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C^*-ALGEBRAS FROM SMALE SPACES

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ABSTRACT. We consider the C^*-algebras constructed from certain hyperbolic dynamical systems. The construction, due to Ruelle, generalizes the C^*-algebras of Cuntz and Krieger. We discuss relations between the C^*-algebras, show the existence of natural asymptotically abelian systems and investigate the K-theory and E-theory of these C^*-algebras.

1. **Introduction.** In [14], David Ruelle constructed C^*-algebras from certain hyperbolic dynamical systems including Smale spaces. Special cases are the topological Markov chains where these C^*-algebras were earlier constructed by Cuntz and Krieger [6,8]. Thus, Ruelle's algebras may be viewed as "higher dimensional" analogues of the Cuntz-Krieger algebras (—the O_A's as well as other algebras appearing in [6,8]). This paper is an attempt to continue these investigations.

Roughly speaking, a Smale space is a compact metric space (X,d) with a homeomorphism ϕ of X so that, locally, X can be written as a product of two subsets. Moreover, on the first subset ϕ is (exponentially) contracting and on the second ϕ^{-1} is contracting. One is then interested in three equivalence relations on the points of X determined as follows. For x, y in X, they are equivalent if the distance between $\phi^n(x)$ and $\phi^n(y)$, their n-th iterates, tends to zero as n goes to plus infinity, minus infinity and both plus and minus infinity. These are referred to as stable, unstable and asymptotic equivalence. Locally, the first two are given in the local product structure. The third can actually be represented by certain local maps called *conjugating homeomorphisms* arising directly from the Smale space structure. These dynamical notions are presented in Section 2. These are taken more or less directly from Ruelle's papers [13, 14] (except for two technical lemmas), but we present them here for completeness.

We consider the C^*-algebras of these equivalence relations which we denote by S, U and A, respectively. In [14], the emphasis is on the C^*-algebra A. The point of [14] is to relate Gibbs states of the dynamical system with KMS states on the C^*-algebra. Here, we make use of the fact that the original homeomorphism induces *-automorphisms, α_s, α_u and α_a, of S, U and A, respectively. We show that the action of α_a on A is asymptotically abelian. This result along with other basic properties of the C^*-algebras is developed in Section 3.

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In Section 4, we consider the K-theory for our C^*-algebras. The asymptotically abelian action provides us with various elements in the Connes-Higson E-theory [5]. In particular, the K_0-group of one of our C^*-algebras (the mapping cylinder for (A, α_a)) is actually a ring. Moreover, this C^*-algebra has a natural trace and the induced map from K_0 to the reals is actually a ring homomorphism precisely because our original system is strong mixing (with respect to the measure of maximum entropy).

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2. **Dynamics.** We describe Smale spaces and certain results we will need later. We will also present several examples. We follow the two papers of Ruelle [13, 14] with some minor changes of notation.

Let (X, d) be a compact metric space and let ϕ be a homeomorphism of X. Rather than begin with the rigourous (and perhaps confusing) treatment, we will proceed heuristically. We suppose that, locally, X is a product space; for every x in X, we have two sets, $V^S(x,\epsilon)$, $V^U(x,\epsilon)$, where $\epsilon > 0$ is some small parameter. These are subsets of X and their intersection is $\{x\}$. Moreover, their cartesian product is homeomorphic to a neighbourhood of x. This decomposition should be invariant under ϕ in the sense that $\phi\big(V^S(x,\epsilon)\big)$ and $V^S\big(\phi(x),\epsilon\big)$ should agree in some neighbourhood of $\phi(x)$, as should $\phi\big(V^U(x,\epsilon)\big)$ and $V^U\big(\phi(x),\epsilon\big)$. Most importantly $\phi \mid V^S(x,\epsilon)$ should be contracting, as is $\phi^{-1} \mid V^U(x,\epsilon)$.

Postponing our rigourous definition further, let us look at some examples.

1. SUBSHIFTS OF FINITE TYPE (SFT). Let n be a positive integer and let A be a fixed $n \times n$ matrix whose entries are zeros and ones. We will assume A is primitive; *i.e.* for some k, A^k has no zero entries. Let $\{1,\ldots,n\}^{\mathbb{Z}}$ be the space of doubly infinite sequences of $\{1,\ldots,n\}$ with the product topology. Define

$$X = \{(x_i)_{i=-\infty}^{\infty} \in \{1,\ldots,n\}^{\mathbb{Z}} \mid A_{x_i x_{i+1}} = 1, \text{ for all } i \text{ in } \mathbb{Z}\},$$

and

$$\phi(x)_i = x_{i-1}, \quad i \in \mathbb{Z},\ x \in X.$$

We use the metric

$$d(x,y) = \sum_{i\in\mathbb{Z}} 2^{-|i|}|x_i - y_i|.$$

To see the local product structure here, consider

$$V^S(x,\epsilon) = \{y \in X \mid x_i = y_i, \text{ for all } i \leq 0\}$$
$$V^U(x,\epsilon) = \{y \in X \mid x_i = y_i, \text{ for all } i \geq 0\}.$$

It's fairly easy to see that there is a natural homeomorphism between $V^U(x,\epsilon) \times V^S(x,\epsilon)$ and

$$\{y \in X \mid x_0 = y_0\}$$

which is a neighbourhood of x. Moreover, for $y, y' \in V^S(x,\epsilon)$

$$d\big(\phi(y),\phi(y')\big) = \frac{1}{2}\, d(y,y')$$

and for z,z' in $V^U(x,\epsilon)$

$$d\big(\phi^{-1}(z),\phi^{-1}(z')\big) = \frac{1}{2}\, d(z,z').$$

We leave it to the reader to observe that $V^S\big(\phi(x),\epsilon\big)$ and $\phi\big(V^S(x,\epsilon)\big)$ are not equal but "agree in a neighbourhood of $\phi(x)$."

2. ANOSOV DIFFEOMORPHISMS. Let M be a compact Riemannian manifold. An Anosov diffeomorphism is a smooth map $\phi: M \longrightarrow M$ such that $TM = E \oplus F$, where E, F are sub-bundles of TM, each invariant under $T\phi$ and such that, for some constants C and $0 < \delta < 1$, we have

$$\|(T\phi)^k v\| \leq C\delta^k \|v\|, \quad v \in E,\ k = 1,2,3,\ldots$$
$$\|(T\phi)^{-k} w\| \leq C\delta^k \|w\|, \quad w \in F,\ k = 1,2,3,\ldots.$$

The sets $V^S(x,\epsilon)$ and $V^U(x,\epsilon)$ are obtained by integrating E and F, locally. We refer the reader to [2] and [16] for further discussion.

Let us examine a prototype more closely. Let $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ and view A as a linear isomorphism of $\mathbb{R}^2$. As A preserves the integer lattice $\mathbb{Z}^2$, we may pass to a diffeomorphism ϕ of the quotient $\mathbb{R}^2/\mathbb{Z}^2 \cong \mathbb{T}^2 = M$. Now A has eigenvalues $\lambda = (3-\sqrt{5})/2 < 1$ and $\lambda^{-1} > 1$. The decomposition of TM into $E \oplus F$ is obtained by decomposing $\mathbb{R}^2$ into the eigenspaces of A. The sets $V^S(x,\epsilon)$ and $V^U(x,\epsilon)$ can be seen in M as line segments through x determined by the eigenvectors.

3. SOLENOIDS. We describe one specific example only. Regard S^1 as the unit circle in the complex plane and $\phi_0: S^1 \longrightarrow S^1$ be the map $\phi_0(z) = z^2$. Let X be the inverse limit of the system

$$X_0 = S^1 \underset{\phi_0}{\longleftarrow} X_1 = S^1 \underset{\phi_0}{\longleftarrow} X_2 = S^1 \longleftarrow \cdots.$$

Concretely, we can describe X as

$$\{(z_0, z_1, z_2, \ldots) \mid z_i \in S^1, z_{i+1}^2 = z_i, i = 0,1,2,\ldots\}.$$

Let $\pi: X \longrightarrow S^1$ denote the map $\pi(z_0, z_1, z_2, \ldots) = z_0$. Also, define $\phi: X \longrightarrow X$ by $\phi(z_0, z_1, z_2, \ldots) = (z_0^2, z_0, z_1, \ldots)$ so that $\phi^{-1}(z_0, z_1, z_2, \ldots) = (z_1, z_2, z_3, \ldots)$. It is easy to see that, for any z_0 in S^1,

$$\pi^{-1}\{z_0\} \cong \prod_{n=1}^{\infty} \{-1,+1\} = \Sigma.$$

Moreover π is a fibration; for any $(z_n)_1^\infty$, in X, there is a neighbourhood which is homeomorphic to

$$\big\{z \in S^1 \;\big|\; |z - z_0| < \epsilon\big\} \times \Sigma.$$

It is also easy to see that $\pi \circ \phi = \phi_0 \circ \pi$. Fix $x = (x_n)_1^\infty$ in X, which is identified with $\big(x_0, (\delta_n)_1^\infty\big)$ in the product space above. Let

$$V^S(x,\epsilon) = \{x_0\} \times \Sigma \quad \text{and}$$
$$V^U(x,\epsilon) = \big\{z \in S^1 \;\big|\; |z - x_0| < \epsilon\big\} \times \{(\delta_n)_1^\infty\}.$$

(Or rather, $V^S(x,\epsilon)$ and $V^U(x,\epsilon)$ are the sets in X identified with these.) We leave it to the reader to verify that these sets satisfy the desired properties.

Let us return to our attempt to define a Smale space in the general setting. If x and y are sufficiently close then their local product neighbourhoods will "agree" where they overlap. The intersection of $V^S(x,\epsilon)$ and $V^U(y,\epsilon)$ will be a single point which we denote by $[x,y]$ (—having nothing to do with commutators).

Notice that with this definition, we may characterize $V^S(x,\epsilon)$ as those points z such that $[x,z] = z$. The rigourous definition begins by hypothesizing the existence of the map $[\cdot,\cdot]$ and obtaining the $V^S(x,\epsilon)$ and $V^U(x,\epsilon)$ as above.

We say that (X,d,ϕ) is a *Smale space* if there is $0 < \lambda_0 < 1$, $\epsilon_0 > 0$ and a continuous function

$$[\;,\;]\colon \big\{(x,y) \mid x,y \in X, d(x,y) < \epsilon_0\big\} \longrightarrow X$$

satisfying the following. First we require

$$[x,x] = x$$
$$\big[[x,y],z\big] = [x,z]$$
$$\big[x,[y,z]\big] = [x,z]$$

for x, y, z in X, whenever both sides of the equation are defined. We let

$$V^S(x,\epsilon) = \{y \in X \mid [x,y] = y \text{ and } d(x,y) < \epsilon\},$$
$$V^U(x,\epsilon) = \{y \in X \mid [y,x] = y \text{ and } d(x,y) < \epsilon\},$$

for any $0 < \epsilon \le \epsilon_0$. We also require

$$[\phi(x),\phi(y)] = \phi([x,y]),$$

whenever both sides of the equation are defined. Finally, we assume that

$$d\big(\phi(y),\phi(z)\big) \le \lambda_0\, d(y,z), \quad y,z \in V^S(x,\epsilon)$$
$$d\big(\phi^{-1}(y),\phi^{-1}(z)\big) \le \lambda_0\, d(y,z), \quad y,z \in V^U(x,\epsilon).$$

Briefly referring back to example 1, we let $\epsilon_0 = \frac{1}{2}$, $\lambda_0 = \frac{1}{2}$. Note that if $d\big((x_i)_i,(y_i)_i\big) < \epsilon_0$, then $x_0 = y_0$. We define

$$[(x_i),(y_i)]_j = \begin{cases} x_j & \text{for } j \le 0 \\ y_j & \text{for } j \ge 0 \end{cases}$$

if $d(x,y) < \epsilon_0$.

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It follows from the definitions that, for any x in X,

$$[\,,\,]\colon V^U(x,\epsilon_0/2)\times V^S(x,\epsilon_0/2)\longrightarrow X$$

is a homeomorphism onto a neighbourhood of x in X. It can also be shown that, for any $0<\epsilon<\epsilon_0$,

$$V^S(x,\epsilon)=\big\{y\in X \mid d\big(\phi^n(x),\phi^n(y)<\epsilon, \text{ for all } n=0,1,2,\ldots\big\}$$
$$V^U(x,\epsilon)=\big\{y\in X \mid d\big(\phi^n(x),\phi^n(y)\big)<\epsilon, \text{ for all } n=0,-1,-2,\ldots\big\}$$

and that, for x,y with $d(x,y)<\epsilon_0$,

$$V^S(x,\epsilon_0)\cap V^U(y,\epsilon_0)=\{[x,y]\}.$$

These last observations show that $[\,,\,]$, if it exists, depends only on (X,d,ϕ).

We will also assume throughout that our Smale space is irreducible in the sense that it is topologically mixing; that is, for every pair of open sets U and V, there is $N\geq 1$, such that for all $n\geq N$, $\phi^n(U)\cup V$ is non-empty. For more on this issue, we refer the reader to the discussion of Smale's spectral decomposition in [13].

For a Smale space as above, there is a unique ϕ-invariant probability measure which maximizes the entropy of the transformation ϕ. We denote this measure by μ and refer to it as the Bowen measure [15]. The idea of the proof (which is due to Sinai originally) is to "code" the system by using Markov partitions. The existence of Markov partitions follows from the definition of Smale space. As shown in Theorem 1 [15], much more is true. Fixing x in X, the map $[\,,\,]$ defines a homeomorphism between $V^U(x,\epsilon)\times V^S(x,\epsilon)$ and a neighbourhood of x in X. Restricting μ to this set and identifying the set with $V^U(x,\epsilon)\times V^S(x,\epsilon)$ *via* $[\,,\,]$, the measure μ is a product measure $\mu_u^x\times\mu_s^x$. Here the measures μ_u^x and μ_s^x depend on x. However, Theorem 1 of [15] asserts that these may be chosen such that

(i) for x and y sufficiently close, and ϵ, ϵ' small, $z\longrightarrow[y,z]$ defines a homeomorphism from $V^S(x,\epsilon)$ into $V^S(y,\epsilon')$ which carries μ_s^x to μ_s^y. Similarly $z\longrightarrow[z,y]$ defines a homeomorphism from $V^U(x,\epsilon)$ into $V^U(y,\epsilon')$ which carries μ_u^x to μ_u^y.

(ii)

$$\mu_s^{\phi(x)}\circ\phi=\lambda^{-1}\mu_s^x$$

on the appropriate domain and

$$\mu_u^{\phi(x)}\circ\phi=\lambda\mu_u^x,$$

where $\lambda>1$ and $\log(\lambda)$ is the topological entropy of (X,ϕ) [18].

We now describe conjugating homeomorphisms for the Smale space (X,d,ϕ). First of all, we say x and y in X are *conjugate* or *asymptotic* if

$$\lim_{|n|\longrightarrow\infty} d\big(\phi^n(x),\phi^n(y)\big)=0.$$

Fix such a pair (x,y). We will define a map γ from a neighbourhood of x to one of y which maps x to y and so that z and $\gamma(z)$ are asymptotic, for all z in the domain of γ. First, find

$n_0 \geq 1$ so that $d\big(\phi^n(x), \phi^n(y)\big) < \epsilon_0$ for all $|n| \geq n_0$. Next, choose ϵ sufficiently small that $\lambda_0^{-n_0}\epsilon < \epsilon_0$. This means that, for all $n = -n_0, \ldots, n_0$,

$$\phi^n\big(V^S(x,\epsilon)\big) \subseteq V^S\big(\phi^n(x), \epsilon_0\big)$$
$$\phi^n\big(V^U(x,\epsilon)\big) \subseteq V^U\big(\phi^n(x), \epsilon_0\big).$$

Consider the composition of the following three maps: (let $n = n_0$)

$$z \in V^S(x,\epsilon) \longrightarrow \phi^{-n}(z) \in V^S\big(\phi^{-n}(x), \epsilon_0\big),$$
$$\phi^{-n}(z) \in V^S\big(\phi^{-n}(x), \epsilon_0\big) \longrightarrow [\phi^{-n}(y), \phi^{-n}(z)] \in V^S\big(\phi^{-n}(y), \epsilon_0\big),$$
$$[\phi^{-n}(y), \phi^{-n}(z)] \in V^S\big(\phi^{-n}(y), \epsilon_0\big) \longrightarrow \phi^n[\phi^{-n}(y), \phi^{-n}(z)] \in V^S(y, \epsilon_0).$$

Each is a homeomorphism onto its image. This is the "stable coordinate" of the map γ. The "unstable coordinate" is obtained in a similar way. To write γ, we take z close to x, take its stable and unstable coordinates (namely $[x,z]$ and $[z,x]$) apply these maps to both and recover a point near y from its stable and unstable coordinates. Specifically,

$$\gamma(z) = \Big[\phi^{-n}\big[\phi^n[z,x], \phi^n(y)\big], \phi^n\big[\phi^{-n}(y), \phi^{-n}[x,z]\big]\Big].$$

It is easy to verify that γ is defined in a neighbourhood of x, that $\gamma(x) = y$ and that

$$\lim_{|n|\to\infty} d\Big(\phi^n\big(\gamma(z)\big), \phi^n(z)\Big) = 0$$

and the limit is uniform over z in the domain of γ, which we denote O_γ.

The following facts are consequences of the hypothesis of topological mixing—proofs can be found in [13]. By 7.16(b) of [13], the asymptotic equivalence class of any point in X is countable and dense in X. Also each conjugating homeomorphism leaves invariant μ. As noted before, the Smale space structure provides a coding by Markov partitions. This means that our Smale space is metrically isomorphic with a subshift of finite type. Since the Smale space is topologically mixing, so is the subshift. This implies that the subshift, hence the Smale space, are both strong mixing (with respect to μ) [13, 18].

In addition to asymptotic equivalence, we will be interested in stable and unstable equivalence. Two points x and y are stably equivalent if

$$\lim_{n\to+\infty} d\big(\phi^n(x), \phi^n(y)\big) = 0$$

and unstably equivalent if

$$\lim_{n\to-\infty} d\big(\phi^n(x), \phi^n(y)\big) = 0.$$

We denote the stable and unstable equivalence classes of x by $V^S(x)$ and $V^U(x)$. Note that it follows from the definitions that $V^S(x) \cap V^U(x)$ is the set of points asymptotic with x, which we denote $V(x)$. It is easy to see, using the contracting property of ϕ, that $V^S(x, \epsilon_0)$

is contained in $V^S(x)$. In fact, if $\phi^n(y)$ is in $V^S\big(\phi^n(x), \epsilon_0\big)$, for some positive n, then y is in $V^S(x)$. So we have

$$\phi^{-n}\Big(V^S\big(\phi^n(x), \epsilon_0\big)\Big) \subseteq V^S(x),$$

for all $n = 1, 2, 3, \ldots$. After taking the union over n above, the reverse inclusion also holds. This can be seen most clearly in the case x is a fixpoint of ϕ. If y is in $V^S(x)$, it means that the forward orbit of y tends to x. For some N, $d\big(\phi^n(y), x\big) < \epsilon_0$, for all $n \geq N$. Consider the stable and unstable co-ordinates of $\phi^N(y)$, $[x, \phi^N(y)]$ and $[\phi^N(y), x]$. If the unstable part is not equal to x, then the expanding nature of ϕ on $V^U(x, \epsilon_0)$ will force $d\big(\phi^n(y), x_0\big) > \epsilon_0$ for some $n > N$, a contradiction. This can be made into a rigourous proof that $[\phi^N(y), x] = x$ and hence $\phi^N(y)$ is in $V^S(x, \epsilon_0)$. In general, we have

$$V^S(x) = \bigcup_{n\geq 0} \phi^{-n}\Big(V^S\big(\phi^n(x), \epsilon_0\big)\Big)$$

$$V^U(x) = \bigcup_{n\geq 0} \phi^{n}\Big(V^U\big(\phi^{-n}(x), \epsilon_0\big)\Big).$$

We now have three equivalence relations which we want to consider as groupoids (see [11]). Thus, we need topologies on all three and Haar systems for each.

First define

$$G_s^0 = \{(x,y) \in X \times X \mid y \in V^S(x, \epsilon_0)\}$$

$$G_u^0 = \{(x,y) \in X \times X \mid y \in V^U(x, \epsilon_0)\}$$

and then let

$$G_s^n = (\phi \times \phi)^{-n}(G_s^0)$$

$$G_u^n = (\phi \times \phi)^{n}(G_u^0)$$

for each $n = 1, 2, 3, \ldots$. Each G_s^n, G_u^n is given the relative topology of $X \times X$ and

$$G_s = \bigcup_{n=1}^{\infty} G_s^n$$

$$G_u = \bigcup_{n=1}^{\infty} G_u^n$$

are given the inductive limit topology.

As we noted above, these are the stable and unstable equivalence relations. We can also define

$$G_a^n = G_s^n \cap G_u^n, \quad n = 0, 1, 2, \ldots$$

and let

$$G_a = \bigcup_{n\geq 0} G_a^n,$$

with each G_a^n given the relative topology of $X \times X$ and G_a the inductive limit topology. The last agrees with the topology on G_a given by Ruelle in [14].

As for Haar systems for G_s, G_u and G_a, we proceed as follows. As in [14], G_a is r-discrete and counting measure is a Haar system. Let us consider G_s. Fix x in X. Let δ_x denote point mass at x. We define a measure on G_s^0 by $\delta_x \times \mu_s^x$, and then on G_s^n by

$$\lambda^{-n}\delta_{\phi^n(x)} \times \mu_s^{\phi^n(x)} \circ (\phi \times \phi)^n.$$

The fact that any two of these measures agree on their common domain of definition follows from (c) of Theorem 1 of [15], which is our condition (ii) mentioned earlier. In this way we obtain a measure μ_s^x on G_s. It is easy to verify that $\{\mu_s^x \mid x \in X\}$ forms a Haar system for G_s and

$$\mu_s^{\phi(x)} \circ (\phi \times \phi) = \lambda^{-1}\mu_s^x.$$

The Haar system $\{\mu_u^x \mid x \in X\}$ for G_u is obtained in a similar way and

$$\mu_u^{\phi(x)} \circ (\phi \times \phi) = \lambda\mu_u^x.$$

Later, we will need the following technical results.

LEMMA 2.1. *Let γ be a conjugating homeomorphism for (X,ϕ) with domain O_γ and let ϵ be so that $0 < \epsilon < \epsilon_0$. Then there is a positive integer N so that, if $n \geq N$ and x,y in X lie in $\phi^{-n}(O_\gamma)$ with y in $V^S(x,\epsilon)$, then*

$$\phi^{-n}\gamma\phi^n(y) = [\phi^{-n}\gamma\phi^n(x), y].$$

PROOF. First we use the fact that $d\big(\phi^n\gamma(z), \phi^n(z)\big)$ tends to zero uniformly for z in O_γ. We find N so

$$d\big(\phi^n\gamma(z), \phi^n(z)\big) < \epsilon_0 - \epsilon$$

for all z in O_γ and $|n| \geq N$. It follows that, for $n \geq N$ and z in O_γ,

$$\phi^n\gamma(z) \in V^S\big(\phi^n(z), \epsilon_0 - \epsilon\big)$$
$$\phi^{-n}\gamma(z) \in V^U\big(\phi^{-n}(z), \epsilon_0 - \epsilon\big).$$

Let $x' = \phi^{-n}\gamma\phi^n(x)$, for $n \geq N$ fixed. Note that x' is in $V^U(x, \epsilon_0 - \epsilon)$ and $\phi^{2n}(x')$ is in $V^S\big(\phi^{2n}(x), \epsilon_0 - \epsilon\big)$. Let γ' be the conjugating map taking $\phi^n(x)$ to $\phi^n(x')$. Of course $\gamma\phi^n(x) = \phi^n(x')$, by definition of x', so by the uniqueness property of conjugating maps described in [14], $\gamma = \gamma'$ on their common domain. By hypothesis $\phi^n(y)$ is in O_γ; we show that $\phi^n(y)$ is in $O_{\gamma'}$. Since y is in $V^S(x,\epsilon)$, $\phi^n(y)$ is in $V^S\big(\phi^n(x),\epsilon\big)$ and $[\phi^n(y), \phi^n(x)] = \phi^n(x)$. Also, x' is in $V^U(x, \epsilon_0 - \epsilon)$ and y is in $V^S(x,\epsilon)$ so

$$d(x', y) \leq d(x', x) + d(x, y) < \epsilon_0$$

and so $[x', y]$ exists. Since $[x', y]$ is in $V^S(x', \epsilon_0)$, $\phi^i([x,y])$ is in $V^S\big(\phi^(ix'), \epsilon_0\big)$ for all $i \geq 0$. A direct computation using the definition of γ' shows $\gamma'\big(\phi^n(y)\big)$ exists and equals $\phi^n([x,y])$. The conclusion follows at once. ∎

The next result shows that for given conjugating maps γ_1 and γ_2, the maps $\phi^{-n}\gamma_1\phi^n$ and $\phi^m\gamma_2\phi^{-m}$ will commute as m, n tend to plus infinity. This also appears in Krieger's work on subshifts of finite type [8]. The difference here is that the conjugating maps appear directly from the Smale space structure as well as this property. Secondly, unlike the situation for subshifts of finite type, our conjugating maps are only defined locally.

LEMMA 2.2. *Let γ_1, γ_2 be conjugating maps and let $K_1 \subseteq O_{\gamma_1}$, $K_2 \subseteq O_{\gamma_2}$ be compact. Then there is a positive integer N so that for all $m, n \geq N$ we have*

(i) if $x \in \phi^{-n}(K_1)$, $\phi^{-n}\gamma_1\phi^n(x) \in \phi^m(K_2)$, then $x \in \phi^m(O_{\gamma_2})$, $\phi^m\gamma_2\phi^{-m}(x) \in \phi^{-n}(O_{\gamma_1})$

(ii) if $x \in \phi^m(K_2)$, $\phi^m\gamma_2\phi^{-m}(x) \in \phi^{-n}(K_1)$, then $x \in \phi^{-n}(O_{\gamma_1})$, $\phi^{-n}\gamma_1\phi^n(x) \in \phi^m(O_{\gamma_2})$

and in either case,

$$(\phi^m\gamma_2\phi^{-m})(\phi^{-n}\gamma_1\phi^n)(x) = (\phi^{-n}\gamma_1\phi^n)(\phi^m\gamma_2\phi^{-m})(x).$$

PROOF. We first choose $\epsilon > 0$ (and $\epsilon < \epsilon_0$) so that all x with ϵ of K_i are in O_{γ_1}, for $i = 1, 2$. We choose N sufficiently large so as to satisfy the conclusion of the previous lemma for both γ_1 and γ_2 and so that, for $|k| \geq N$

$$d\big(\phi^k\gamma_1(z_1), \phi^k(z_1)\big) < \epsilon$$
$$d\big(\phi^k\gamma_2(z_2), \phi^k(z_2)\big) < \epsilon$$

for all z_1 in O_{γ_1}, z_2 in O_{γ_2}.

For x, m, n as in (i) it follows that $\phi^{-m}(x)$ is within ϵ of $\phi^{-m}\phi^{-n}\gamma_1\phi^n(x) \in K_2$ and so $\phi^{-m}(x) \in O_{\gamma_2}$. Similarly, $\phi^n\phi^m\gamma_2\gamma^{-m}(x)$ is within ϵ of $\phi^n(x) \in K$ and so is in O_{γ_1}. Property (ii) is checked in a similar way.

Let $y = \phi^m\gamma_2\phi^{-m}(x)$, so for all $k \geq 0$

$$d\big(\phi^k(y), \phi^k(x)\big) = d\big(\phi^{k+m}\gamma_2\phi^{-m}(x), \phi^{k+m}\phi^{-m}(x)\big) < \epsilon$$

and so y is in $V^S(x, \epsilon)$. Therefore, we may apply Lemma 2.1 to compute

$$\begin{aligned}(\phi^{-n}\gamma_1\phi^n)(\phi^m\gamma_2\phi^{-m})(x) &= (\phi^{-n}\gamma_1\phi^n)(y)\\ &= [\phi^{-n}\gamma_1\phi^n(x), y]\\ &= [\phi^{-n}\gamma_1\phi^n(x), \phi^m\gamma_2\phi^{-m}(x)].\end{aligned}$$

A similar application of Lemma 2.1 shows

$$(\phi^m\gamma_2\phi^{-m})(\phi^{-n}\gamma_1\phi^n)(x) = [\phi^{-n}\gamma_1\phi^n(x), \phi^m\gamma_2\phi^{-m}(x)]$$

and we are done. ∎

3. **C^*-algebras.** From the Smale space (X, d, ϕ) we have constructed the groupoids (of equivalence relations) G_a, G_s and G_u, each with a Haar system. Again, we remark that G_a is an r-discrete groupoid; *i.e.* $\Delta = \{(x, x) \mid x \in X\}$ is an open subset of G_a. We let A, S and U denote the C^*-algebras associated with G_a, G_s and G_u, respectively [11]. (The choice of notation is to suggest "asymptotic," "stable" and "unstable" C^*-algebras. There is a slight problem since the term "stable C^*-algebra" already has a distinct meaning [10].

Caution should be used, for example, when one observes that in all of our examples both the stable and unstable C^*-algebras are stable.

For convenience, we regard the convolution algebra of continuous complex-valued functions on G_a, denoted $C_c(G_a)$, as a subalgebra of A. Similarly, we have $C_c(G_s) \subseteq S$, $C_c(G_s) \subseteq U$. Also, since Δ is open in G_a, the C^*-algebra of continuous complex-valued functions on X, $C(X)$, is a C^*-subalgebra of A.

We remark that in our examples, the groupoids G_a, G_s and G_u are amenable [11]. I do not know if this is true in general. By virtue of II.4.6 of [11], and the fact noted in Section 2 the G_a-equivalence classes are dense in X, the reduced groupoid C^*-algebra, $C^*_{\mathrm{red}}(G_a)$, is simple.

We begin with some basic properties of A, S and U.

THEOREM 3.1. *The C^*-algebras A and $S \otimes_{\max} U$ are strongly Morita equivalent.*

PROOF. Let H denote the product groupoid $G_s \times G_u$. Then, we have

$$C^*(H) \cong C^*(G_s) \otimes_{\max} C^*(G_u) = S \otimes U.$$

The unit space of H is $X \times X$ and the diagonal Δ is an abstract transversal in the sense of Muhly *et al.* [9]. Using the notation of [9], H^{Δ}_{Δ} is clearly isomorphic to G_a, so the result follows by Theorem 2.8 of [9]. ∎

The original homeomorphism ϕ preserves the equivalence relations we are considering and induces $*$-automorphisms α_a, α_s and α_u of A, S and U, respectively. Explicitly, we note that

$$\begin{aligned} \alpha_a(f) &= f \circ (\phi^{-1} \times \phi^{-1}), \quad f \in C_c(G_a) \\ \alpha_s(g) &= \lambda g \circ (\phi^{-1} \times \phi^{-1}), \quad g \in C_c(G_s) \\ \alpha_u(h) &= \lambda^{-1} h \circ (\phi^{-1} \times \phi^{-1}), \quad h \in C_c(G_u), \end{aligned}$$

where $\log \lambda$ is the topological entropy of ϕ as before.

THEOREM 3.2. *The C^*-dynamical system (A, α_a) is asymptotically abelian; that is, for all a, b in A,*

$$0 = \lim_{|n| \to \infty} \|[\alpha_a^n(a), b]\| = \lim_{|n| \to \infty} \|\alpha_a^n(a)b - b\alpha_a^n(\alpha)\|.$$

PROOF. We will show that for a, b in A,

$$\lim_{m,n \to +\infty} \|[\alpha_a^{-n}(a), \alpha_a^m(b)]\| = 0$$

and the conclusion follows. Also, it suffices to consider $a = f$, $b = g$ in $C_c(G_a)$ which is dense in A. We can cover the supports of f and g by finitely many open sets of the form

$$\{(x, \gamma(x)) \mid x \in O_\gamma\},$$

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for some conjugating map γ [14]. Then by using a partition of unity, we may express f and g as finite sums of functions each of whose support is contained in a set as above. In this way, we see it suffices to consider the case where we have two conjugating maps $\gamma_1, \ \gamma_2$, compact sets $K_1 \subseteq O_{\gamma_1}, K_2 \subseteq O_{\gamma_2}$ and f and g are supported in $\{(x, \gamma_1(x)) \mid x \in K_1\}$ and $\{(x, \gamma_2(x)) \mid x \in K_2\}$, respectively.

Let $\epsilon > 0$ be given. Choose N sufficiently large so as to satisfy the conclusion of Lemma 2.2. Since f and g are uniformly continuous, there is a $\delta > 0$ so that for any x, y with $d(x,y) < \delta$, we have

$$\begin{aligned} |f(x, \gamma_1(x)) - f(y, \gamma_1(y))| &< \epsilon/2 \sup|g| \\ |g(x, \gamma_2(x)) - g(y, \gamma_2(y))| &< \epsilon/2 \sup|f|. \end{aligned}$$

Also choose N sufficiently large so that

$$\begin{aligned} d(\phi^n(x), \phi^n\gamma_1(x)) &< \delta \\ d(\phi^n(y), \phi^n\gamma_2(y)) &< \delta \end{aligned}$$

for all $|n| \geq N$, x in O_{γ_1} and y in O_{γ_2}.

Let us compute the products $\alpha_a^{-n}(f)\alpha_a^m(g)$ and $\alpha_a^m(g)\alpha_a^{-n}(f)$ at a point (x,y) in G_a, for $n \geq N$. We denote the respective values by c_1 and c_2 for convenience. So we have

$$\begin{aligned} c_1 &= \alpha_a^{-n}(f)\alpha_a^m(g)(x,y) \\ &= \sum f(\phi^n(x), \phi^n(z)) g(\phi^{-m}(z), \phi^{-m}(y)) \end{aligned}$$

where the sum is taken over all z in $V(x)$. Immediately, we see the sum reduces to a single term, when $z = \phi^{-n}\gamma_1\phi^n(x)$. More precisely $c_1 = 0$ unless x is in $\phi^{-n}(K_1)$, $z = \phi^{-n}\gamma_1\phi^n(x)$ is in $\phi^m(K_2)$ and $y = (\phi^m\gamma_2\phi^{-m})(\phi^{-n}\gamma_1\phi^n)(x)$. Similarly, $c_2 = 0$ unless x is in $\phi^m(K_2)$, $z' = \phi^m\gamma_2\phi^{-m}(x)$ is in $\phi^{-n}(K_1)$ and $y = \phi^{-n}\gamma_1\phi^n\phi^m\gamma_2\phi^{-n}(x)$. By Lemma 2.2, we need only compare c_1 and c_2 for x in $\phi^{-n}(O_{\gamma_1})$ and $\phi^m(O_{\gamma_2})$, $\phi^{-n}\gamma_1\phi^n(x)$ in $\phi^m(O_{\gamma_2})$, and $\phi^m\gamma_2\phi^{-m}(x)$ in $\phi^{-n}(O_{\gamma_1})$. For such x, we have

$$d(\phi^n(x), \phi^n\phi^m\gamma_2\phi^{-m}(x)) = d(\phi^{n+m}\phi^{-m}(x), \phi^{n+m}\gamma_2\phi^{-m}(x)) < \delta$$

since $m + n \geq N$, $\phi^{-m}(x) \in O_{\gamma_2}$ and our choice of N. Similarly, we also have

$$d(\phi^{-m}(x), \phi^{-m}\phi^{-n}\gamma_1\phi^n(x)) < \delta.$$

Finally, we can compute, using $z = \phi^{-n}\gamma_1\phi^n(x)$ and $z' = \phi^m\gamma_2\phi^{-m}(x)$,

$$\begin{aligned} |c_1 - c_2| &= \big|f(\phi^n(x), \phi^n(z)) g(\phi^{-m}(z), \phi^{-m}(y)) \\ &\qquad - g(\phi^{-m}(x), \phi^{-m}(z')) f(\phi^n(z'), \phi^n(y))\big| \\ &\leq \big|f(\phi^x(x), \phi^n(z)) g(\phi^{-m}(z), \phi^{-m}(y)) \\ &\qquad - f(\phi^n(z'), \phi^n(y)) g(\phi^{-m}(z), \phi^{-m}(y))\big| \\ &\qquad + \big|f(\phi^n(z'), \phi^n(y)) g(\phi^{-m}(z), \phi^{-m}(y)) \\ &\qquad\qquad - f(\phi^n(z'), \phi^n(y)) g(\phi^{-n}(x), \phi^{-m}(z'))\big| \\ &\leq \sup|g| \, \big|f(\phi^n(x), \gamma_1\phi^n(x)) - f(\phi^n(z'), \gamma_1\phi^n(z'))\big| \\ &\qquad + \sup|f| \, \big|g(\phi^{-m}(z), \gamma_2\phi^{-m}(z)) - g(\phi^{-m}(x), \gamma_2\phi^{-m}(x))\big| < \epsilon. \end{aligned}$$

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We have shown that, for a given x, there is at most one y for which either c_1 or c_2 is non-zero and in either case $|c_1 - c_2| < \epsilon$. So for fixed x,

$$\sum_{y \in V(x)} \big|[\alpha_a^{-n}(f), \alpha_a^m(g)](x,y)\big| < \epsilon$$

and so

$$\big\|[\alpha_a^{-n}(f), \alpha_a^m(g)]\big\|_{I,d} < \epsilon$$

—see page 50 of [11]. A similar argument deals with the I,r-norm and so by definition

$$\big\|[\alpha_a^{-n}(f), \alpha_a^m(g)]\big\| < \epsilon$$

as desired. ∎

Some of the following is already in [14] (see Lemmas 2.1 and 2.2) but we provide a proof for completeness.

THEOREM 3.3. *The formula*

$$\mathrm{Tr}(f) = \int_X f(x,x)\, d\mu(x)$$

defines a trace on the algebras $C_c(G_a)$, $C_c(G_s)$ and $C_c(G_u)$. This extends to a bounded trace on A.

Moreover, we have

$$\mathrm{Tr} \circ \alpha_s = \lambda\, \mathrm{Tr}$$
$$\mathrm{Tr} \circ \alpha_u = \lambda^{-1}\, \mathrm{Tr} \quad \textit{and}$$
$$\mathrm{Tr} \circ \alpha_a = \mathrm{Tr}.$$

PROOF. The last three formulas follow from the definitions. The fact that Tr extends to a linear functional on A can be seen by realizing it as the composition of two maps

$$C_c(G) \longrightarrow C(X) \longrightarrow \mathbb{C}$$

the first given by restriction to Δ (which is identified with X) and the second by integration. Proposition II.4.8 of [11] asserts that the first map extends to a continuous map on A and we are done.

It remains for us to verify the trace properties. First, we consider $C_c(G_a)$. Arguing as in the proof of Theorem 3.2, we may assume that f and g are of the form considered there. Then, we have

$$\mathrm{Tr}(f \cdot g) = \int_{x \in X} \sum_{z \in V(x)} f(x,z) g(z,x)\, d\mu(x).$$

The sum is zero except for those x in K_1 with $\gamma_1(x)$ in K_2 and $\gamma_2\gamma_1(x) = x$. We denote this set by L_1 and so

$$\mathrm{Tr}(fg) = \int_{L_1} f\big(x, \gamma_1(x)\big) g\big(\gamma_1(x), x\big)\, d\mu(x).$$

Similarly, let L_2 denote the set of x in K_2 with $\gamma_2(x)$ in K_1 and $\gamma_1\gamma_2(x) = x$ and we have

$$\mathrm{Tr}(gf) = \int_{L_2} g\big(x,\gamma_2(x)\big) f\big(\gamma_2(x),x\big)\, d\mu(x).$$

It is straightforward to verify that $\gamma_1(L_1) = L_2$ and $\gamma_1^{-1} = \gamma_2$ and then $\mathrm{Tr}(fg) = \mathrm{Tr}(gf)$ follows from the invariance of μ under the conjugating homeomorphisms as noted in Section 2.

Let us now deal with $C_c(G_s)$ and since $\mathrm{Tr}\circ\alpha_s = \lambda\,\mathrm{Tr}$, we can replace f and g by $(\alpha_s)^n(f)$ and $(\alpha_s)^n(g)$, for any n. Therefore, without loss of generality we may assume that the support of f is contained in

$$K_0 = \{(x,y) \in G_s \mid d(x,y) < \epsilon\}$$

where $\epsilon > 0$ is any fixed constant. Further, for any x_0 in X the map

$$V^U(x_0,\epsilon_0/2) \times V^S(x_0,\epsilon_0/2) \times V^S(x_0,\epsilon_0/2) \longrightarrow G_s$$

defined by sending (x_1,x_2,x_3) to $([x_1,x_2],[x_1,x_3])$ is a homeomorphism onto a neighbourhood of (x_0,x_0) in G_s. By compactness, we may cover Δ by finitely many such neighbourhoods, then choose ϵ small enough so that K_0 is also covered by these neighbourhoods. Finally using a partition of unity we can reduce to the case where f is supported in such a neighbourhood. In this case, we may rewrite the integral for $\mathrm{Tr}(fg)$ changing variables via the homeomorphism above so that

$$\mathrm{Tr}(fg) = \iiint f(x_1,x_2,x_3)g(x_1,x_3,x_2)\, d\mu_u(x_1)\, d\mu_s(x_2)\, d\mu_s(x_3)$$

where the integral is over

$$(x_1,x_2,x_3) \in V^U(x_0,\epsilon_0/2) \times V^S(x_0,\epsilon_0/2) \times V^S(x_0,\epsilon_0/2).$$

We have used the fact that under $[\ ,\ ]$, μ becomes $\mu_u \times \mu_s$. A similar computation can be made for $\mathrm{Tr}(gf)$ and it is clear that they are equal. ∎

There is one more important relation between A, S and U; there are natural $*$-homomorphisms from A into the multiplier algebras $M(S)$ and $M(U)$. We refer the reader to [10] for a treatment of multiplier algebras.

THEOREM 3.4. *For f in $C_c(G_a)$, g in $C_c(G_s)$ and h in $C_c(G_u)$, define*

$$\big(\rho_s(f)g\big)(x,y) = \sum_{z\in V(x)} f(x,z)g(z,y)$$
$$\big(g\rho_s(f)\big)(x,y) = \sum_{z\in V(x)} g(x,z)f(z,y)$$

for (x,y) in G_s and

$$\big(\rho_u(f)h\big)(x,y) = \sum_{z\in V(x)} f(x,z)h(z,y)$$
$$\big(h\rho_u(f)\big)(x,y) = \sum_{z\in V(x)} h(x,z)f(z,x)$$

for (x,y) *in* G_u. *Then,* ρ_s *and* ρ_u *extend to* $*$*-homomorphisms*

$$\rho_s: A \longrightarrow M(S)$$
$$\rho_u: A \longrightarrow M(U).$$

PROOF. This is actually shown in II.2.4 of [11]. While it is not true that G_a is a *closed* subgroupoid of G_s and G_u, the inclusion maps $G_a \subseteq G_s$ and $G_a \subseteq G_u$ are continuous and proper and that is all that is required in II.2.4. ■

If we let α_s and α_u also denote the natural extensions of α_s and α_u to $M(S)$ and $M(U)$, respectively, then it is easy to verify that

$$\rho_s \circ \alpha_a = \alpha_s \circ \rho_s$$
$$\rho_u \circ \alpha_a = \alpha_u \circ \rho_u.$$

We will not prove the following—its proof is similar to that of Theorem 3.2. Note though, that the limits are one-sided.

THEOREM 3.5. *For a in A, b in S and c in U, we have*

$$\lim_{n \to +\infty} \left\| \left[\rho_s\left(\alpha_a^{-n}(a)\right), b\right] \right\| = 0$$
$$\lim_{n \to +\infty} \left\| \left[\rho_u\left(\alpha_a^{n}(a)\right), c\right] \right\| = 0.$$

We introduce yet another C^*-algebra which will be important for our later discussion involving E-theory. We denote this by C_a and define it to be the mapping cylinder for (A, α); specifically,

$$C_a = \left\{ f: [0,1] \longrightarrow A \mid f \text{ is continuous and } f(1) = \alpha_a\big(f(0)\big) \right\}.$$

There is a natural action of $\mathbb{R}$ on C, also denoted by α, defined by

$$(\alpha_t f)(x) = \alpha_a^{\lfloor t+s \rfloor}\big(f(t+s-\lfloor t+s \rfloor)\big),$$

for f in C, t in $\mathbb{R}$ and s in $[0,1]$, where $\lfloor \cdot \rfloor$ denotes the greatest integer function.

THEOREM 3.6. *The system* (C_a, α) *is asymptotically abelian; that is, for* f, g *in* C

$$0 = \lim_{|t| \to \infty} \|[\alpha_t(f), g]\|.$$

PROOF. Let $\epsilon > 0$ be given. Since f and g are uniformly continuous, we may partition the interval $[0,1]$ by points $0 = s_0 < s_1 < \cdots < s_m = 1$ so that, for s in $[s_i, s_{i+1}]$, $i = 0, \ldots, m-1$,

$$\|f(s) - f(s_i)\| < \epsilon/5\|g\|$$
$$\|g(s) - g(s_i)\| < \epsilon/5\|f\|.$$

Since (A, α_a) is asymptotically abelian, we may find N so that, for $|n| \geq N$,

$$\left\| \left[\alpha_a^n\big(f(s_i)\big), g(s_j) \right] \right\| < \epsilon/5$$

for all $i,j = 0, 1, \ldots, m$.

For any $|t| \geq N+1$ and s in $[0,1]$, let k denote $\lfloor t+s \rfloor$ and s' denote $t+s-\lfloor t+s \rfloor$. Note that $|k| \geq N$ and for some i, j, $s \in [s_i, s_{i+1}]$ and $s' \in [s_j, s_{j+1}]$ so we have

$$\begin{aligned}
\|[\alpha_t(f), g](s)\| &= \left\| \left[\alpha_a^k\big(f(s')\big), g(s) \right] \right\| \\
&\leq \left\| \alpha_a^k\big(f(s')\big) g(s) - \alpha_a^k\big(f(s')\big) g(s_i) \right\| \\
&\quad + \left\| \alpha_a^k\big(f(s')\big) g(s_i) - \alpha_a^k\big(f(s_j)\big) g(s_i) \right\| \\
&\quad + \left\| \left[\alpha_a^k\big(f(s_j)\big), g(s_i) \right] \right\| \\
&\quad + \left\| g(s_i) \alpha_a^k\big(f(s_j)\big) - g(s_i) \alpha_a^k\big(f(s')\big) \right\| \\
&\quad + \left\| g(s_i) \alpha_a^k\big(f(s')\big) - g(s) \alpha_a^k\big(f(s')\big) \right\| < \epsilon.
\end{aligned}$$

■

There is a natural trace on C_a, which we denote by Tr, defined by

$$\operatorname{Tr}(f) = \int_0^1 \operatorname{Tr}\big(f(s)\big)\, ds.$$

Note that Tr is α-invariant. Also, if f is a projection in C_a (or $M_n(C_a)$), then $\{f(s) \mid 0 \leq s \leq 1\}$ is a continuous path of projections in A which are therefore all unitarily equivalent and so all have the same trace. Hence, we have

$$\operatorname{Tr}(f) = \int_0^1 \operatorname{Tr}\big(f(s)\big)\, ds = \operatorname{Tr}\big(f(0)\big).$$

4. ***K*-theory and *E*-theory.** In this section, we discuss the K-theory of the C^*-algebras constructed in Section 3. One of the principal tools will be the E-theory of Connes and Higson [5].

Let us make some preliminary remarks about $K_*(C_a)$. There is an obvious map e_0: $C_a \longrightarrow A$ defined by $e_0(f) = f(0)$, for f in C_a, and we have a short exact sequence

$$0 \longrightarrow C_0(0,1) \otimes A \longrightarrow C_a \xrightarrow{e_0} A \longrightarrow 0.$$

The six-term exact sequence for K-groups can be used to produce the following exact sequence [4].

$$\begin{array}{ccccc}
K_0(A) & \xrightarrow{\operatorname{id} - (\alpha_a)_*} & K_0(A) & \longrightarrow & K_1(C_a) \\
{\scriptstyle (e_0)_*} \uparrow & & & & \downarrow {\scriptstyle (e_0)_*} \\
K_0(C_a) & \longleftarrow & K_1(A) & \xleftarrow[\operatorname{id} - (\alpha_a)_*]{} & K_1(A)
\end{array}$$

We also remark that

$$K_i(C_a) \cong K_{i+1}(C_a \times_{\alpha_a} \mathbb{R}) \cong K_{i+1}(A \times_{\alpha_a} \mathbb{Z});$$

the first isomorphism being Connes' analogue of the Thom isomorphism [4] and the second resulting from $C_a \times_{\alpha} \mathbb{R}$ and $A \times_{\alpha_a} \mathbb{Z}$ being strongly Morita equivalent [9].

THEOREM 4.1. *$K_*(C_a)$ has a natural $\mathbb{Z}_2$-graded ring structure. In particular, $K_0(C_a)$ is an ordered ring.*

PROOF. The asymptotically abelian action of Theorem 3.5 provides the ring structure as described in [5]. ∎

Since we will require it later, let us explicitly write the formula for the product on $K_0(C_a)$. Suppose $p = (p_{ij})$ is a projection in $M_m(C_a)$ and $q = (q_{k\ell})$ is a projection in $M_n(C_a)$. For each t in $\mathbb{R}$, we will construct a_t in $M_{mn}(C_a)$. Rather than indexing the matrices in the conventional way, it will be more convenient to use pairs of entries from $\{1,\ldots,m\} \times \{1,\ldots,n\}$. We define $(a_t)_{(i,k),(j,\ell)} = \alpha_t(p_{ij})q_{k\ell}$. It follows from α_t being asymptotically abelian that

$$\lim_{t\to+\infty} \|a_t^2 - a_t\| = 0.$$

Let χ denote the following function on $\mathbb{C}$,

$$\chi(z) = \begin{cases} 1 & \text{if } \mathrm{Re}(z) > \frac{1}{2} \\ 0 & \text{if } \mathrm{Re}(z) < \frac{1}{2}. \end{cases}$$

For t sufficiently large, the spectrum of a_t lies in the domain of χ and $\chi(a_t)$ is a projection in $M_{mn}(C_a)$. Moreover, the function sending t to $\chi(a_t)$ (for t large) is continuous so by the homotopy invariance of K-theory

$$\lim_{t\to+\infty} [\chi(a_t)]_0$$

exists in $K_0(C_a)$ and this is the product of the classes of p and q.

The trace on C_a induces a group homomorphism $\mathrm{Tr}: K_0(C_a) \longrightarrow \mathbb{R}$ which we wish to show is actually a ring homomorphism.

LEMMA 4.2. *For f, g in $C_c(G)$, we have*

$$\lim_{n\to\infty} \mathrm{Tr}\big(\alpha_a^n(f)g\big) = \mathrm{Tr}(f)\,\mathrm{Tr}(g).$$

PROOF. As in the proof of Theorem 3.2, it suffices to consider f and g as there, arising from $K_1, \gamma_1, K_2, \gamma_2$. First suppose $\gamma_2 \neq \mathrm{id}$. Then we have $\mathrm{supp} g \cap \Delta = \emptyset$ and so

$$\mathrm{Tr}(g) = \int g(x,x)\,d\mu(x) = 0.$$

Also, we have

$$\mathrm{Tr}\big(\alpha_a^n(f)g\big) = \int_X \sum_{y\in V(x)} f\big(\phi^{-n}(x), \phi^{-n}(y)\big) g(y,x)\,d\mu(x).$$

As before the sum reduces to a single term with $y = \phi^n\gamma_1\phi^{-n}(x)$. Indeed, the integrand is zero except on the set

$$\{x \in X \mid x \in \phi^n(K_1), \phi^n\gamma_1\phi^{-n}(x) \in K_2, \gamma_2\phi^n\gamma_1\phi^{-n}(x) = x\}.$$

We will show that this set is empty for n sufficiently large and so

$$\mathrm{Tr}\big(\alpha_a^n(f)g\big) = 0 = \mathrm{Tr}(f)\,\mathrm{Tr}(g),$$

as desired.

First, there is $\delta > 0$ so that $d\big(x, \gamma_2(x)\big) \geq \delta$ for all x in K_2 using the fact that, if $\gamma_2(x) = x$ for some x in K_2 then $\gamma_2 = \mathrm{id}$. For n sufficiently large

$$d\big(\phi^n(z), \phi^n\gamma_1(z)\big) < \delta$$

for all z in O_{γ_1}. Applying this to $z = \phi^{-n}(x)$ we see

$$d\big(x, \phi^n\gamma_1\phi^{-n}(x)\big) < \delta$$
$$d\big(\gamma_2\phi^n\gamma_1\phi^{-n}(x), \phi^n\gamma_1\phi^{-n}(x)\big) \geq \delta$$

and so $\gamma_2\phi^n\gamma_1\phi^{-n}(x) = x$ is impossible.

In a similar way, we have the result if $\gamma_1 \neq \mathrm{id}$ and we are left to consider the case $\gamma_1 = \gamma_2 = \mathrm{id}$. Here, we have

$$\mathrm{Tr}\big(\alpha_a^n(f)g\big) = \int_X f\big(\phi^{-n}(x), \phi^{-n}(x)\big) g(x,x)\, d\mu(x)$$

and as n tends to infinity, this has limit

$$\int_X f(x,x)\, d\mu(x) \cdot \int_X g(x,x)\, d\mu(x) = \mathrm{Tr}(f)\,\mathrm{Tr}(g)$$

since ϕ is strong mixing with respect to μ [18]. ∎

We remark that an alternate proof would be to show that Tr is a factor state and then appeal to 7.13.4 of [10].

THEOREM 4.3. *$\hat{\mathrm{Tr}}: K_0(C_a) \longrightarrow \mathbb{R}$ is a ring homomorphism.*

PROOF. It suffices to consider projections p in $M_m(C_a)$ and q in $M_n(C_a)$ and show that

$$\hat{\mathrm{Tr}}\,([p]_0 \cdot [q]_0) = \mathrm{Tr}(p) \cdot \mathrm{Tr}(q).$$

Let a_t be as defined earlier so

$$[p]_0 \cdot [q]_0 = \lim_{t \to +\infty} [\chi(a_t)]_0.$$

Since $\|a_t^2 - a_t\|$ tends to zero as t goes to infinity, we have

$$\lim_{t \to \infty} \|\chi(a_t) - a_t\| = 0.$$

Furthermore, as $\chi(a_t)$ is a projection,

$$\mathrm{Tr}\big(\chi(a_t)\big) = \mathrm{Tr}\big(\chi(a_t)(0)\big) = \mathrm{Tr}\Big(\chi\big(a_t(0)\big)\Big).$$

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Finally, putting this all together with Lemma 4.2, we obtain

$$\begin{aligned}
\hat{\mathrm{Tr}}([p]_0[q]_0) &= \lim_{t\to+\infty} \mathrm{Tr}\big(\chi(a_t)\big) \\
&= \lim_{t\to+\infty} \mathrm{Tr}\Big(\chi\big(a_t(0)\big)\Big) \\
&= \lim_{t\to+\infty} \mathrm{Tr}\big(a_t(0)\big) \\
&= \lim_{\ell\to+\infty} \mathrm{Tr}\big(a_\ell(0)\big) \\
&= \lim_{\ell\to+\infty} \sum_{i=1}^{m}\sum_{k=1}^{n} \mathrm{Tr}\big(a_\ell(0)_{(i,k),(i,k)}\big) \\
&= \lim_{\ell\to+\infty} \sum_{i,k} \mathrm{Tr}\Big(\alpha_a^\ell\big(p_{ii}(0)\big)q_{kk}(0)\Big) \\
&= \sum_{i,k} \mathrm{Tr}\big(p_{ii}(0)\big)\,\mathrm{Tr}\big(q_{kk}(0)\big) \\
&= \hat{\mathrm{Tr}}\,[p]_0 \cdot \hat{\mathrm{Tr}}\,[q]_0.
\end{aligned}$$

∎

The choice of letting t go to plus infinity is rather arbitrary; using t tending to minus infinity gives the opposite ring structure. Having decided on this, we have several other asymptotic homomorphisms.

(i) $C_a \otimes A \longrightarrow A$ by $f\otimes b \longrightarrow e_0\big(\alpha_{-t}(f)\big)b$, $t \geq 0$,

(ii) $A\otimes C_a \longrightarrow A$ by $a\otimes f \longrightarrow ae_0\big(\alpha_t(f)\big)$, $t\geq 0$,

making $K_*(A)$ a (graded) $K_*(C_a)$-bimodule. Furthermore, we also have

(iii) $C_a\otimes S \longrightarrow S$ by $f\otimes a \longrightarrow \rho_s\Big(e_0\big(\alpha_{-t}(f)\big)\Big)a$, $t\geq 0$

making $K_*(S)$ a left $K_*(C_a)$-module and

(iv) $U\otimes C_a \longrightarrow U$ by $b\otimes f \longrightarrow b\rho_u\Big(e_0\big(\alpha_t(f)\big)\Big)$, $t \geq 0$

making $K_*(U)$ a right $K_*(C_a)$-module.

We conclude to this section by returning to our examples.

SUBSHIFTS OF FINITE TYPE. Let A be an $n\times n$ matrix with non-negative integer entries (slightly more general than we considered earlier). Let (X,ϕ) be the associated subshift of finite type. As described in [6,8], the C^*-algebras A, S and U are all approximately finite-dimensional or AF-algebras. Their respective K_0-groups are computed as inductive limits of the systems

$$\mathbb{Z}^n\otimes\mathbb{Z}^n \xrightarrow{A^T\otimes A} \mathbb{Z}^n\otimes\mathbb{Z}^n \xrightarrow{A^T\otimes A} \cdots$$

$$\mathbb{Z}^n \xrightarrow{A^T} \mathbb{Z}^n \xrightarrow{A^T} \mathbb{Z}^n \longrightarrow \cdots$$

$$\mathbb{Z}^n \xrightarrow{A} \mathbb{Z} \xrightarrow{A} \mathbb{Z}^n \longrightarrow \cdots$$

In each case, the groups $\mathbb{Z}^n$ are given the standard or simplicial order and the limit is taken in the category of ordered abelian groups [7]. In the case of $K_0(A)$, we may re-interpret the result as follows. Use the natural identification of

$$\mathbb{Z}^n\otimes\mathbb{Z}^n \cong M_n(\mathbb{Z}) = H_k \quad \text{for all } k = 1,2,3,\ldots.$$

Then the map $A^T \otimes A$ from H_k to H_{k+1} becomes

$$i_k(T) = ATA.$$

Also define $\alpha_k: H_k \longrightarrow H_{k+1}$ by

$$\alpha_k(T) = A^2 T.$$

Letting H denote the limit of the system (H_k, i_k), one can show $K_0(A) \cong H$ and the α_k's induce an automorphism α of H which coincides with $(\alpha_a)_*$. Using the six term exact sequence at the start of this section and the fact $K_1(A) = 0$ since it is an AF-algebra, one sees that

$$K_0(C_a) \cong \ker(\alpha: H \longrightarrow H)$$
$$K_1(C_a) \cong \operatorname{coker}(\alpha: H \longrightarrow H).$$

Using standard methods from algebra, one can show that $K_0(C_a)$ can also be described as follows. In the ring $M_n(\mathbb{Z})$, let $Z(A)$ denote the centralizer of A. Then $K_0(C_a)$ is obtained by inverting A in $Z(A)$ (see [12]). In the case

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

the reader can easily verify that $K_0(C_a)$ is non-commutative. (Also, see [7].)

Let us return to the specific Anosov example of Section 2. The eigenvectors of the matrix $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ associated with eigenvalues $\lambda = (3 - \sqrt{5})/2$ and λ^{-1} are $(1, \theta)$ and $(1, -\theta^{-1})$, where $\theta = (1 + \sqrt{5})/2$. One then sees that the groupoids G_s and G_u are those associated with the Kronecker foliations of the two-torus associated with angles θ and $-\theta^{-1}$ (or rather $2\pi\theta$ and $-2\pi\theta^{-1}$). Thus, we have

$$S \cong A_\theta \otimes K, \quad U \cong A_{-\theta^{-1}} \otimes K,$$

where A_θ, $A_{-\theta^{-1}}$ are the irrational rotation C^*-algebras associated with θ and $-\theta^{-1}$ and K denote the compact operators.

We remark that the stable manifold theorem [2] asserts that for a general Anosov diffeomorphism ϕ of M, the stable set, $V^S(x)$, of a point x is always a one-to-one immersed copy of $\mathbb{R}^k$ (where $k = \dim E$). So stable equivalence actually gives a foliation of M (smoothness is not always present) without holonomy. Thus the foliation C^*-algebra [3] coincides with the C^*-algebra of the equivalence relation. Takai [17] has conjectured

$$K_i(S) \cong K^{i+k}(M), \quad K_i(U) \cong K^{i+n-k}(M),$$

for $i = 0, 1$, where $n = \dim M$, $k = \dim E$, $n - k = \dim F$. In the example above the foliations are actually given by flows, so the result is true by virtue of Connes' analogue of the Thom isomorphism [4].

Finally, let us consider the example of the "twice-around" solenoid of Section 2. Let

$$D = \{\exp(2\pi i k 2^{-\ell}) \mid k \in \mathbb{Z}, \ell \in N\} \subseteq S^1$$

be the dyadic roots of unity. It is easy to check that points x, y in X are stably equivalent if and only if $\pi(x) = \pi(y)\,d$, for some d in D. From this, one can show that

$$\begin{aligned} S &\cong \Big(C(S^1) \otimes K\big(L^2(\Sigma)\big)\Big) \times D \\ &\cong \big(C(S^1) \times D\big) \otimes K. \end{aligned}$$

As for unstable equivalence, there is a natural flow F on X such that $\pi \circ F_t(x) = \exp(2\pi i t)\pi(x)$. The orbits of this flow are exactly the unstable equivalence classes and, moreover,

$$U \cong C(X) \times_F \mathbb{R}.$$

This flow has a natural transversal $\pi^{-1}\{1\} \cong \Sigma$ and the first return map, F_1, is the 2^∞-odometer [18]. Therefore, using results of Rieffel (which can be found in [9]), we have

$$U \cong C(X) \times_F \mathbb{R} \cong \big(C(\Sigma) \times_{F_1} \mathbb{Z}\big) \otimes K.$$

It is interesting to note that while U and S are $*$-isomorphic to each other and to the stabilized Bunce-Deddens algebra of type 2^∞ [1], one seems to be the Fourier transformed version of the other—$\hat{D} \cong \Sigma$, $\widehat{S^1} \cong \mathbb{Z}$.

In [19], Williams gives a more general construction for one-dimensional Smale spaces. These are to the "2^∞-example" above as shifts of finite type are to the full 2-shift.

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