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CORRIGENDUM: LIFTINGS OF JORDAN AND SUPER JORDAN PLANES

NICOLÁS ANDRUSKIEWITSCH[1], IVÁN ANGIONO[1] AND ISTVÁN HECKENBERGER[2]

[1] *FaMAF-CIEM (CONICET), Universidad Nacional de Córdoba, Medina Allende s/n, Ciudad Universitaria, 5000 Córdoba, República Argentina* (nicolas.andruskiewitsch@unc.edu.ar, ivan.angiono@unc.edu.ar)
[2] *Fachbereich Mathematik und Informatik, Philipps-Universität Marburg, Hans-Meerwein-Strasse, D-35032 Marburg, Germany* (heckenberger@mathematik.uni-marburg.de)

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Abstract We complete the classification of the pointed Hopf algebras with finite Gelfand-Kirillov dimension that are liftings of the Jordan plane over a nilpotent-by-finite group, correcting the statement in [N. Andruskiewitsch, I. Angiono and I. Heckenberger, Liftings of Jordan and super Jordan planes, Proc. Edinb. Math. Soc., II. Ser. 61(3) (2018), 661–672.].

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1. Introduction

In the paper [1] we stated the classification of the pointed Hopf algebras with finite Gelfand-Kirillov dimension that are liftings of either the Jordan plane or the super Jordan plane over a nilpotent-by-finite group. But we overlooked one possibility, namely to deform degree one relations and therefore the classification in *loc. cit.* of liftings of Jordan planes is not complete. Here we fill the gap. It turns out that the missed example is essentially a Hopf algebra introduced by C. Ohn in 1992, see [3].

Throughout $\Bbbk$ is an algebraically closed field of characteristic 0. Recall that $\mathcal{V}(1, 2)$ is the braided vector space with basis x_1, x_2 and braiding c given by $c(x_i \otimes x_1) = x_1 \otimes x_i$, $c(x_i \otimes x_2) = (x_1 + x_2) \otimes x_i$, $i = 1, 2$. Here is the revised version of [1, Proposition 4.2].

Proposition 1. *Let G be a nilpotent-by-finite group and let H be a pointed Hopf algebra with finite* GKdim *such that*

- $\circ$ *$G(H) \simeq G$ and*
- $\circ$ *the infinitesimal braiding of H is isomorphic to $\mathcal{V}(1, 2)$.*

Then there exists a Jordanian YD-triple $\mathcal{D} = (g, \chi, \eta)$ *for* $\Bbbk G$ *such that either*

(I) $H \simeq \mathfrak{U}(\mathcal{D}, 0)$ *or* $H \simeq \mathfrak{U}(\mathcal{D}, 1)$, *introduced in* [1, §4.1]; *or*

(II) $\chi = \varepsilon$ *and there exists* $\xi \in \mathrm{Der}_{\varepsilon,\varepsilon}(\Bbbk G, \Bbbk)$, $\xi \neq 0$, *such that* $H \simeq \mathfrak{U}_\xi(\mathcal{D}, 0)$ *or* $H \simeq \mathfrak{U}_\xi(\mathcal{D}, 1)$ *see Definition* 9; *or*

(III) $\chi = \varepsilon$ *and* $H \simeq \mathfrak{U}^{jordan}(\mathcal{D})$, *see Definition* 11.

Conversely, any of these Hopf algebras is pointed and has finite GKdim, actually $\mathrm{GKdim}\, \Bbbk G + 2$. See Lemmas 10, 12 and [1, Proposition 4.2]. Notice that if $\chi = \varepsilon$ and $\xi = 0$, then $\mathfrak{U}_0(\mathcal{D}, \lambda) \simeq \mathfrak{U}(\mathcal{D}, \lambda)$, introduced in [1, §4.1].

The subspace of $(g, 1)$ skew-primitive elements in a Hopf algebra in case (I) is decomposable as G-module, while in (II) is decomposable as $\langle g \rangle$-module but it is an indecomposable G-module, and in (III) it is an indecomposable $\langle g \rangle$-module. Thus Hopf algebras from different cases could not be isomorphic. Whether Hopf algebras in the same case are isomorphic is treated as in [1, §4.1].

This note is organized as follows. In § 1.1 the minimal Hopf algebra missing in [1, Proposition 4.2] and its relation with [3] are described. In § 1.2 we discuss the gap. Proposition 1 is proved in § 1.3.

Notation

We keep the notations from [1]. Let G be a group, let $\Bbbk G$ be its group algebra and let $\widehat{G}$ be its group of characters. Given $\chi \in \widehat{G}$, recall that

$$\mathrm{Der}_{\chi,\chi}(\Bbbk G, \Bbbk) = \{\eta \in (\Bbbk G)^* : \eta(ht) = \chi(h)\eta(t) + \chi(t)\eta(h) \quad \forall h, t \in G\}.$$

A collection $\mathcal{D} = (g, \chi, \eta) \in Z(G) \times \widehat{G} \times \mathrm{Der}_{\chi,\chi}(\Bbbk G, \Bbbk)$ is a *YD-triple* for $\Bbbk G$ if $\eta(g) = 1$. Then the vector space $\mathcal{V}_g(\chi, \eta)$ with a basis $(x_i)_{i \in \mathbb{I}_2}$ belongs to ${}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$, with the coaction $\delta(x_i) = g \otimes x_i$, $i \in \mathbb{I}_2$, and the action given by

$$h \cdot x_1 = \chi(h)x_1, \quad h \cdot x_2 = \chi(h)x_2 + \eta(h)x_1, \quad h \in \Bbbk G.$$

When $\chi(g) = 1$ we say that $\mathcal{D} = (g, \chi, \eta)$ is a *Jordanian* YD-triple.

Let L be a Hopf algebra. The Δ, ε and $\mathcal{S}$ denote respectively the comultiplication, the counit and the antipode. The group of group-like elements is denoted by $G(L)$. Also the space of (g, h)-primitive elements is $\mathcal{P}_{g,h}(L) = \{\ell \in L : \Delta(\ell) = \ell \otimes h + g \otimes \ell\}$, where $g, h \in G(L)$, and $\mathcal{P}(L) = \mathcal{P}_{1,1}(L)$ is the space of primitive elements. The adjoint action of $G(L)$ on L is denoted by $g \cdot \ell := g\ell g^{-1}$, $g \in G(L)$, $\ell \in L$.

1.1. The Jordanian enveloping algebra of $\mathfrak{sl}(2)$

Let $\widetilde{\mathfrak{U}}^{\mathtt{jordan}}$ be the algebra generated by a_1, a_2, g, g^{-1} with defining relations

$$g^{\pm 1} g^{\mp 1} = 1, \quad g a_1 = a_1 g + (g - g^2), \; g a_2 = a_2 g + a_1 g. \tag{1.1}$$

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It is easy to see that $\widetilde{\mathfrak{U}}^{\mathtt{jordan}}$ is a Hopf algebra by imposing $g \in G(\widetilde{\mathfrak{U}}^{\mathtt{jordan}})$ and $a_1, a_2 \in \mathcal{P}_{g,1}(\widetilde{\mathfrak{U}}^{\mathtt{jordan}})$. We introduce

$$z = a_1a_2 - a_2a_1 - \frac{a_1^2}{2} + a_2 + \frac{1}{2}a_1 \in \widetilde{\mathfrak{U}}^{\mathtt{jordan}} \tag{1.2}$$

Lemma 2. *The element z belongs to $\mathcal{P}_{g^2,1}(\widetilde{\mathfrak{U}}^{\mathtt{jordan}})$ and commutes with g.*

Proof. We compute

$$\begin{aligned}\Delta(z) &= a_1a_2 \otimes 1 + a_1\, g \otimes a_2 + ga_2 \otimes a_1 + g^2 \otimes a_1a_2\\ &\quad - a_2a_1 \otimes 1 - a_2\, g \otimes a_1 - ga_1 \otimes a_2 - g^2 \otimes a_2a_1\\ &\quad - \frac{1}{2}a_1^2 \otimes 1 - \frac{1}{2}(a_1\, g + ga_1) \otimes a_1 - \frac{1}{2}g^2 \otimes a_1^2\\ &\quad + a_2 \otimes 1 + g \otimes a_2 + \frac{1}{2}a_1 \otimes 1 + \frac{1}{2}g \otimes a_1\\ &= z \otimes 1 + g^2 \otimes z + (a_1\, g - ga_1 + g - g^2) \otimes a_2\\ &\quad + \left(ga_2 - a_2\, g - \frac{1}{2}(a_1\, g + ga_1) + \frac{1}{2}g - \frac{1}{2}g^2\right) \otimes a_1\\ &= z \otimes 1 + g^2 \otimes z;\end{aligned}$$

here $ga_2 - a_2\, g - \frac{1}{2}(a_1\, g + ga_1) + \frac{1}{2}g - \frac{1}{2}g^2 = \frac{1}{2}(a_1\, g - ga_1 + (g - g^2)) = 0$.

It remains to prove that $\gamma(z) = 0$, where $\gamma \in \operatorname{End}_{\Bbbk}(\widetilde{\mathfrak{U}}^{\mathtt{jordan}})$ is given by $\gamma(x) = gxg^{-1} - x$, for all $x \in \widetilde{\mathfrak{U}}^{\mathtt{jordan}}$. Note that

$$\gamma(xy) = \gamma(x)(\gamma(y) + y) + x\gamma(y) \text{ for all } x, y \in \widetilde{\mathfrak{U}}^{\mathtt{jordan}}.$$

From (1.1) we have that

$$\gamma(a_1) = 1 - g, \quad \gamma(a_2) = a_1. \tag{1.3}$$

Therefore,

$$\begin{aligned}\gamma(z) &= \gamma(a_1a_2 + \left(a_2 + \frac{1}{2}a_1)(1 - a_1)\right)\\ &= \gamma(a_1)(\gamma(a_2) + a_2) + a_1\gamma(a_2)\\ &\quad + \gamma\left(a_2 + \frac{1}{2}a_1\right)(\gamma(1 - a_1) + 1 - a_1) + \left(a_2 + \frac{1}{2}a_1\right)\gamma(1 - a_1).\end{aligned}$$

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By using (1.3) we obtain that

$$\begin{aligned}\gamma(z) &= (1-g)(a_2+a_1)+a_1^2\\ &\quad + \left(a_1+\frac{1}{2}(1-g)\right)(g-a_1)+\left(a_2+\frac{1}{2}a_1\right)(g-1)\\ &= a_2+a_1-(a_2g+2a_1g+g-g^2)+a_1^2+\left(a_2+\frac{1}{2}a_1\right)(g-1)\\ &\quad + a_1g-a_1^2+\frac{1}{2}(g-g^2)+\frac{1}{2}a_1(g-1)+\frac{1}{2}(g-g^2).\end{aligned}$$

Now it follows easily that $\gamma(z)=0$. □

The Jordanian enveloping algebra of $s\ell(2)$ is

$$\mathfrak{U}^{\mathtt{jordan}} := \widetilde{\mathfrak{U}}^{\mathtt{jordan}}/\langle z\rangle. \tag{1.4}$$

By Lemma 2, $\mathfrak{U}^{\mathtt{jordan}}$ is a Hopf algebra quotient of $\widetilde{\mathfrak{U}}^{\mathtt{jordan}}$. By abuse of notation the images of g, a_1, a_2 in $\mathfrak{U}^{\mathtt{jordan}}$ are denoted by the same symbols.

Remark 3. For each $\lambda\in\Bbbk$ let

$$\mathfrak{U}^{\mathtt{jordan}}_{\lambda} := \widetilde{\mathfrak{U}}^{\mathtt{jordan}}/\langle z-\lambda(1-g^2)\rangle. \tag{1.5}$$

Then $\mathfrak{U}^{\mathtt{jordan}}_{\lambda}$ is a Hopf algebra, since $z-\lambda(1-g^2)\in\mathcal{P}_{g^2,1}(\widetilde{\mathfrak{U}}^{\mathtt{jordan}})$.

Let us now fix λ, $\mu\in\Bbbk$. Let U be the algebra

$$U=\Bbbk\langle g,g^{-1},a_1,a_2\rangle/\langle gg^{-1}-1,g^{-1}g-1\rangle.$$

Then U has a unique Hopf algebra structure such that g, $g^{-1}\in G(U)$ and a_1, $a_2\in\mathcal{P}_{g,1}(U)$. Moreover, there exists a well-defined Hopf algebra map

$$\varphi_{\lambda,\mu}: U\to\mathfrak{U}^{\mathtt{jordan}}_{\lambda},\quad g\mapsto g,\ a_1\mapsto a_1,\ a_2\mapsto a_2+\mu(1-g).$$

It is easily checked that

$$ga_1-a_1g-g+g^2,\quad ga_2-(a_2+a_1)g\in\ker\varphi_{\lambda,\mu}.$$

Moreover, for $z\in U$ defined as in (1.2) we obtain that

$$\varphi_{\lambda,\mu}(z)-z=a_1\mu(1-g)-\mu(1-g)a_1+\mu(1-g)=\mu(1-g^2).$$

Since $z=\lambda(1-g^2)\in\mathfrak{U}^{\mathtt{jordan}}_{\lambda}$, we conclude that $\varphi_{\lambda,\mu}$ induces a surjective Hopf algebra map

$$\varphi_{\lambda,\mu}:\mathfrak{U}^{\mathtt{jordan}}_{\lambda+\mu}\to\mathfrak{U}^{\mathtt{jordan}}_{\lambda}.$$

It follows that $\varphi_{0,\lambda}:\mathfrak{U}^{\mathtt{jordan}}_{\lambda}\to\mathfrak{U}^{\mathtt{jordan}}$ is a Hopf algebra isomorphism.

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Remark 4. For any $\hbar \in \Bbbk$, the Hopf algebra $U_\hbar$ was introduced by Christian Ohn in [3]; this is the algebra generated over $\Bbbk$ by K, Y, $T^{\pm 1}$ with relations:

$$TT^{-1} = T^{-1}T = 1, \quad [K, T] = T^2 - 1, \quad [Y, T] = -\frac{\hbar}{2}(KT + TK), \tag{1.6}$$

$$[K, Y] = -\frac{1}{2}(YT + TY + YT^{-1} + T^{-1}Y), \tag{1.7}$$

with the Hopf algebra structure of $U_\hbar$ determined by $T \in G(U_\hbar)$ and $X, Y \in \mathcal{P}_{T^{-1},T}(U_\hbar)$. It is easy to see that the the Hopf algebras $U_\hbar$ with $\hbar \neq 0$ are all isomorphic so we fix one of them. The appellative *Jordanian* was introduced by Alev and Dumas to the best of our knowledge. We claim that $\mathfrak{U}_\lambda^{\mathtt{jordan}}$ is isomorphic to the Hopf subalgebra $\mathfrak{U}$ of $U_\hbar$ generated by

$$x = KT^{-1}, \quad y = YT^{-1}, \; g = T^{-2}; \tag{1.8}$$

we choose these variables to have $x, y \in \mathcal{P}_{g,1}(\mathfrak{U})$. Now (1.6) implies

$$g \cdot x = x + 2(1 - g), \quad g \cdot y = y - 2\hbar(x + (1 - g)). \tag{1.9}$$

We perform a new change of variables:

$$a_1 = \frac{1}{2}x, \quad a_2 = -\frac{1}{4\hbar}y - \frac{1}{4}x;$$

these new variables satisfy (1.1). Now (1.7) translates succesively into

$$xy - yx = -2y - \hbar x^2 + \frac{\hbar}{4}(1 - g^2)$$

and then into

$$z = -\frac{1}{32}(1 - g^2).$$

That is, $\mathfrak{U} \simeq \mathfrak{U}_{-\frac{1}{32}}^{\mathtt{jordan}}$.

Remark 5. The algebra $U_\hbar$ can be described as an iterated Ore extension:

$$U_\hbar = \Bbbk[T^{\pm}][x\,;\delta][y\,;\sigma, D] \tag{1.10}$$

with δ a derivation of $\Bbbk[T^{\pm}]$, σ an automorphism of $\Bbbk[T^{\pm}][x\,;\delta]$ and D a σ-derivation of $\Bbbk[T^{\pm}][x\,;\delta]$ defined by:

$$xT = Tx + \underbrace{(T - T^{-1})}_{=\delta(T)} \tag{1.11}$$

$$yT = \underbrace{T}_{=\sigma(T)} y + \underbrace{(-\hbar Tx - \frac{\hbar}{2}(T - T^{-1}))}_{=D(T)} \tag{1.12}$$

$$yx = \underbrace{(x + 2)}_{=\sigma(x)} y + \underbrace{\hbar x^2 - \frac{\hbar}{4}(1 - T^{-4})}_{=D(x)}. \tag{1.13}$$

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Proposition 6. *There exist a derivation δ_1 of $R := \Bbbk[g, g^{-1}]$, a derivation δ_2 of $S := R[a_1; \mathrm{id}, \delta_1]$ and an automorphism σ of S such that $\mathfrak{U}^{jordan}$ is isomorphic to the Ore extension $S[a_2; \sigma, \delta_2]$.*

Hence $\mathfrak{U}^{jordan}$ is a noetherian domain of Gelfand-Kirillov 3, and the monomials $g^j a_1^{i_1} a_2^{i_2}$ form a PBW-basis of $\mathfrak{U}^{jordan}$.

Proof. We leave the verification of the first claim to the reader as a long but straightforward exercise: the derivations $\delta_1 : R \to R$, $\delta_2 : S \to S$ satisfy

$$\delta_1(g) = g^2 - g, \quad \delta_2(g) = -a_1\, g, \delta_2(a_1) = \frac{1}{2}a_1(1 - a_1),$$

and σ is given by $\sigma(g) = g$, $\sigma(a_1) = a_1 + 1$. The rest is standard. □

Corollary 7. *The Hopf algebra $\mathfrak{U}^{jordan}$ is pointed and $\operatorname{gr}\mathfrak{U}^{jordan}$ is isomorphic to the bosonization of the Jordan plane by the group algebra of the infinite cyclic group.* □

1.2. The gap and how to fix it

We fix a group G. Let H be a pointed Hopf algebra with coradical filtration $(H_n)_{n\in\mathbb{N}_0}$ such that $G(H) \simeq G$. Then $H_1/H_0 \simeq V\#\Bbbk G$, where $V \in {}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$ is the infinitesimal braiding of H. For $g \in G$, the space of $(g, 1)$ skew-primitives $\mathcal{P}_{g,1}(H)$ satisfies

$$\mathcal{P}_{g,1}(H) \cap H_0 = \Bbbk(1 - g) \text{ and } \mathcal{P}_{g,1}(H)/\left(\mathcal{P}_{g,1}(H) \cap H_0\right) \simeq V_g.$$

Now assume that $V \simeq \mathcal{V}_g(\chi, \eta)$ for a YD-triple $\mathcal{D} = (g, \chi, \eta)$ over $\Bbbk G$. Thus $V = V_g$ and we have an exact sequence of G-modules

$$0 \longrightarrow \Bbbk(1-g) \longrightarrow \mathcal{P}_{g,1}(H) \xrightarrow{\varpi} \mathcal{V}_g(\chi,\eta) \longrightarrow 0.$$

Since $g \in Z(G)$, one has $\Bbbk(1 - g) \subset \mathcal{P}_{g,1}(H)^{\varepsilon}$. Hence $\chi \neq \varepsilon$ implies that

$$\mathcal{P}_{g,1}(H) \simeq \Bbbk(1-g) \oplus \mathcal{V}_g(\chi,\eta)$$

and we have a morphism of Hopf algebras $\pi : \mathcal{T}(\mathcal{V}_g(\chi, \eta)) \to H$, where $\mathcal{T}(\mathcal{V}_g(\chi, \eta)) = T(\mathcal{V}_g(\chi, \eta))\#\Bbbk G$. In particular the proof of [1, Prop. 4.3] goes over without changes.

We assume for the rest of this Section that the infinitesimal braiding V of H is isomorphic to $\mathcal{V}_g(\varepsilon, \eta)$ for a YD-triple $\mathcal{D} = (g, \varepsilon, \eta)$ as Yetter-Drinfeld module over $\Bbbk G$. Under this assumption, $\mathcal{P}_{g,1}(H)$ might be indecomposable.

Example 8. The indecomposability of $\mathcal{P}_{g,1}(H)$ could happen in other situations. Here is a simple example. Let A be the algebra generated by a, $\gamma^{\pm 1}$, where γ^{-1} is the inverse of γ and the relation $\gamma a \gamma^{-1} = a + (1 - \gamma)$ holds, so that A is not commutative. Then A is a pointed Hopf algebra by declaring that γ is a group-like and a a $(\gamma, 1)$ skew-primitive element. Observe that $\mathcal{P}_{g,1}(A)$ is indecomposable. Let $\Gamma \simeq \mathbb{Z}$. It can be shown that $\operatorname{gr} A \simeq T(V) \otimes \Bbbk\Gamma$, where V has dimension 1 and is the infinitesimal braiding of A. But $\mathcal{P}_{g,1}(A)$ is indecomposable and there is no surjective morphism of Hopf algebras $T(V) \otimes \Bbbk\Gamma \to A$.

Back to our situation, let us pick $a_1, a_2 \in \mathcal{P}_{g,1}(H)$ such that $\varpi(a_j) = x_j$, $j = 1, 2$ and set $a_0 = 1 - g$. Then there are $\zeta \in \mathrm{Der}_{\varepsilon,\varepsilon}(\Bbbk G, \Bbbk)$ and a linear map $\xi : \Bbbk G \to \Bbbk$ such that the action of $h \in G$ on $\mathcal{P}_{g,1}(H)$ is given in the basis (a_0, a_1, a_2) by

$$\|h\| = \begin{pmatrix} 1 & \zeta(h) & \xi(h) \\ 0 & 1 & \eta(h) \\ 0 & 0 & 1 \end{pmatrix}. \tag{1.14}$$

Notice that $\mathrm{Der}_{\varepsilon,\varepsilon}(\Bbbk G, \Bbbk) = \mathrm{Hom}_{\mathrm{gps}}(G, (\Bbbk, +))$ and that ξ is a kind of differential operator of degree 2, meaning that

$$\xi(hk) = \xi(h) + \zeta(h)\eta(k) + \xi(k) \text{ for all } h, k \in G. \tag{1.15}$$

Thus if $\zeta \neq 0$, then the claim [1, Prop. 4.2, page p. 669, line 8] is not true. To correct this we consider the subalgebra A generated by g and $\mathcal{P}_{g,1}(H)$, a Hopf subalgebra of H. The action of g on $\mathcal{P}_{g,1}(H) = \mathcal{P}_{g,1}(A)$ in the basis (a_0, a_1, a_2) is given by

$$\|g\| = \begin{pmatrix} 1 & \zeta(g) & \xi(g) \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}. \tag{1.16}$$

As $g \in Z(G)$, we have that $\xi(gh) = \xi(hg)$ for all $h \in G$, so (1.15) says that

$$\zeta(h) = \eta(h)\zeta(g) \text{ for all } h \in G. \tag{1.17}$$

We consider two cases:

(A) $\zeta(g) = 0$. Then $\zeta = 0$ by (1.17) and $\xi \in \mathrm{Der}_{\varepsilon,\varepsilon}(\Bbbk G, \Bbbk)$ by (1.15).

(B) $t := \zeta(g) \neq 0$, the Jordanian case. In the basis $(a_0, t^{-1}a_1, t^{-1}a_2 - t^{-2}\xi(g)a_1)$, the action of g is given by $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$. We still denote the new basis by (a_0, a_1, a_2); that is, we may assume that $\zeta(g) = 1$, $\xi(g) = 0$. By (1.17), $\zeta = \eta$, and by (1.15), $\xi(hk) = \xi(h) + \eta(h)\eta(k) + \xi(k)$ for all $h, k \in G$.

We shall see that the following Hopf algebras exhaust the case (A).

Definition 9. Let $\mathcal{D} = (g, \varepsilon, \eta)$ be a YD-triple, $\xi \in \mathrm{Der}_{\varepsilon,\varepsilon}(\Bbbk G, \Bbbk)$ and $\lambda \in \Bbbk$. We define $\mathfrak{U}_\xi(\mathcal{D}, \lambda)$ as the algebra generated by $h \in G$, a_1, a_2 with defining relations being those of G and

$$ha_1 - a_1 h, \quad h \in G; \tag{1.18}$$

$$ha_2 - (a_2 + \eta(h)a_1 + \xi(h)(1 - g))h, \quad h \in G; \tag{1.19}$$

$$a_1a_2 - a_2a_1 - \frac{a_1^2}{2} - \lambda(1 - g^2). \tag{1.20}$$

As we said already, $\mathfrak{U}_0(\mathcal{D}, \lambda) \simeq \mathfrak{U}(\mathcal{D}, \lambda)$, introduced in [1, §4.1].

Lemma 10. *$\mathfrak{U}_\xi(\mathcal{D}, \lambda)$ is a Hopf algebra with comultiplication determined by*

$$G(\mathfrak{U}_\xi(\mathcal{D},\lambda)) = G \text{ and } a_1, a_2 \in \mathcal{P}_{g,1}(\mathfrak{U}_\xi(\mathcal{D},\lambda)).$$

Thus $\mathfrak{U}_\xi(\mathcal{D}, \lambda)$ is pointed. The set $\{a_1^m a_2^n h \,|\, m, n \in \mathbb{N}_0, \ h \in G\}$ is a basis of $\mathfrak{U}_\xi(\mathcal{D}, \lambda)$; $\operatorname{gr} \mathfrak{U}_\xi(\mathcal{D}, \lambda) \simeq \mathscr{B}(\mathcal{V}(1, 2))\#\Bbbk G$ and

$$\operatorname{GKdim} \mathfrak{U}_\xi(\mathcal{D}, \lambda) = \operatorname{GKdim} \Bbbk G + 2.$$

In particular, if G is nilpotent-by-finite, then $\operatorname{GKdim} \mathfrak{U}_\xi(\mathcal{D}, \lambda) < \infty$.

Proof. Left to the reader. □

We shall see that the following Hopf algebras exhaust the case (B).

Definition 11. Let $\mathcal{D} = (g, \varepsilon, \eta)$ be a YD-triple and define $\xi \in (\Bbbk G)^*$ by $\xi(h) = \frac{1}{2}(\eta(h)^2 - \eta(h))$, $h \in G$. We introduce $\mathfrak{U}^{\texttt{jordan}}(\mathcal{D})$ as the algebra generated by $h \in G$, a_1, a_2 with defining relations those of G, (1.19) and

$$ha_1 - (a_1 + \eta(h)(1 - g))h, \quad h \in G. \tag{1.21}$$

$$a_1a_2 - a_2a_1 - \frac{a_1^2}{2} + a_2 + \frac{1}{2}a_1. \tag{1.22}$$

Observe that ξ, needed in (1.19), satisfies (1.15) with $\zeta = \eta$. The proof of the following Lemma is also standard.

Lemma 12. *$\mathfrak{U}^{jordan}(\mathcal{D})$ is a Hopf algebra with structure determined by*

$$G(\mathfrak{U}^{jordan}(\mathcal{D})) = G \text{ and } a_1, a_2 \in \mathcal{P}_{g,1}(\mathfrak{U}^{jordan}(\mathcal{D})).$$

Thus $\mathfrak{U}^{jordan}(\mathcal{D})$ is pointed. The set $\{a_1^m a_2^n h \,|\, m, n \in \mathbb{N}_0, \ h \in G\}$ is a basis of $\mathfrak{U}^{jordan}(\mathcal{D})$; $\operatorname{gr} \mathfrak{U}^{jordan}(\mathcal{D}) \simeq \mathscr{B}(\mathcal{V}(1, 2))\#\Bbbk G$ and

$$\operatorname{GKdim} \mathfrak{U}^{jordan}(\mathcal{D}) = \operatorname{GKdim} \Bbbk G + 2.$$

In particular, if G is nilpotent-by-finite, then $\operatorname{GKdim} \mathfrak{U}^{jordan}(\mathcal{D}) < \infty$.

1.3. Proof of proposition 1

Let G be a nilpotent-by-finite group and let H be a pointed Hopf algebra with finite GKdim such that $G(H) \simeq G$ and the infinitesimal braiding V of H is isomorphic to $\mathcal{V}(1, 2)$. By [1, Lemma 2.3], there exists a unique YD-triple $\mathcal{D} = (g, \chi, \eta)$ such that $V \simeq \mathcal{V}_g(\chi, \eta)$ in ${}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$. By [1, Lemma 3.7], $\operatorname{gr} H \simeq \mathscr{B}(\mathcal{V}(1, 2))\#\Bbbk G$, hence H is generated by $\mathcal{P}_{g,1}(H)$ and G as algebra.

If $\chi \neq \varepsilon$, then the proof of [1, Prop. 4.1] implies that H is isomorphic either to $\mathfrak{U}(\mathcal{D}, 0)$ or $\mathfrak{U}(\mathcal{D}, 1)$, the Hopf algebras introduced in [1, §4.1].

Assume that $\chi = \varepsilon$. Pick a basis $(a_0 = 1 - g, a_1, a_2)$ such that any $h \in G$ acts on $\mathcal{P}_{g,1}(H)$ by (1.14) where $\zeta \in \operatorname{Der}_{\varepsilon,\varepsilon}(\Bbbk G, \Bbbk)$ and $\xi \in (\Bbbk G)^*$ satisfies (1.15). Let A be the subalgebra generated by $\mathcal{P}_{g,1}(H)$. As explained above we consider two cases.

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Case (A): $\zeta(g) = 0$, thus $\zeta = 0$. Even if [1, Proposition 4.2] does not apply in general since we may have $\xi \neq 0$, it does apply to A up to changing the base to $(a_0, a_1, \widetilde{a}_2)$ where $\widetilde{a}_2 := a_2 - \xi(g)a_1$, see (1.14). Call the new basis again (a_0, a_1, a_2) by abuse of notation. Hence $A \simeq \mathfrak{U}(\mathcal{D}', \lambda)$ where $\mathcal{D}' = (g, \chi_{|\langle g\rangle}, \eta_{|\langle g\rangle})$ is a YD-triple over the subgroup $\langle g\rangle$ of G and $\lambda \in \{0, 1\}$. In particular the following equality holds in H:

$$a_2a_1 = a_1a_2 - \tfrac{1}{2}a_1^2 + \lambda(1 - g^2).$$

We first claim that A is stable under the action of G. Indeed let G act on the free algebra generated by $g^{\pm 1}$, a_1, a_2, where G acts trivially on g, and by (1.14) on a_1, a_2. As g is central, the action of each $h \in G$ preserves the defining ideal of A, so G acts on A.

We next claim that $H \simeq A \rtimes \Bbbk G/I$, where I is the ideal that identifies the two copies of g where $\rtimes$ stands for smash product. Indeed, the inclusions $A \hookrightarrow H$, $\Bbbk G \hookrightarrow H$ induce a Hopf algebra map $\psi : A \rtimes \Bbbk G/I \to H$. As $\operatorname{gr} H \simeq \mathscr{B}(V)\#\Bbbk G$, H is generated by a_1, a_2 and G, so ψ is surjective. On the other hand, $(A \rtimes \Bbbk G/I)_1$ is spanned by the set $\{1 \otimes h,\ a_1 \otimes h,\ a_2 \otimes h | h \in G\}$. The image of this set under ψ is linearly independent, which implies that $\psi_{|(A\rtimes\Bbbk G/I)_1}$ is injective. By [2, 5.3.1], ψ is injective, and the claim follows. As a consequence, the set $\{a_1^m a_2^n h \,|\, m, n \in \mathbb{N}_0,\ h \in G\}$ is a basis of H.

Finally, we see that there is a Hopf algebra map $\mathfrak{U}_\xi(\mathcal{D}, \lambda) \to H$; since this map sends a basis to a basis, we conclude that $H \simeq \mathfrak{U}_\xi(\mathcal{D}, \lambda)$.

Case (B). $\zeta(g) \neq 0$. As discussed above, we may assume that $\zeta = \eta$. Recall that we are assuming that $\operatorname{GKdim} H < \infty$. We claim that

- **(i)** There exists a Hopf algebra isomorphism $A \simeq \mathfrak{U}^{\mathtt{jordan}}$, cf. (1.4).
- **(ii)** $\xi(h) = \frac{1}{2}(\eta(h)^2 - \eta(h))$ for all $h \in G$.
- **(iii)** A is stable under the adjoint action of G and $H \simeq A \rtimes \Bbbk G/I$, where I is the ideal that identifies the two copies of g.
- **(iv)** The set $\{a_1^m a_2^n h \,|\, m, n \in \mathbb{N}_0,\ h \in G\}$ is a basis of H and $H \simeq \mathfrak{U}^{\mathtt{jordan}}(\mathcal{D})$.

(**i**): It is easy to see that there exists a Hopf algebra surjective map $\widetilde{\pi} : \widetilde{\mathfrak{U}}^{\mathtt{jordan}} \to A$, which applies g, a_1, a_2 to the corresponding elements of A. Hence $\widetilde{\pi}(z) \in \mathcal{P}_{g^2,1}(A)$, by Lemma 2. Now, as $g \neq g^2$ and $\operatorname{gr} H \simeq \mathscr{B}(V)\#\Bbbk G$, we have that $\mathcal{P}_{g^2,1}(H) = \mathcal{P}_{g^2,1}(H) \cap H_0 = \Bbbk(1 - g^2)$; thus there exists $\lambda \in \Bbbk$ such that $\widetilde{\pi}(z) = \lambda(1 - g^2)$, which implies that $\widetilde{\pi}$ factors through a map $\pi : \mathfrak{U}_\lambda^{\mathtt{jordan}} \twoheadrightarrow A$. The set $\{g^k, a_1g^k, a_2g^k : k \in \mathbb{Z}\}$ is linearly independent in H, so $\pi_{|(\mathfrak{U}_\lambda^{\mathtt{jordan}})_1}$ is injective. By [2, 5.3.1], π is an isomorphism. Up to composing with $\varphi_{0,\lambda}$, see Remark 3, we may assume that $\lambda = 0$.

(**ii**): Given $h \in G$, let $\gamma_h \in \operatorname{End}_\Bbbk H$ be given by

$$\gamma_h(x) = hxh^{-1} - x \text{ for all } x \in H.$$

Note that $\gamma_h(xy) = \gamma_h(x)(\gamma_h(y) + y) + x\gamma_h(y)$ for all $x, y \in H$. From (1.14),

$$\gamma_h(a_1) = \eta(h)(1 - g), \quad \gamma_h(a_2) = \eta(h)a_1 + \xi(h)(1 - g). \tag{1.23}$$

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Therefore,

$$\begin{aligned}\gamma_h(z) &= \eta(h)(1-g)(\eta(h)a_1+\xi(h)(1-g)+a_2)+a_1(\eta(h)a_1+\xi(h)(1-g))\\ &\quad+\left(\eta(h)a_1+\xi(h)(1-g)+\frac{1}{2}\eta(h)(1-g)\right)(-\eta(h)(1-g)+1-a_1)\\ &\quad-\left(a_2+\frac{1}{2}a_1\right)\eta(h)(1-g)=\left(\frac{1}{2}\eta(h)-\frac{1}{2}\eta(h)^2+\xi(h)\right)(1-g^2).\end{aligned}$$

By (**i**), $z=0$, so $\gamma_h(z)=0$. Thus, $\xi(h)=\frac{1}{2}(\eta(h)^2-\eta(h))$.

(**iii**): Let G act on the free algebra generated by $g^{\pm 1}$, a_1, a_2, where G acts trivially on g, and by (1.14) on a_1, a_2. Each $h\in G$ fixes the defining relations $gg^{-1}-1$, $g^{-1}g-1$, $ga_1-a_1g-g+g^2$, z, and

$$h\cdot(ga_2-a_2g-a_1g)=ga_2-a_2g-a_1g+\eta(h)(ga_1-a_1g-(1-g)g),$$

so the action descends to A. The proof of (**iv**) is as in Case (A). □

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