega)\| \leqslant \|f(\cdot,\omega)\|_{C(S^n_\infty)} = \sup_{s\in S^n_\infty}|f(\boldsymbol{s},\omega)| \leqslant |f(\mathbf{1},\omega)|+\lambda.$$

https://doi.org/10.4153/S0008439518000036 Published online by Cambridge University Press

Since $\left|f(\mathbf{1},\omega)\right| + \lambda$ is an integrable function of ω, we conclude that $\|\tilde{f}\|$ is integrable, hence $\tilde{f}$ is Bochner integrable by Theorem 2.1(iii). It now follows from (8) that

$$\left\| \int f(\boldsymbol{x},\omega)\,d\mu(\omega) - \int g(\boldsymbol{x},\omega)\,d\mu(\omega) \right\| \leqslant \int \left\| f(\boldsymbol{x},\omega) - g(\boldsymbol{x},\omega)\right\| \,d\mu(\omega) < \varepsilon. \tag{13}$$

Finally, combining (7), (12), and (13), we get

$$\left\| F(\boldsymbol{x}) - \int f(\boldsymbol{x},\omega)\,d\mu(\omega)\right\| < 2\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, this proves the theorem. ■

Some of the work on this paper was done during a visit of the second author to the University of Alberta. We would like to thank the referee whose helpful remarks and suggestions considerably improved this paper.

References

[AA02] Y. A. Abramovich and C. D. Aliprantis, *An invitation to operator theory*. Graduate Studies in Mathematics, 50, American Mathematical Society, Providence, RI, 2002.

[AB06] C. D. Aliprantis and O. Burkinshaw, *Positive operators*. Springer, Dordrecht, 2006. Reprint of the 1985 original.

[DGTJ84] W. J. Davis, D. J. H. Garling, and N. Tomczak-Jaegermann, *The complex convexity of quasinormed linear spaces*. J. Funct. Anal. 55(1984), no. 1, 110–150.

[DU77] J. Diestel and J. J. Uhl Jr., *Vector measures*. Mathematical Surveys, 15, American Mathematical Society, Providence, RI, 1977.

[Kal12] N. J. Kalton, *Hermitian operators on complex Banach lattices and a problem of Garth Dales*. J. Lond. Math. Soc. (2) 86(2012), no. 3, 641–656.

[LT79] J. Lindenstrauss and L. Tzafriri, *Classical Banach spaces. II*. Function spaces, Springer-Verlag, Berlin, 1979.

Department of Mathematical and Statistical Sciences, University of Alberta, Edmonton, AB, T6G 2G1, Canada
e-mail: troitsky@ualberta.ca

Department of Mathematics and Computer Science, İstanbul Kültür University, Bakırköy 34156, İstanbul, Turkey
e-mail: m.turer@iku.edu.tr