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NONEXISTENCE OF ALMOST-QUATERNION SUBSTRUCTURES ON THE COMPLEX PROJECTIVE SPACE

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ABSTRACT. It is shown that there are no almost-quaternion substructures on the complex projective space $P_n(\mathbb{C})$.

In [2], we have shown that there are no almost-quaternion substructures on even dimensional projective spaces $P_{2n}(\mathbb{C})$ for $n \neq 1$. In [1], which appeared after the submission of our paper [2], the following theorem was proved:

THEOREM 1. (Glover, Homer and Stong): *If n is even, the tangent bundle $\tau(P_n(\mathbb{C}))$ does not split into a nontrivial Whitney sum of complex subbundles. If n is odd, $\tau(P_n(\mathbb{C}))$ splits only into a complex line bundle and its complement.*

In the light of Theorem 1, the nonexistence of almost-quaternion substructures can be proved for all projective spaces, odd or even dimensional. Thus, we have the following theorem:

THEOREM 2. *There are no almost-quaternion substructures on the complex projective space $P_n(\mathbb{C})$.*

It is the purpose of this note to prove Theorem 2. As in [2], by an almost-quaternion k-substructure on $P_n(\mathbb{C})$ we mean a $4k$-dimensional subbundle ξ of the tangent bundle $\tau(P_n(\mathbb{C}))$ which is invariant under the standard almost-complex structure of $P_n(\mathbb{C})$, together with two orthogonal (continuous) almost-complex substructure maps F, G defined on the total space of ξ, satisfying the extra condition $FG = -GF$, and such that F is the restriction of the standard almost-complex structure map of $P_n(\mathbb{C})$ to the total space of ξ.

Thus, there exists an almost-quaternion k-substructure on $P_n(\mathbb{C})$ if and only if the structure group of $\tau(P_n(\mathbb{C}))$ can be reduced from $U(n)$ to $\mathrm{Sp}\,(k) \times U(n-2k)$.

To prove Theorem 2 we need the following lemma:

LEMMA 1. *Let K be a CW-complex, and let $\xi = (E, K, p, \mathbb{C}^{2n})$ be a complex vector bundle with a given Hermitian metric on it. If the structure group $U(2n)$ of ξ can be reduced to* Sp (n), *then all odd dimensional Chern classes of ξ vanish.*

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PROOF. The proof is similar to that of Corollary 41.9 of [3]. Let α^q be the associated bundle of ξ with fibre $\mathrm{Sp}(n)/\mathrm{Sp}(q)$, and let β^{2q} be the corresponding unitary bundle with fibre $U(2n)/U(2q)$ under the imbedding $\mathrm{Sp}(n) \subset U(2n)$. Since $\mathrm{Sp}(n)/\mathrm{Sp}(q)$ is $(4q + 2)$ − connected, α^q has a cross section over the $(4q + 3)$-skeleton K^{4q+3} of K by 29.2 of [3]. Then β^{2q} has a cross section over K^{4q+3}. Since $U(2n)/U(2q)$ is $4q$-connected, it follows by 35.5 of [3] that the primary obstruction of β^{2q} is zero. But, the primary obstruction of β^{2q} is precisely the $(2q + 1)^{\text{th}}$ Chern class of ξ. This proves the lemma.

PROOF OF THEOREM 2. For n even and $n \neq 2$ the result has already been proved in corollary 5.1 of [2]. For $n = 2$, the result follows from Lemma 1, because the total Chern class of $P_2(\mathbb{C})$ is $(1 + a)^3$ where a is a suitably chosen generator $H^2(P_n(\mathbb{C}), \mathbb{Z})$.

Assume $n = 2m + 1$ for some $m \geq 1$. Since the existence of an almost-quaternion substructure on $P_n(\mathbb{C})$ implies the splitting of the tangent bundle of $P_n(\mathbb{C})$ into non-trivial Whitney sum of complex subbundles, it follows by Theorem 1 that there are no almost-quaternion k-substructures on $P_{2m+1}(\mathbb{C})$ unless $k = m$.

Now, assume there exists an almost-quaternion m-substructure on $P_{2m+1}(\mathbb{C})$. Let α be the underlying $4m$-dimensional subbundle of $\tau(P_n(\mathbb{C}))$. The bundle α together with the almost-complex structure F defined in the introduction can be considered as a complex $2m$-bundle ξ. Its complement in $\tau(P_n(\mathbb{C}))$ is a complex line bundle and by the remark after the statement of Theorem 1.1 in [1], it has to be isomorphic to $\eta \otimes \eta$ where η is the Hopf bundle. Thus we have

$$c_1(\xi) + c_1(\eta \otimes \eta) = c_1(\tau) \text{ in } H^*(P_n(\mathbb{C}); \mathbb{Z}),$$

where τ is $\tau(P_n(\mathbb{C}))$ considered as a complex bundle, and $c_1(\xi)$, $c_1(\eta \otimes \eta)$, $c_1(\tau)$ are respective first Chern classes. If $a = -c_1(\eta)$, then

$$c_1(\tau) = \binom{2m+2}{1} \quad a = (2m + 2)a$$

$$c_1(\eta \otimes \eta) = c_1(\eta) + c_1(\eta) = -2a$$

Therefore,

$$c_1(\xi) = (2m + 2)a + 2a = (2m + 4)a$$

which is nonzero. By Lemma 1, this is a contradiction.

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