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DIAMETRICALLY CONTRACTIVE MAPPINGS

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A contractive mapping on a complete metric space may fail to have a fixed point. Diametrically contractive mappings are introduced and it is shown that a diametrically contractive self-mapping of a weakly compact subset of a Banach space always has a fixed point.

1. INTRODUCTION

Let (M, d) be a complete metric space. Recall that a contraction on M is a self-mapping T of M such that

$$d(Tx, Ty) \leqslant \alpha d(x, y) \text{ for all } x, y \in M \tag{1}$$

where $\alpha \in [0, 1)$ is a constant. Recall also that a mapping $T : M \to M$ is contractive if

$$d(Tx, Ty) < d(x, y) \text{ for all } x, y \in M,\ x \neq y. \tag{2}$$

It is well-known that Banach's contraction mapping principle ensures that any contraction T on M has a unique fixed point u and for each $x \in M$, $T^n x \to u$. However a contractive mapping may fail to have a fixed point as shown in the following examples.

EXAMPLE 1.1. Let $M = \mathbb{R}$ be equipped with the usual distance and define $T : M \to M$ by $Tx = \ln(1 + e^x)$ for $x \in M$. Then it is easy to see that T is contractive and fixed point free.

EXAMPLE 1.2. (see [3]) Let $X = C[0, 1]$, $M = \{f \in X : 0 \leqslant f(t) \leqslant 1, t \in [0, 1], f(0) = 0, f(1) = 1\}$, and $T : M \to M$ defined by $Tf(x) = xf(x)$ for $x \in [0, 1]$. Then it is easily seen that T is contractive and fixed point free. Note that in this example, M is a closed bounded convex subset of X, but not weakly compact.

Some known partial results are summarised below (see [**2**, p. 18]).

PROPOSITION 1.3. *Let (M, d) be a complete metric space and let $T : M \to M$ be a contractive mapping.*

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(i) *If, for some $x_0 \in M$, the sequence $\{T^n x_0\}$ has a convergent subsequence, then T has a unique fixed point.*

(ii) *If $\overline{TM}$ is compact (that is, T is a compact mapping), then T has a unique fixed point u and for each $x \in M$, $T^n x \to u$.*

Contractive mappings have not been paid much attention; indeed the following fundamental question has been standing open so far.

QUESTION 1.4. ([1]) *Let M be a weakly compact subset of a Banach space and let $T : M \to M$ be a contractive mapping. Does T have a fixed point?*

In this note, we shall present a partial answer to Question 1.4. More precisely, we shall introduce diametrically contractive mappings and show that a diametrically contractive self-mapping of a weakly compact subset of a Banach space always has a fixed point. It is however unclear whether weak compactness can be weakened to boundedness and weak closedness. Moreover, if, in the framework of complete metric spaces, we strengthen diametrical contractivity to asymptotically diametrical contractivity, then T has a fixed point provided it has a bounded trajectory.

2. Diametrically Contractive Mappings

Since a contractive mapping may fail to have a fixed point even in one-dimensional space, we shall strengthen the contractive condition to hope for the existence of a fixed point.

DEFINITION 2.1: A mapping T on a complete metric space (M, d) is said to be diametrically contractive if

(3) $$\delta(TA) < \delta(A) \text{ for all closed subsets } A \subset M \text{ such that } 0 < \delta(A) < \infty.$$

(Here $\delta(A) := \sup\{d(x,y) : x, y \in A\}$ is the diameter of $A \subset M$.)

PROPOSITION 2.2. *Some basics on the relations between contractive and diametrically contractive mappings are collected below.*

(a) *A diametrically contractive mapping is contractive.*

(b) *In a finite-dimensional Banach space, a contractive mapping is also diametrically contractive.*

(c) *In an infinite-dimensional Banach space, a contractive mapping may fail to be diametrically contractive.*

PROOF: To see (a), take the set A in (3) to be the subset of two arbitrarily distinct points x and y to get the contractivity condition (2).

To see (b), let X be finite-dimensional and $T : M \to M$ be contractive. Given any closed subset A of M such that $0 < \delta(A) < \infty$. Since A is indeed compact, we have that

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$T(A)$ is compact since T is continuous. Hence we can take $x, y \in A$, $x \neq y$, such that $\delta\big(T(A)\big) = d(Tx, Ty)$. Using the contractivity of T, we get $\delta\big(T(A)\big) < d(x, y) \leqslant \delta(A)$. Consequently, T is diametrically contractive.

Finally to see (c), we show that the mapping T given in Example 1.2 works; that is, T is not diametrically contractive. Indeed, let $A = \overline{\{f_n : n \geqslant 1\}}$, where $f_n(x) = x^n$ for $n \geqslant 1$. Then, observing that the function

$$h(t) := t^{t/(1-t)}(1-t), \quad 0 < t < 1$$

is decreasing on $(0,1)$ and $\lim\limits_{t \to 0^+} h(t) = 1$, we get

$$\begin{aligned}
\sup_{f,g \in A} d(Tf, Tg) &= \sup_{n > m \geqslant 1} \max_{t \in [0,1]} (x^{m+1} - x^{n+1}) \\
&= \sup_{n > m \geqslant 1} \left(\frac{m+1}{n+1}\right)^{((m+1)/(n+1))/(1-(m+1)/(n+1))} \left(1 - \frac{m+1}{n+1}\right) \\
&= \sup_{0 < t < 1} t^{t/(1-t)}(1-t) \\
&= \lim_{t \to 0^+} t^{t/(1-t)}(1-t) \\
&= 1.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\sup_{f,g \in A} d(f, g) &= \sup_{n > m \geqslant 1} \max_{t \in [0,1]} (x^m - x^n) \\
&= \sup_{n > m \geqslant 1} \left(\frac{m}{n}\right)^{(m/n)/(1-m/n)} \left(1 - \frac{m}{n}\right) \\
&= \sup_{0 < t < 1} t^{t/(1-t)}(1-t) \\
&= 1.
\end{aligned}$$

Hence $\sup\limits_{f,g \in A} d(Tf, Tg) = \sup\limits_{f,g \in A} d(f, g)$ and T is not diametrically contractive. ∎

We next present the main result of this note.

Theorem 2.3. *Let M be a weakly compact subset of a Banach space X and let $T : M \to M$ be a diametrically contractive mapping. Then T has a fixed point.*

Proof: Let

$$\mathcal{F} = \{A \subset M : A \text{ a weakly compact nonempty subset of } M \text{ and } T\text{-invariant}\}.$$

(Recall that A is T-invariant if $T(A) \subset A$.) Since $M \in \mathcal{F}$, $\mathcal{F} \neq \emptyset$. $\mathcal{F}$ is partially ordered by

$$A_1, A_2 \in \mathcal{F}, \ A_1 \prec A_2 \ \Leftrightarrow \ A_1 \subset A_2.$$

Every chain $\mathcal{J}$ in $\mathcal{F}$ has the finite intersection property (that is, the intersection of any finitely many members in $\mathcal{J}$ is nonempty). It follows from the weak compactness of M

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that the intersection $B := \cap\{A : A \in \mathcal{J}\} \neq \emptyset$ belongs to $\mathcal{F}$. Obviously, this B is a lower bound for $\mathcal{J}$. So Zorn's lemma is applicable to get a minimal element, denoted A, in $\mathcal{F}$. Put $A_0 = \overline{TA}^w$. (Here $\overline{K}^w$ denotes the weak closure of a subset $K \subset X$.) Since A is weakly compact and T-invariant, we see that $A_0 \subset A$; hence $TA_0 \subset TA \subset \overline{TA}^w = A_0$; that is, A_0 is also T-invariant and $A_0 \in \mathcal{F}$. By minimality of A, we must have $A_0 = A$; or $A = \overline{TA}^w$. Since the norm of X is lower weakly semicontinuous, we deduce that $\delta(A) = \delta(\overline{TA}^w) = \delta(TA)$. So by the diametrical contractivity of T, we must have $\delta(A) = 0$; that is, A consists of exactly one point, ξ (say). Since A is also T-invariant, $T\xi = \xi$. ∎

REMARK 2.4. The proof above to Theorem 2.3 is not constructive. It is unknown whether, under the assumption of Theorem 2.3, each trajectory $\{T^n x\}$ converges to the fixed point ξ.

If we replace the diametrical contractivity by the stronger asymptotically diametrical contractivity introduced below, then T always has a fixed point provided it has a bounded orbit.

DEFINITION 2.5: A mapping T on a complete metric space (M,d) is said to be asymptotically diametrically contractive if

$$\delta_a(\{TA_n\}) < \delta_a(\{A_n\}) \tag{4}$$

for all decreasing sequences $\{A_n\}$ of closed bounded subsets of M with $\delta_a(\{A_n\}) > 0$. (Here $\delta_a(\{A_n\}) := \lim_{n\to\infty} \delta(A_n)$ is called the asymptotic diameter of the sequence $\{A_n\}$.)

Note that asymptotically diametrical contractivity implies diametrical contractivity. Note also that if (M,d) is a compact metric space, then contractivity implies asymptotically diametrical contractivity. To see this, take a decreasing sequence $\{A_n\}$ of closed subsets of M such that $\delta_a(\{A_n\}) > 0$. Since each A_n is now compact, we have x_n, $y_n \in A_n$ such that $d(Tx_n, Ty_n) = \delta(TA_n)$, and with no loss of generality, we may assume $x_n \to \overline{x}$ and $y_n \to \overline{y}$. It then follows that

$$\delta_a(\{TA_n\}) = \lim_{n\to\infty} \delta(TA_n) = \lim_{n\to\infty} d(Tx_n, Ty_n) = d(T\overline{x}, T\overline{y}).$$

If $\overline{x} = \overline{y}$, then (4) holds trivially; so assume $\overline{x} \neq \overline{y}$. Since T is contractive,

$$d(T\overline{x}, T\overline{y}) < d(\overline{x}, \overline{y}) = \lim_{n\to\infty} d(x_n, y_n) \leqslant \lim_{n\to\infty} \sup\{d(x,y) : x, y \in A_n\} = \delta_a(\{A_n\}).$$

So (4) holds again and T is therefore asymptotically diametrically contractive.

THEOREM 2.6. *Let (M,d) be a complete metric space and $T : M \to M$ be a asymptotically diametrically contractive mapping. Assume T has a bounded orbit $\{T^n x_0\}_{n=0}^{\infty}$ for some $x_0 \in M$. Then T has a unique fixed point, ξ (say) and for each $x \in M$, $\{T^n x\}_{n=0}^{\infty}$ converges to ξ.*

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PROOF: Observe that the boundedness of one orbit $\{T^n x_0\}$ of T implies, due to the contractivity of T, that every orbit $\{T^n x\}$ of T is bounded. Put

$$A_n = \{T^k x : k \geqslant n\}, \quad n \geqslant 0.$$

Notice that $A_n = TA_{n-1}$ for $n \geqslant 1$. Since T is continuous, it follows that

$$\delta_a(\{\overline{A}_n\}) = \delta_a(\{A_n\}) = \delta_a(\{TA_{n-1}\}) = \delta_a(\{TA_n\}) = \delta_a(\{T\overline{A}_n\}).$$

Therefore the asymptotically diametrical contractivity condition (4) implies that

$$\delta_a(\{A_n\}) = 0$$

which is equivalent to the condition that $\{T^n x\}$ is Cauchy and hence convergent. Let $\xi = \lim\limits_{n\to\infty} T^n x$. Then ξ is a fixed point of T by the continuity of T and the uniqueness follows from the contractivity of T. ∎

REMARK 2.7. In Theorem 2.6, the assumption that T has a bounded orbit can not be removed. Indeed, take the same fixed point free contractive mapping T as given in Example 1.1, we have, for any $x \in \mathbb{R}$, $T^n x = \ln(n + e^x)$ for $n \geqslant 0$; thus every orbit $\{T^n x\}$ is unbounded. Note also that T satisfies the asymptotic regularity condition: for each $x \in \mathbb{R}$, $\lim\limits_{n\to\infty} d(T^{n+1}x, T^n x) = 0$ which is an immediate consequence of the calculation

$$|T^{n+1}x - T^n x| = \ln\Big(1 + \frac{1}{n + e^x}\Big).$$

This shows that asymptotic regularity is insufficient to guarantee the existence of a fixed point of a contractive mapping.

In contrast to Question 1.4, we raise the following

QUESTION 2.8. *Let M be a closed bounded subset of a Banach space and let $T : M \to M$ be a diametrically contractive mapping. Does T have a fixed point?*

REMARK 2.9. If X is a reflexive Banach space and M is a closed bounded convex subset of X, then Theorem 2.3 ensures that every diametrically contractive mapping $T : M \to M$ has a fixed point. The following converse question would be interesting.

QUESTION 2.10. *Suppose X is a real Banach space with the property that for each closed bounded convex subset M of X, every diametrically contractive mapping $T : M \to M$ has a fixed point. Is X reflexive?*

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