REMARKS ON THE DIFFERENTIAL FORMS OF THE FIRST KIND ON ALGEBRAIC VARIETIES

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§1. A differential form ω on a complete variety $\mathbf{U}^n$ is said to be of the first kind if it is finite at every simple point of any variety which is birationally equivalent to $\mathbf{U}$. Let k be a common field of definition for $\mathbf{U}$ and ω, and let $\mathbf{P}$ be a generic point of $\mathbf{U}$ over k. If ω is of the first kind, then $\omega(\mathbf{P})$ is of course a differential form of the first kind belonging to the extension $k(\mathbf{P})$ of k. With respect to the converse, we prove the following

THEOREM 1. *Let k be a field of definition for a complete variety $\mathbf{U}^n$ and a differential form ω on $\mathbf{U}$, and let $\mathbf{P}$ be a generic point of $\mathbf{U}$ over k. Let k be a perfect*[1] *field or more generally let k have a perfect*[1] *subfield which is a field of definition for $\mathbf{U}$. If $\omega(\mathbf{P})$ is a differential form of the first kind belonging to the extension $k(\mathbf{P})$ of k, then ω is of the first kind.*[2]

Proof. Let V be a variety which is birationally equivalent to $\mathbf{U}$ and let K be a field of definition of the birational correspondence between $\mathbf{U}$ and V. We may assume without loss of generality that K is algebraically closed and contains k and that $\mathbf{P}$ is a generic point of $\mathbf{U}$ over K. We want to show that ω is finite at every simple point of V. It suffices to show that $\omega(\mathbf{P})$, considered as the differential form belonging to the extension $K(\mathbf{P})$ of K, is of the first kind or that $\omega(\mathbf{P})$ is finite at every prime divisor $\mathfrak{P}$ in the sense of Zariski of $K(\mathbf{P})$ (= valuation of $K(\mathbf{P})$ of dimension $n-1$ over K), namely, that $\omega(\mathbf{P})$ is of the form

$$\omega(\mathbf{P}) = \sum z_{\alpha\beta}\ldots dy_\alpha dy_\beta \ldots \;;$$

$z_{\alpha\beta}\ldots$, y_α, y_β, etc. being in the valuation ring of $\mathfrak{P}$.[3]

We first prove

LEMMA. *Let K be a field, k a subfield of K; let (x) be a set of quantities, such that K and $k(x)$ are independent over k. Then if v is a valuation of $K(x)$*

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[1] If we omit the condition of perfectness this theorem does not hold in general.

[2] If the problem of the reduction of the singularity over perfect field is solved affirmatively, this theorem is an immediate consequence of theorem 1 of S. Koizumi's paper; On the differential forms of the first kind on algebraic varieties, Journal of the Mathematical Society of Japan, Vol. 2. However it would not be meaningless to give a simple direct proof.

[3] See Y. Kawahara, On the differential forms on algebraic varieties, this journal, Vol. 4, Theorem 1.

of dimension s over K, the induced valuation v' of $k(x)$ is of dimension not smaller than s over k.

Proof. Let R denote the valuation ring of v in $K(x)$, A its valuation ideal, and let R' denote the valuation ring of v' in $k(x)$, A' its valuation ideal. As v is of dimension s, there are s elements $y_1, \ldots, y_s$ in R which are algebraically independent mod A over K. Let K_0 be a finite extension field of k, such that all $y_1, \ldots, y_s$ belong to $K_0(x)$. Then v induces the valuation of $K_0(x)$ of dimension $\geqq s$ over K_0. Therefore we may assume that K is a finite extension field of k.

$$R/A \supseteqq R'/A' \supseteqq k,$$
$$R/A \supseteqq \quad K \quad \supseteqq k.$$

Let the dimension of K over k be t. Then R/A is of dimension $s+t$ over k. On the other hand the dimension of R/A over R'/A' is $\leqq t$. For, if $Z_1, \ldots, Z_{t+1}$ are $t+1$ elements in R, then as the dimension of $K(x)$ over $k(x)$ is t, there is an algebraic relation among them:

$$\sum a_{r_1 \ldots r_{t+1}} Z_1^{r_1} \ldots Z_{t+1}^{r_{t+1}} = 0$$

where all $a_{r_1 \ldots r_{t+1}}$ belong to $k(x)$. We may assume that all $a_{r_1 \ldots r_{t+1}}$ belong to R' and there is an element among them which does not belong to A'. Considering this relation mod A, we see that the dimension of R/A over R'/A' is $\leqq t$. Therefore the dimension of R'/A' over k is $\geqq s$.

From this lemma we see that $\mathfrak{P}$ induces in $k(\mathbf{P})$ the valuation $\mathfrak{p}$ of dimension at least $n-1$. As $\omega(\mathbf{P})$ is the differential form of the first kind belonging to the extension $k(\mathbf{P})$ of k, $\omega(\mathbf{P})$ is finite at $\mathfrak{p}$; hence $\omega(\mathbf{P})$ is finite at $\mathfrak{P}$. This completes the proof of Theorem 1.

§ **2.** We prove the following

THEOREM 2. *Let U^n be a projective model without singular point and let ω be a differetial form on U^n, defined over k. Let U'^{n-1} be the generic hyperplane section of U^n (over k) on which ω induces the differential form ω' of the first kind. Then ω is of the first kind.*[4]

Proof. Let $\mathbf{U}'$ be the intersection of $\mathbf{U}^n$ and a hyperplane $\mathbf{H}$ defined by a homogeneous equation

$$\sum_{i=0}^{N} u_i X_i = 0$$

in $\mathbf{P}^N$. where $u_0, u_1, \ldots, u_N$ are algebraically independent over k. Let $\mathbf{W}^{n-1}$ be a subvariety of $\mathbf{U}^n$ which is algebraic over k and let $\mathbf{W}'^{n-2}$ be a component of $\mathbf{W} \cap \mathbf{H}$, which is contained in $\mathbf{U}'$.

[4] This theorem has been proved also by S. Koizumi.

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Without loss of generality we may assume that $\mathbf{W}^{n-1}$ has a representative $\mathbf{W}_0^{n-1}$. Put $K=k(u_0, \ldots, u_N)$ and let $P=(x)$ be a generic point of U_0' over $\bar{K}$ and Q a generic point of $\mathbf{W}_0'$ over $\bar{K}$. Then P is also a generic point of U_0 over k and Q is a generic point of W_0 over $\bar{k}$. Further we may assume that $x_1, \ldots, x_n$ is a set of uniformizing parameters for U_0 at Q. Then it is easily seen that $x_2, \ldots, x_n$ is a set of uniformizing parameters for U_0' at Q. For simplicity we assume that ω is a simple differential form. We may treat the case of the differential form of the higher degrees analogously.

Let ω be defined by $\omega(P)=\sum_{i=1}^{n} a_i dx_i$, $a_i \in k(P)$. Then ω' is defined over $\bar{K}$ by $\omega'(P)=\sum_{i=1}^{n} a_i dx_i$, where $\sum_{i=1}^{n} a_i dx_i$ is considered as the differential form belonging to the extension $\bar{K}(P)$ of $\bar{K}$. Now since

$$u_0+u_1x_1+\ldots+u_Nx_N=0,$$
$$\sum_{j=1}^{N} u_j dx_j = 0, \text{ i.e.}$$
$$-u_1dx_1=\sum_{k=2}^{N} u_k dx_k.$$

If we put $dx_k=\sum_{i=1}^{n} b_{ki}dx_i$, $b_{ki}\in k(P)$, $k=n+1, \ldots. N$, we get

$$-u_1dx_1=\sum_{i=2}^{n} u_i dx_i+\sum_{k=n+1}^{N} u_k dx_k$$
$$=\sum_{i=2}^{n}\Big(u_i+\sum_{k=n+1}^{N} u_k b_{ki}\Big)dx_i+\sum_{k=n+1}^{N} u_k b_{k1} dx_1.$$
$$-dx_1=\sum_{i=2}^{n}\frac{\left(u_i+\sum_{k=n+1}^{N} u_k b_{ki}\right)}{\left(u_1+\sum_{k=n+1}^{N} u_k b_{k1}\right)}dx_i,$$
$$\omega'(P)=\sum_{i=2}^{n}\left(a_i-a_1\frac{\left(u_i+\sum_{k=n+1}^{N} u_k b_{ki}\right)}{\left(u_1+\sum_{k=n+1}^{N} u_k b_{k1}\right)}\right)dx_i.$$

By the assumptions that ω' is of the first kind and $x_2, \ldots, x_n$ form a set of uniformizing parameters,

$$A_i=a_i-a_1\frac{u_i+\sum_k u_k b_{ki}}{u_1+\sum_k u_k b_{k1}}$$

is in the specialization ring of Q in $\bar{K}(P)$, therefore A_i has a finite specialization over $P\to Q$ with respect to $\bar{K}$, and hence it has a finite specialization over $P\to Q$ with respect to $k(u_1, \ldots, u_N)$. Now as P is a generic point of U_0^n over $k(u_1, \ldots, u_N)$ and Q is a generic point of W_0^{n-1} over $\bar{k}(u_1, \ldots, u_N)$, either a or $1/a$ is in the specialization ring $\mathfrak{O}$ of Q in $k(u_1, \ldots, u_N)(P)$, where a is an arbitrary element in $k(u_1, \ldots, u_N)(P)$. Therefore A_i must be in the specialization ring $\mathfrak{O}$; moreover since $1/(u_1 + \sum_{k=n+1}^{N} u_k b_{k1})$ is in $\mathfrak{O}$,

$$A_i/(u_1 + \sum_k u_k b_{k1}) = a_i u_1 + a_1 u_i + \sum_{k=n+1}^{N} u_k \ (a_i b_{k1} + a_1 b_{ki})$$

is in $\mathfrak{O}$ for $i = 2, \ldots, n$, where a_i and b_{kj} are in $k(P)$. As $u_1, \ldots, u_N$ are algebraically independent over $k(P)$, a_i and a_1 must belong to the specialization ring of Q in $k(P)$.[5] This shows that $\omega(P)$ is finite at Q. Since ω is finite at the generic point of every $(n-1)$-dimensional subvariety of U, ω is of the first kind.[6]

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[5] See A. Weil's book, Foundations of Algebraic Geometry, Prop. 8 in Chapter IV.
[6] See Prop. 4 of Koizumi's paper loc. cit. 2).

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