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ON QUASISIMILARITY FOR ANALYTIC TOEPLITZ OPERATORS

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ABSTRACT. Let f be a function in H^∞. We show that if f is inner or if the commutant of the analytic Toeplitz operator T_f is equal to that of T_b for some finite Blaschke product b, then any analytic Toeplitz operator quasisimilar to T_f is unitarily equivalent to T_f.

1. **Introduction.** It is not yet known whether two quasisimilar (or similar) analytic Toeplitz operators are necessarily unitarily equivalent (cf. [2], [5], [10]). In this note we give some conditions for quasisimilar analytic Toeplitz operators which imply their unitary equivalence.

Let $\mathscr{H}_1$ and $\mathscr{H}_2$ be Hilbert spaces. A (bounded linear) operator $X:\mathscr{H}_1 \to \mathscr{H}_2$ is called a *quasiaffinity* if it has trivial kernel and dense range, that is, if $\ker X = \{0\}$ and $(\operatorname{ran} X)^- = \mathscr{H}_2$. Operators T_1 and T_2 acting on $\mathscr{H}_1$ and $\mathscr{H}_2$ respectively are said to be *quasisimilar* if there exist quasiaffinities $X:\mathscr{H}_1 \to \mathscr{H}_2$ and $Y:\mathscr{H}_2 \to \mathscr{H}_1$ such that $XT_1 = T_2X$ and $YT_2 = T_1Y$, and this relation of T_1 and T_2 is denoted by $T_1 \sim T_2$. If T_1 and T_2 are unitarily equivalent, we write $T_1 \cong T_2$.

For f in H^∞ of the open unit disc $\mathbf{D}$, the analytic Toeplitz operator T_f is the operator on the Hardy space H^2 defined by $T_fh = fh$. If f is inner and nonconstant, then T_f is a unilateral shift and its multiplicity is equal to $\dim(H^2 \ominus fH^2)$. Quasisimilar unilateral shifts have the same multiplicity, and therefore they are unitarily equivalent. Thus, if both f and g are inner and $T_f \sim T_g$, then $T_f \cong T_g$. It was shown by Conway [3] that if f is a single Blaschke factor (i.e., T_f is a unilateral shift of multiplicity one) and $g \in H^\infty$, then $T_f \sim T_g$ implies $T_f \cong T_g$, and this result was extended in [13] to the case where f is a finite Blaschke product. In this note we show that if f is inner and g is a function in H^∞ with $\|g\|_\infty \leqq 1$ for which there exists a nonzero operator X such that $XT_g = T_fX$, then g is inner. The result is applied to show that whenever f is inner, for any $g \in H^\infty$, the relation $T_f \sim T_g$ implies $T_f \cong T_g$, and to prove a conjecture given by Wu [15]. We also show that if f is in H^∞ and

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there exists a finite Blaschke product b such that $\{T_f\}' = \{T_b\}'$, where for an operator A, $\{A\}'$ denotes the commutant of A, then for any $g \in H^\infty$, $T_f \sim T_g$ implies $T_f \cong T_g$. This result partially generalizes results of Cowen [5] and Seddighi [10]. From a result of Clary [1] or Deddens [6], it is known that quasisimilar analytic Toeplitz operators have equal spectra. Our result, together with a result of Cowen [4] or Thomson [14] on the commutants of analytic Toeplitz operators, shows that quasisimilar analytic Toeplitz operators also have equal essential spectra.

2. **Results.** We first consider analytic Toeplitz operators which are quasi-similar to unilateral shifts. If f is in H^∞ and there is a nonzero operator X such that $XT_u = T_fX$ for some inner function u, then by [6] we have $f(\mathbf{D}) \subseteq \sigma(T_u) = (u(\mathbf{D}))^-$ and so $||f||_\infty \leqq 1$. Conversely, if $f \in H^\infty$ is nonconstant and $||f||_\infty \leqq 1$, then for any nonconstant inner function u, there is a nonzero operator X such that $XT_u = T_fX$ [7]. However, we have the following result.

THEOREM 1. *Let f be a function in H^∞ with $||f||_\infty \leqq 1$ and let u be an inner function. If there is a nonzero operator X such that $XT_f = T_uX$, then f is inner.*

PROOF. If u is constant; $u(z) \equiv \lambda$ for some scalar λ with $|\lambda| = 1$, then $XT_f = T_uX = \lambda X$. Since $X \neq 0$, it follows that $\bar{\lambda}$ is an eigenvalue of T_f^*. Then, since T_f^* is a contraction and $|\lambda| = 1$, λ is an eigenvalue of T_f (cf. [11, Proposition I.3.1]), which implies $f = \lambda$, that is, f is a constant inner function.

Now, suppose that u is nonconstant. Then clearly f is nonconstant. Let $\alpha = \{e^{it}: |f(e^{it})| = 1\}$. We have to show $m(\partial\mathbf{D}\backslash\alpha) = 0$ where m denotes the normalized Lebesgue measure on the unit circle $\partial\mathbf{D}$. Since T_u is isometric and $XT_f = T_uX$, for $h \in H^2$ and $n = 1, 2, \ldots$, we have

$$||Xh|| = ||T_u^nXh|| = ||XT_f^nh|| \leqq ||X||\,||T_f^nh||.$$

But, since $|f| \leqq 1$ a.e. on $\partial\mathbf{D}$,

$$\lim_{n\to\infty} ||T_f^nh||^2 = \lim_{n\to\infty} \int |f|^{2n}|h|^2dm = \int \chi_\alpha|h|^2dm = ||\chi_\alpha h||^2$$

for $h \in H^2$, where χ_α is the characteristic function of α. Therefore it follows that $||Xh|| \leqq ||X||\,||\chi_\alpha h||$ for all $h \in H^2$, so we obtain the operator $Y: \mathscr{M} \to H^2$, where $\mathscr{M} = (\chi_\alpha H^2)^- \subseteq L^2$, such that $Y(\chi_\alpha h) = Xh$ for $h \in H^2$. Let M_f be the normal operator of multiplication on L^2 by f. Clearly the subspace $\mathscr{M}$ is invariant for M_f and $M_f|\mathscr{M}$ is isometric. We have for $h \in H^2$

$$\begin{aligned} Y(M_f|\mathscr{M})(\chi_\alpha h) &= Y(\chi_\alpha fh) = X(fh) \\ &= XT_fh = T_uXh = T_uY(\chi_\alpha h), \end{aligned}$$

so $Y(M_f|\mathscr{M}) = T_uY$. It follows from [1] that $\sigma(T_u|(\text{ran } Y)^-) \subseteq \sigma(M_f|\mathscr{M})$. But, since $Y \neq 0$ and u is nonconstant, $T_u|(\text{ran } Y)^-$ is a nonzero unilateral shift, hence we have $\mathbf{D}^- \subseteq \sigma(M_f|\mathcal{M})$ and so $M_f|\mathcal{M}$ is not unitary. This implies that $m(\partial\mathbf{D}\backslash\alpha) = 0$, that is, f is inner. Indeed, if $m(\partial\mathbf{D}\backslash\alpha) \neq 0$, then Szegö's theorem (cf. [2, Theorem IV.5.13]) shows $\mathcal{M} = (\chi_\alpha H^2)^- = \chi_\alpha L^2$, hence $\mathcal{M}$ reduces the normal operator M_f and the isometry $M_f|\mathcal{M}$ is unitary.

Let S_i ($i = 1, 2$) be a unilateral shift of multiplicity n_i and let X be an operator satisfying $XS_1 = S_2X$. It is easily seen that if X has dense range, then $n_1 \geqq n_2$. It is also known [12] that if X is injective, then $n_1 \leqq n_2$. Thus, if X is a quasi-affinity, then $n_1 = n_2$, so S_1 and S_2 are unitarily equivalent.

Parts of the following corollary were shown in [3] when u is a single Blaschke factor and in [13] when u is a finite Blaschke product (cf. also [16]).

COROLLARY 1. *Let u be an inner function and $f \in H^\infty$. Then the following conditions are equivalent.*

(i) $T_f \cong T_u$.

(ii) $T_f \sim T_u$.

(iii) *There are operators X and Y having dense range such that $XT_f = T_uX$ and $YT_u = T_fY$.*

(iv) *There are injections X and Y such that $XT_f = T_uX$ and $YT_u = T_fY$.*

(v) $\|f\|_\infty \leqq 1$ *and there is a quasiaffinity X such that $XT_f = T_uX$.*

PROOF. By [6], the existence of the operator Y in (iii) or (iv) shows $\|f\|_\infty \leqq 1$. Thus the implications (iii) $\Rightarrow$ (i), (iv) $\Rightarrow$ (i) and (v) $\Rightarrow$ (i) follow from Theorem 1 and the above facts on the multiplicty of unilateral shifts. The other implications are trivial.

The following corollary was conjectured by Wu [15] and proved in [16] for isometries V with dim ker $V^* < \infty$.

COROLLARY 2. *Let V be an isometry on a separable Hilbert space $\mathcal{H}$. If $T \in$ Alg V and $T \sim V$, then $T \cong V$. Here for an operator X,* Alg X *is the weakly closed algebra generated by X and I.*

PROOF. If V is unitary, then T is normal and so the result follows from the well-known fact that quasisimilar normal operators are unitarily equivalent. Thus we assume that V is non-unitary, hence we can write $V = U \oplus T_u$ on $\mathcal{H} = \mathcal{G} \oplus H^2$ where U is a unitary operator and u is a nonconstant inner function (i.e., T_u is a unilateral shift). Since $T \in$ Alg V and Alg $V \subseteq$ Alg $U \oplus$ Alg T_u, we have $T = A \oplus T_f$ where $A \in$ Alg U and $f \in H^\infty$. (Note that Alg U and Alg T_u consist of normal operators and of analytic Toeplitz operators, respectively.) Let X and Y be quasiaffinities such that $XT = VX$ and $YV = TY$. Since V is not unitary, the relation $XT = VX$ implies that T_f is not normal (cf. [2, Proposition III.11.7]) and therefore T_f is a pure subnormal operator (cf. [15, Lemma 4.4]). Thus it follows from [2, Proposition III.14.11] and its proof that $A \cong U$, $X\mathcal{G} \subseteq \mathcal{G}$ and $Y\mathcal{G} \subseteq \mathcal{G}$. Let $X_1 = PX|H^2$ and

$Y_1 = PY|H^2$ where P denotes the projection of $\mathscr{H} = \mathscr{G} \oplus H^2$ onto H^2. Clearly $X_1 T_f = T_u X_1$ and $Y_1 T_u = T_f Y_1$. Since X has dense range and $X\mathscr{G} \subseteq \mathscr{G}$, X_1 has dense range too. Similarly Y_1 has dense range. Thus it follows from Corollary 1 that $T_f \cong T_u$, hence $T \cong V$.

In [5] Cowen proved that if analytic Toeplitz operators T_f and T_g satisfying $\{T_f\}' = \{T_u\}'$ and $\{T_g\}' = \{T_v\}'$ for some inner functions u and v are similar, then they are unitarily equivalent. On the other hand, Seddighi [10] proved that if analytic Toeplitz operators T_f and T_g respectively generate the same weak* closed algebras as T_u and T_v for some inner functions u and v, then $T_f \sim T_g$ implies $T_f \cong T_g$. We have the following result.

THEOREM 2. *Let $f \in H^\infty$ and assume that there is a finite Blaschke product b such that $\{T_f\}' = \{T_b\}'$. If $g \in H^\infty$ and $T_f \sim T_g$, then $T_f \cong T_g$.*

The following lemma is known (cf. the proof of [16, Proposition 2.1]), but we include its proof for completeness. For an operator A, let $\{A\}''$ denote the double commutant of A.

LEMMA. *If S is a unilateral shift of finite multiplicity, then for any quasiaffinity $X \in \{S\}'$ there is a quasiaffinity $Y \in \{S\}'$ such that $YX \in \{S\}''$.*

PROOF. We may suppose that S is the operator on the $\mathbf{C}^n$-valued Hardy space H_n^2 defined by $(Sh)(z) = zh(z)$, $z \in \mathbf{D}$, where $n(<\infty)$ is the multiplicity of S. Then $\{S\}'$ consists of all multiplication operators on H_n^2 by $n \times n$ matrix valued, bounded analytic functions on $\mathbf{D}$. Thus X is the multiplication operator on H_n^2 by some $n \times n$ matrix valued, bounded analytic function F; $(Xh)(z) = F(z)h(z)$ for $z \in \mathbf{D}$ and $h \in H_n^2$. Since X has dense range, F is outer, hence by [11, Proposition V.6.1 and Corollary V.6.3] $d(z) := \det F(z)$ $(z \in \mathbf{D})$ is an outer function in H^∞ and there is an $n \times n$ matrix valued, bounded analytic function G such that $G(z)F(z) = F(z)G(z) = d(z)I$, $z \in \mathbf{D}$. Let Y be the multiplication operator on H_n^2 by G. Then $Y \in \{S\}'$, and we have $YX = XY = M_d$ where M_d is the multiplication operator by d. Hence $YX \in \{S\}''$. Since d is outer, M_d is a quasiaffinity. Therefore it follows from $YX = M_d = XY$ that Y is a quasiaffinity. Thus Y is the required operator.

PROOF OF THEOREM 2. Let X and Y be quasiaffinities such that $XT_f = T_g X$ and $YT_g = T_f Y$. Then, since the quasiaffinity YX belongs to $\{T_f\}' = \{T_b\}'$ and T_b is a unilateral shift of finite multiplicity, by Lemma there is a quasiaffinity $Z \in \{T_b\}'$ such that $ZYX \in \{T_b\}'' = \{T_f\}''$. Thus, by replacing Y by ZY, we can assume that the quasiaffinities X and Y satisfy $YX \in \{T_f\}''$. Under this assumption, we see that for all $A \in \{T_f\}''$ the operator XAY belongs to $\{T_g\}''$, so XAY is an analytic Toeplitz operator. Indeed, if $A \in \{T_f\}''$, then for any $B \in \{T_g\}'$, since $YBX \in \{T_f\}'$ and $YX \in \{T_f\}''$ by our assumption, we have

$YXAYBX = YBXAYX$ and therefore $XAYB = BXAY$ because X and Y are quasiaffinities. This shows $XAY \in \{T_g\}''$.

For $n = 0, 1, 2, \ldots$, since $T_b^n \in \{T_b\}'' = \{T_f\}''$, by the above fact XT_b^nY is an analytic Toeplitz operator, hence there is $w_n \in H^\infty$ such that $XT_b^nY = T_{w_n}$. Then, noting that YX and T_b commute, we have

$$T_{w_1}^n = (XT_bY)^n = (XY)^{n-1}XT_b^nY = T_{w_0}^{n-1}T_{w_n},$$

so that $w_1^n = w_0^{n-1}w_n$ for $n \geqq 1$. We also have

$$||w_n||_\infty = ||T_{w_n}|| = ||XT_b^nY|| \leqq ||X||\,||Y||$$

for every n (because b is inner). Therefore it follows that

$$|w_1|^{n/(n-1)} = |w_0|\,|w_n|^{1/(n-1)} \leqq |w_0|\,(\,||X||\,||Y||\,)^{1/(n-1)}$$

a.e. on $\partial\mathbf{D}$ for $n \geqq 2$, and letting $n \to \infty$ we get $|w_1| \leqq |w_0|$ a.e. on $\partial\mathbf{D}$. But, since $T_{w_0} = XY$ has dense range, w_0 is outer. Therefore there is $v \in H^\infty$ such that $w_1 = w_0v$ and $||v||_\infty \leqq 1$ (cf. [9, Proposition 6.22]). Then we have

$$XT_bY = T_{w_1} = T_{w_0}T_v = XYT_v$$

and therefore the injectivity of X implies $T_bY = YT_v$. Hence it follows from the implication (v) $\Rightarrow$ (i) in Corollary 1 that v is inner and $T_b \cong T_v$.

Now since $T_f \in \{T_f\}'' = \{T_b\}''$, there is $h \in H^\infty$ such that $f = h \circ b$ (cf. [4, Theorem 1 and 2]). Let p_n be the n-th Cesàro mean of h. Then, since $p_n \to h$ weak* in H^∞, $T_{p_n\circ b} \to T_{h\circ b}$ and $T_{p_n\circ v} \to T_{h\circ v}$ weakly (cf. [11, Theorem III.2.1]). Therefore it follows from $T_bY = YT_v$ that $T_fY = T_{h\circ b}Y = YT_{h\circ v}$. Also, if U is a unitary operator satisfying $T_bU = UT_v$, then $T_fU = UT_{h\circ v}$. Hence the relation $T_b \cong T_v$ implies $T_f \cong T_{h\circ v}$. But $YT_g = T_fY = YT_{h\circ v}$, so we have $T_g = T_{h\circ v}$ by the injectivity of Y. Thus $g = h \circ v$ and $T_f \cong T_g$.

It was proved by Cowen [4] that if $f \in H^\infty$ and for some scalar λ the inner factor of $f - \lambda$ is a (nonconstant) finite Blaschke product, then there is a finite Blaschke product b such that $\{T_f\}' = \{T_b\}'$ (cf. also Thomson [14], Deddens and Wong [8]). Thus we have the following result.

Corollary 3. *Let $f \in H^\infty$ and assume that for some scalar λ the inner factor of $f - \lambda$ is a finite Blaschke product. If $g \in H^\infty$ and $T_f \sim T_g$, then $T_f \cong T_g$.*

It is known (cf. [1], [6]) that quasisimilar analytic Toeplitz operators have equal spectra.

Corollary 4. *Quasisimilar analytic Toeplitz operators have equal essential spectra.*

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Proof. Let T_f and T_g be quasisimilar analytic Toeplitz operators. By the result of [1] or [6], we have only to consider the case when $\sigma(T_f) \neq \sigma_e(T_f)$ (= the essential spectrum of T_f) or $\sigma(T_g) \neq \sigma_e(T_g)$. Suppose that $\sigma(T_f) \neq \sigma_e(T_f)$, so there is a scalar λ such that $T_f - \lambda I$ is a non-invertible Fredholm operator. Then, as noted in [14, Corollary 2], the inner factor of $f - \lambda$ is a finite Blaschke product. Therefore it follows from Corollary 3 that $T_f \cong T_g$ and so $\sigma_e(T_f) = \sigma_e(T_g)$. Similarly, if $\sigma(T_g) \neq \sigma_e(T_g)$, then $\sigma_e(T_f) = \sigma_e(T_g)$.

We note that the proof of Theorem 2 shows the following result.

Proposition. *Let $f \in H^\infty$ and assume that $\{T_f\}' = \{T_u\}'$ for some inner function u. If $g \in H^\infty$ and there are quasiaffinities X and Y such that $XT_f = T_gX$, $YT_g = T_fY$ and $YX \in \{T_f\}''$, then $T_f \cong T_g$.*

References

1. W. S. Clary, *Equality of spectra of quasisimilar hyponormal operators*, Proc. Amer. Math. Soc. **53** (1975), pp. 88-90.
2. J. B. Conway, *Subnormal operators*, Pitman, Boston, 1981.
3. J. B. Conway, *On quasisimilarity for subnormal operators, II*, Canad. Math. Bull. **25** (1982), pp. 37-40.
4. C. C. Cowen, *The commutant of an analytic Toeplitz operator*, Trans. Amer. Math. Soc. **239** (1978), pp. 1-31.
5. C. C. Cowen, *On equivalence of Toeplitz operators*, J. Operator Theory **7** (1982), pp. 167-172.
6. J. A. Deddens, *Intertwining analytic Toeplitz operators*, Michigan Math. J. **18** (1971), pp. 243-246.
7. J. A. Deddens, *Analytic Toeplitz and composition operators*, Canad. J. Math. **24** (1972), pp. 859-865.
8. J. A. Deddens and T. K. Wong, *The commutant of analytic Toeplitz operators*, Trans. Amer. Math. Soc. **184** (1973), pp. 261-273.
9. R. G. Douglas, *Banach algebra techniques in operator theory*, Academic Press, New York, 1972.
10. K. Seddighi, *On quasisimilarity for Toeplitz operators*, Canad. Math. Bull. **28** (1985), pp. 107-112.
11. B. Sz.-Nagy and C. Foias, *Harmonic analysis of operators on Hilbert space*, North-Holland, Amsterdam, 1970.
12. B. Sz.-Nagy and C. Foias, *Jordan model for contractions of class $C_{\cdot 0}$*, Acta Sci. Math. **36** (1974), pp. 305-322.
13. K. Takahashi, *On quasiaffine transforms of unilateral shifts*, Proc. Amer. Math. Soc. **100** (1987), pp. 683-687.
14. J. E. Thomson, *The commutant of a class of analytic Toeplitz operators, II*, Indiana Univ. Math. J. **25** (1976), pp. 793-800.
15. P. Y. Wu, *Hyponormal operators quasisimilar to an isometry*, Trans. Amer. Math. Soc. **291** (1985), pp. 229-239.
16. P. Y. Wu, *Contractions quasisimilar to an isometry*, preprint.

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