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PROBABILISTIC APPROXIMATION OF A NONLINEAR PARABOLIC EQUATION OCCURRING IN RHEOLOGY

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Abstract

In this paper we are interested in a nonlinear parabolic evolution equation occurring in rheology. We give a probabilistic interpretation to this equation by associating a nonlinear martingale problem with it. We prove the existence of a unique solution, P, to this martingale problem. For any t, the time marginal of P at time t admits a density $\rho(t, x)$ with respect to the Lebesgue measure, where the function ρ is the unique weak solution to the evolution equation in a well-chosen energy space. Next we introduce a simulable system of n interacting particles and prove that the empirical measure of this system converges to P as n tends to ∞. This propagation-of-chaos result ensures that the solution to the equation of interest can be approximated using a Monte Carlo method. Finally, we illustrate the convergence in some numerical experiments.

Keywords: Nonlinear martingale problem; propagation of chaos; stochastic particle method

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1. Introduction

In rheology, modeling the flow of complex fluids is a very intricate problem which to date is far from being solved. Hébraux and Lequeux [4] presented a model which aims at describing the behavior of very concentrated suspensions of soft particles, known as soft glassy materials, under a simple shear flow. This model is obtained by dividing the material into a large number of mesoscopic elements ('blocks') with a given shear stress. From a mathematical point of view, the probability density, $p(t, x)$, for a block to undergo stress x at time t is supposed to satisfy the following evolution equation: for all $(t, x) \in [0, T] \times \mathbb{R}$,

$$\begin{gathered}\frac{\partial p}{\partial t}(t,x) = -b(t)\frac{\partial p}{\partial x}(t,x) + D(p(t))\frac{\partial^2 p}{\partial x^2}(t,x) - \mathbf{1}_{[-1,1]^c}(x)p(t,x) + \frac{2}{\sigma^2}D(p(t))\delta_0(x),\\ p \geq 0, \qquad p(0,x) = \rho_0(x).\end{gathered} \tag{1}$$

Here, for $f \in L^1(\mathbb{R})$, we define

$$D(f) := \frac{\sigma^2}{2}\int_{|x|>1} f(x)\,\mathrm{d}x, \qquad \sigma > 0,$$

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$\mathbf{1}_{[-1,1]^c}$ denotes the characteristic function of the open set $[-1, 1]^c = (-\infty, -1) \cup (1, \infty)$, δ_0 denotes the Dirac delta distribution on $\mathbb{R}$, and ρ_0 is a probability density on the real line. Let us make precise the physical interpretation of the above equation. When a block is sheared, the stress of this block evolves with a variation rate $b(t)$. This variation rate is proportional to the shear rate but does not depend on the value of the stress. In our study, the function b is assumed to be in $L^2([0, T])$. When the modulus of the stress overcomes the critical value of the stress, chosen to equal 1 here, the block becomes unstable and may relax into a state with zero stress after a characteristic relaxation time also chosen to equal 1. This phenomenon induces a rearrangement of the blocks modeled through the diffusion term $D(p(t))\partial^2 p(t, x)/\partial x^2$.

Motivated by the physical interest of this model, Cancès *et al.* [2] have studied the existence and uniqueness of solutions to (1). From an analytic point of view, the difficulty of this study comes from the possibility that the coefficient, $D(p(t))$, of the second-order spatial derivative might vanish. In the case in which the initial density ρ_0 satisfies $D(\rho_0) > 0$ (and under regularity assumptions made precise in Theorem 1, below), Cancès *et al.* [2] were able to control the time evolution to this multiplicative coefficient and prove that (1) admits a unique weak solution ρ in a well-chosen energy space, this solution being such that

$$\inf_{t\in[0,T]} D(\rho(t)) > 0. \tag{2}$$

By a weak solution we mean an integrable function $p\colon [0, T] \times \mathbb{R} \to \mathbb{R}$ such that, for any $C^{1,2}$ function ψ with compact support on $[0, T] \times \mathbb{R}$, for all $t \in [0, T]$,

$$\begin{aligned}\int_{\mathbb{R}} \psi(t, x)p(t, x)\,\mathrm{d}x = \int_{\mathbb{R}} &\psi(0, x)\rho_0(x)\,\mathrm{d}x \\ &+ \int_{[0,t]\times\mathbb{R}} \left(p\frac{\partial\psi}{\partial s} + bp\frac{\partial\psi}{\partial x} + D(p)p\frac{\partial^2\psi}{\partial x^2}\right)(s, x)\,\mathrm{d}s\,\mathrm{d}x \\ &+ \int_{[0,t]\times\mathbb{R}} \mathbf{1}_{\{|x|>1\}}\, p(s, x)(\psi(s, 0) - \psi(s, x))\,\mathrm{d}s\,\mathrm{d}x.\end{aligned}$$

For a mathematical study of the full model obtained by coupling (1) at the microscopic level with the conservation of the momentum at the macroscopic level, we refer the reader to Cancès *et al.* [3].

In this paper we are interested in constructing and proving the convergence of some Monte Carlo approximations of the solution ρ. For this purpose, we first associate a nonlinear martingale problem with (1). Let $D([0, T], \mathbb{R})$ be the space of functions on $[0, T]$ that are right continuous and have left-hand limits. We denote by X the canonical process on $D([0, T], \mathbb{R})$.

Definition 1. We say that a probability measure P on $D([0, T], \mathbb{R})$ with time marginals $(P_t)_{0\le t\le T}$ solves the nonlinear martingale problem (MP) if $P_0(\mathrm{d}x) = \rho_0(x)\,\mathrm{d}x$ and, for all $\phi \in C_b^2(\mathbb{R})$,

$$\begin{aligned}\phi(X_t) - \phi(X_0) - \int_0^t \left(b(s)\phi'(X_s) + \frac{\sigma^2}{2}P_s([-1, 1]^c)\phi''(X_s)\right)\mathrm{d}s \\ - \int_0^t (\phi(0) - \phi(X_s))\mathbf{1}_{\{|X_s|>1\}}\,\mathrm{d}s\end{aligned}$$

is a P-martingale on the time interval $[0, T]$.

This problem is nonlinear since $(\sigma^2/2)P_s([-1,1]^c)$, the diffusion coefficient at time s, involves the time marginal P_s of the solution.

If P solves problem (MP) then, according to Lemma 2(i) below, for all $\psi \in C_b^{1,2}([0,T]\times\mathbb{R})$,

$$\psi(t,X_t)-\psi(0,X_0)-\int_0^t\left(\frac{\partial\psi}{\partial s}(s,X_s)+b(s)\frac{\partial\psi}{\partial x}(s,X_s)+\frac{\sigma^2}{2}P_s([-1,1]^c)\frac{\partial^2\psi}{\partial x^2}(s,X_s)\right)\mathrm{d}s$$
$$-\int_0^t(\psi(s,0)-\psi(s,X_s))\mathbf{1}_{\{|X_s|>1\}}\,\mathrm{d}s$$

is a P-martingale on the time interval $[0,T]$. From the constancy of the expectation of this martingale, we deduce the following link between problem (MP) and (1).

Lemma 1. *If P is a solution to the nonlinear martingale problem (MP), then $t\mapsto P_t$ is a weak solution to the partial differential equation (1).*

In the first section of the paper we prove that problem (MP) admits a unique solution P and that, for any $t\in[0,T]$, $P_t(\mathrm{d}x)=\rho(t,x)\,\mathrm{d}x$, where ρ is the solution to (1) obtained by Cancès *et al.* [2]. Then, in the second section, we introduce the following system of n interacting particles obtained by replacing the nonlinearity by an interaction in the stochastic dynamics associated with the nonlinear martingale problem:

$$Y_t^{i,n}=Y_0^i+\sigma\int_0^t\sqrt{\frac{1}{n}\sum_{j=1}^n\mathbf{1}_{\{|Y_s^{j,n}|>1\}}\vee\frac{1}{n}}\,\mathrm{d}W_s^i+\int_0^t b(s)\,\mathrm{d}s-\int_0^t Y_{s^-}^{i,n}\mathbf{1}_{\{|Y_{s^-}^{i,n}|>1\}}\,\mathrm{d}N_s^i,\qquad 1\le i\le n.$$

Here $(W^i)_{1\le i\le n}$ are n independent Brownian motions, $(N^i)_{1\le i\le n}$ are n independent Poisson processes with (common) intensity 1 and $(Y_0^i)_{1\le i\le n}$ are n independent random variables with (common) density $\rho_0(\mathrm{d}x)$. We assume that $(W^i)_{1\le i\le n}$, $(N^i)_{1\le i\le n}$, and $(Y_0^i)_{1\le i\le n}$ are independent. We now face the probabilistic counterpart of the possibility that $D(p(t))$ might vanish: the empirical probability $(1/n)\sum_{j=1}^n\mathbf{1}_{\{|Y_s^{j,n}|>1\}}$ of the set $[-1,1]^c$ may be equal to 0. This is why we take the supremum of this empirical probability with $1/n$ in the diffusion coefficient of each particle in order to ensure the existence of a unique weak solution to this n-dimensional stochastic differential equation. We prove a propagation-of-chaos result which ensures that $\rho(t,\cdot)$, the solution to (1), can be approximated by $(1/n)\sum_{i=1}^n\delta_{Y_t^{i,n}}$; indeed, we prove that the $\mathcal{P}(D([0,T],\mathbb{R}))$-valued empirical measure $(1/n)\sum_{i=1}^n\delta_{Y^{i,n}}$ converges in probability to P, the unique solution to problem (MP). In the mathematical analysis of the convergence, the main difficulty is that $1/n$, the lower bound of the diffusion coefficient in the system with n particles, vanishes as $n\to\infty$. To overcome this difficulty, we first prove convergence on a small time interval. Then, to iterate the argument, we take advantage of (2), which holds for the solution to (1) given that $D(\rho_0)>0$.

In the third section we present some numerical results obtained by simulation of the system with n particles.

We use the following notation.

- For $\tau>0$, let $L_t^\infty([0,\tau],L_x^1\cap L_x^2)$ denote the space of real-valued functions f defined on $[0,\tau]\times\mathbb{R}$ and satisfying

$$\sup_{t\in[0,\tau]}\int_{\mathbb{R}}|f(t,x)|\,\mathrm{d}x<\infty\quad\text{and}\quad\sup_{t\in[0,\tau]}\int_{\mathbb{R}}|f(t,x)|^2\,\mathrm{d}x<\infty.$$

- By $L_t^2([0,\tau], H_x^1)$, we denote the space of functions f on $[0,\tau]\times\mathbb{R}$ such that the distribution derivative $\partial f/\partial x$ is a function and

$$\int_0^\tau \int_{\mathbb{R}} \left(|f(t,x)|^2 + \left| \frac{\partial f}{\partial x}(t,x) \right|^2 \right) \mathrm{d}x\, \mathrm{d}t < \infty.$$

- We say that a probability density ρ_0 satisfies the condition (H) if

$$\rho_0 \in L^\infty(\mathbb{R}), \qquad \int_{\mathbb{R}} |x|\rho_0(x)\,\mathrm{d}x < \infty, \quad \text{and} \quad D(\rho_0) > 0.$$

- Let C be a constant which may change from line to line.
- For a topological space E, $\mathcal{P}(E)$ denotes the set of probability measures on E endowed with its Borel σ-field.

2. Existence and uniqueness of the martingale problem

2.1. On equation (1)

We now recall existence and uniqueness results for (1) established in Theorem 1.1 of [2].

Theorem 1. *Let the initial density ρ_0 satisfy the condition (H). Then, for every $T > 0$, there exists a unique weak solution ρ to the system (1) in $L_t^\infty([0,T], L_x^1 \cap L_x^2) \cap L_t^2([0,T], H_x^1)$. Moreover, for all $t \in [0,T]$, $\int_{\mathbb{R}} \rho(t,x)\,\mathrm{d}x = 1$ and there exists a positive constant ν such that*

$$\frac{2}{\sigma^2} D(\rho(t)) \geq \nu \quad \textit{for all } t \in [0,T]. \tag{3}$$

In addition,

$$\sup_{t\in[0,T]} \int_{\mathbb{R}} |x|\rho(t,x)\,\mathrm{d}x < \infty. \tag{4}$$

Since, for $\alpha > 1$, denoting $\|f\|_{L_x^2} = (\int_{\mathbb{R}} f^2(x)\,\mathrm{d}x)^{1/2}$ for all square integrable real functions f,

$$\int_{[-\alpha,-1]\cup[1,\alpha]} \rho(t,x)\,\mathrm{d}x \leq 2\sqrt{\alpha-1} \sup_{t\leq T} \|\rho(t,\cdot)\|_{L_x^2},$$

we easily deduce the following corollary.

Corollary 1. *There exists an $\alpha > 1$ satisfying*

$$\int_{|x|>\alpha} \rho(t,x)\,\mathrm{d}x \geq \frac{\nu}{2} \quad \textit{for all } t \in [0,T].$$

2.2. Main results

Theorem 2. *Assume that ρ_0 satisfies condition (H). The nonlinear martingale problem (MP) admits a unique solution P. In addition, for all $t \in [0,T]$, $\rho(t,\cdot)$ is a density of the time marginal P_t with respect to the Lebesgue measure on $\mathbb{R}$.*

For the reader's convenience, the rather technical proof of the following proposition, which ensures that the last statement holds, is postponed to Section 2.3.

Proposition 1. *Assume that ρ_0 satisfies condition (H). If P solves the martingale problem (MP), then, for all $t \in [0, T]$, P_t admits $\rho(t, \cdot)$ as a density with respect to the Lebesgue measure.*

In order to deduce Theorem 2 from Proposition 1, we need to introduce a linear martingale problem.

Definition 2. Let a be a nonnegative function. We say that a probability measure P on $D([0, T], \mathbb{R})$ solves the linear martingale problem (LMP) starting at $\lambda \in \mathcal{P}(\mathbb{R})$ if $P_0 = \lambda$ and, for all $\phi \in C_b^2(\mathbb{R})$,

$$\phi(X_t) - \phi(X_0) - \int_0^t (b(s)\phi'(X_s) + a(s)\phi''(X_s))\,\mathrm{d}s - \int_0^t (\phi(0) - \phi(X_s))\mathbf{1}_{\{|X_s|>1\}}\,\mathrm{d}s$$

is a P-martingale on $[0, T]$.

On a probability space $(\Omega, \mathcal{A}, \mathrm{P})$, let $(W_t)_{t\geq 0}$ be a Brownian motion and $(N_t)_{t\geq 0}$ an independent Poisson process with intensity 1. The stochastic differential equation associated with the linear martingale problem (LMP) starting at λ is

$$Y_t = Y_0 + \int_0^t \gamma(s)\,\mathrm{d}W_s + \int_0^t b(s)\,\mathrm{d}s - \int_0^t Y_{s-}\,\mathbf{1}_{\{|Y_{s-}|>1\}}\,\mathrm{d}N_s \tag{5}$$

where $\gamma(s) = \sqrt{2a(s)}$, Y_0 is distributed according to λ, and Y_0, $(W_t)_{t\geq 0}$, and $(N_t)_{t\geq 0}$ are independent. It is clear that existence and trajectorial uniqueness results hold for this stochastic differential equation.

From [6, Theorems II_9 and II_{13} and Corollary II_{13}], we deduce the first assertion in the following lemma.

Lemma 2. (i) *For any $\lambda \in \mathcal{P}(\mathbb{R})$, the distribution of the unique solution to (5) is the unique solution to the linear martingale problem (LMP) starting at λ, say P.*

(ii) *If, in addition, $\lambda(\mathrm{d}x) = f(x)\,\mathrm{d}x$ with $f \in L^2(\mathbb{R})$ and there exists an interval $[0, \tau]$, $\tau > 0$, such that on $[0, \tau]$ the function a is bounded from below by a positive constant, then, for all $t \in [0, \tau]$, P_t has a density $p(t, x)$ with respect to the Lebesgue measure and the function p belongs to $L_t^\infty([0, \tau], L_x^1 \cap L_x^2) \cap L_t^2([0, \tau], H_x^1)$.*

The proof of the remaining assertion is postponed to Section 2.3.

Proof of Theorem 2. Let us suppose that Proposition 1 holds, and let P and Q denote two solutions to the nonlinear martingale problem (MP). Then both P and Q solve the linear martingale problem (LMP) with diffusion coefficient $a(s) = D(\rho(s))$, starting at $\lambda(\mathrm{d}x) = \rho_0(x)\,\mathrm{d}x$. Since uniqueness holds for this linear martingale problem, $P = Q$, and uniqueness holds for the nonlinear martingale problem (MP).

We still have to prove existence for the nonlinear martingale problem (MP). Let P be the solution to the linear martingale problem introduced above. By (3) and Lemma 2(ii) above, for all t in $[0, T]$ the probability distribution P_t admits a density $p(t, \cdot)$ with respect to the Lebesgue measure and the function p belongs to $L_t^\infty([0, T], L_x^1 \cap L_x^2) \cap L_t^2([0, T], H_x^1)$. Moreover, by reasoning as in the proof of Lemma 1, we find that p is a weak solution to the linear partial differential equation

$$\frac{\partial p}{\partial t}(t,x) = -b(t)\frac{\partial p}{\partial x}(t,x) + a(t)\frac{\partial^2 p}{\partial x^2}(t,x) - \mathbf{1}_{[-1,1]^c}(x)p(t,x) + \frac{2}{\sigma^2}D(p(t))\delta_0(x),$$
$$p(0,x) = \rho_0(x).$$

As ρ satisfies (1) and $a(t) = D(\rho(t))$, ρ also satisfies the above linear partial differential equation. Now, by adapting the ideas of Cancès *et al.* [2] in the proof of uniqueness for (1), we shall prove that $p = \rho$. By subtracting the equation satisfied by ρ from the one satisfied by p, we find that $q = p - \rho$ satisfies the same equation with initial condition $q(0,x) = 0$. Multiplying this equation by q and integrating over $\mathbb{R}$ with respect to x, we formally obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} q^2(t,x)\,\mathrm{d}x + a(t)\int_{\mathbb{R}}\left(\frac{\partial q}{\partial x}(t,x)\right)^2\mathrm{d}x + \int_{|x|>1} q^2(t,x)\,\mathrm{d}x = \frac{2}{\sigma^2}D(q(t))q(t,0). \quad (6)$$

Because of the regularity of the functions p and ρ, this formal computation is rigorous. We next remark that, since $\int_{\mathbb{R}} p(t,x)\,\mathrm{d}x = \int_{\mathbb{R}} \rho(t,x)\,\mathrm{d}x = 1$, we obtain

$$\left|\frac{2}{\sigma^2}D(q(t))\right| = \left|\int_{|x|\le 1} q(t,x)\,\mathrm{d}x\right| \le \sqrt{2}\|q(t,\cdot)\|_{L^2_x},$$

from the Cauchy–Schwarz inequality. Let $H^1(\mathbb{R})$ denote the space of functions f on the real line square integrable together with their distribution derivative f', endowed with norm $\|f\|_{H^1_x} = \sqrt{\int_{\mathbb{R}}(f^2(x) + (f')^2(x))\,\mathrm{d}x}$. Moreover, using the Sobolev embedding of $H^1(\mathbb{R})$ into the space of continuous, bounded functions on $\mathbb{R}$ endowed with the supremum norm, we bound the term on the right-hand side of (6) from above in the following way, for any positive constant ε:

$$\begin{aligned}\left|\frac{2}{\sigma^2}D(q(t))q(t,0)\right| &\le C\|q(t,\cdot)\|_{L^2_x}\|q(t,\cdot)\|_{H^1_x} \\ &\le \frac{C^2\|q(t,\cdot)\|^2_{L^2_x}}{2\varepsilon} + \frac{\varepsilon}{2}\|q(t,\cdot)\|^2_{L^2_x} + \frac{\varepsilon}{2}\left\|\frac{\partial}{\partial x}q(t,\cdot)\right\|^2_{L^2_x}.\end{aligned}$$

Since, by Theorem 1, $\inf_{0\le t\le T} a(t) > 0$, we may choose $\varepsilon/2 \le \inf_{0\le t\le T} a(t)$ and deduce from (6) that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|q(t,\cdot)\|^2_{L^2_x} \le \left(\frac{C^2}{2\varepsilon} + \frac{\varepsilon}{2}\right)\|q(t,\cdot)\|^2_{L^2_x}.$$

Finally, by applying Gronwall's lemma, we find that $\|q(t,\cdot)\|^2_{L^2_x} = 0$, for all $t \in [0,T]$ and, thus, that $q = 0$. This ensures that $a(t) = D(p(t))$. Therefore, P solves the nonlinear martingale problem (MP).

2.3. Proofs of technical results

Proof of Lemma 2(ii). By Lemma 2(i), it is enough to consider the stochastic differential equation (5). For $n \in \mathbb{N}^*$, let $T_n = \inf\{t > 0 \colon N_t = n\}$. The conditional distribution of $(T_1,\dots,T_n)$ given $\{N_t = n\}$ is uniform on the n-dimensional simplex $\Delta_n = \{0 < t_1 < \cdots < t_n < t\}$. Let $Q_{s,t}$ be the density of the random variable $\int_s^t \gamma(r)\,\mathrm{d}W_r + \int_s^t b(r)\,\mathrm{d}r$. Since N is independent of (Y_0, W), for $n \in \mathbb{N}$ the conditional density, $p_n(t,y)$, of Y_t given $\{N_t = n\}$ may be computed by induction on n. For $t > 0$ and $y \in \mathbb{R}$, we have $p_0(t, y) = f * Q_{0,t}(y)$ and, for all $n \geq 1$,

$$p_n(t, y) = \int_0^t \int_{\mathbb{R}} \frac{ns^{n-1}}{t^n} p_{n-1}(s, x)[\mathbf{1}_{\{|x|\leq 1\}} Q_{s,t}(y - x) + \mathbf{1}_{\{|x|>1\}} Q_{s,t}(y)]\,\mathrm{d}x\,\mathrm{d}s.$$

In order to bound the norm of

$$p(t, y) = \sum_{n=0}^{\infty} \mathrm{e}^{-t} \frac{t^n}{n!} p_n(t, y)$$

in H_x^1 and, in particular, to estimate the norm of $\partial p/\partial y$ in L_x^2, the Fourier transform is a very convenient tool.

Setting $\hat{p}_n(t, \zeta) = \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}\zeta y} p_n(t, y)\,\mathrm{d}y$, we have $\hat{p}_0(t, \zeta) = \hat{f}(\zeta)\hat{Q}_{0,t}(\zeta)$ and, for all $n \geq 1$,

$$\hat{p}_n(t, \zeta) = \int_0^t \int_{\mathbb{R}} \frac{ns^{n-1}}{t^n} p_{n-1}(s, x)[\mathbf{1}_{\{|x|\leq 1\}}\, \mathrm{e}^{\mathrm{i}\zeta x} \hat{Q}_{s,t}(\zeta) + \mathbf{1}_{\{|x|>1\}} \hat{Q}_{s,t}(\zeta)]\,\mathrm{d}x\,\mathrm{d}s.$$

Assume that the function γ^2 is bounded from below by $\varepsilon > 0$. Since, for $s \leq t$, $|\hat{Q}_{s,t}(\zeta)| \leq \exp\{-(\zeta^2/2)\varepsilon(t - s)\}$, we have

$$|\hat{p}_0(t, \zeta)| \leq |\hat{f}(\zeta)| \exp\left\{-\frac{\zeta^2}{2}\varepsilon t\right\},$$

$$|\hat{p}_n(t, \zeta)| \leq \frac{n}{t} \int_0^t \int_{\mathbb{R}} p_{n-1}(s, x)|\hat{Q}_{s,t}(\zeta)|\,\mathrm{d}x\,\mathrm{d}s \leq \frac{2n(1 - \exp\{-(\zeta^2/2)\varepsilon t\})}{t\zeta^2\varepsilon}, \qquad n \geq 1.$$

To check that p belongs to $L_t^\infty([0, \tau], L_x^2)$, we combine the Parseval–Plancherel theorem with the bounds on the modulus of the Fourier transform given before, obtaining

$$\begin{aligned}
2\pi \int_{\mathbb{R}} p^2(t, y)\,\mathrm{d}y &\leq \sum_{n=0}^{\infty} \mathrm{e}^{-t} \frac{t^n}{n!} \int_{\mathbb{R}} |\hat{p}_n(t, \zeta)|^2\,\mathrm{d}\zeta \\
&\leq \mathrm{e}^{-t} \int_{\mathbb{R}} |\hat{f}(\zeta)|^2\,\mathrm{d}\zeta + \sum_{n=1}^{\infty} \mathrm{e}^{-t} \frac{t^n}{n!} \int_{\mathbb{R}} \frac{4n^2(1 - \exp\{-(\zeta^2/2)\varepsilon t\})^2}{t^2\zeta^4\varepsilon^2}\,\mathrm{d}\zeta \\
&\leq 2\pi \mathrm{e}^{-t} \|f\|_{L^2}^2 + \sum_{n=1}^{\infty} \mathrm{e}^{-t} \frac{t^n}{n!} \frac{4n^2}{\sqrt{\varepsilon t}} \int_{\mathbb{R}} \frac{(1 - \mathrm{e}^{-x^2/2})^2}{x^4}\,\mathrm{d}x.
\end{aligned}$$

As the right-hand side is bounded uniformly if t belongs to $[0, \tau]$, it follows that $p \in L_t^\infty([0, \tau], L_x^2)$. To check that p belongs to $L_t^2([0, \tau], H_x^1)$, we note that $(\widehat{\partial p/\partial y})(t, \zeta) = \mathrm{i}\zeta\,\hat{p}(t, \zeta)$ and we write

$$\begin{aligned}
2\pi \int_0^\tau \int_{\mathbb{R}} \left|\frac{\partial p}{\partial y}(t, y)\right|^2 \mathrm{d}t\,\mathrm{d}y &= \int_0^\tau \int_{\mathbb{R}} \zeta^2 |\hat{p}(t, \zeta)|^2\,\mathrm{d}t\,\mathrm{d}\zeta \\
&\leq \sum_{n=0}^{\infty} \int_0^\tau \int_{\mathbb{R}} \mathrm{e}^{-t} \frac{t^n}{n!} \zeta^2 |\hat{p}_n(t, \zeta)|^2\,\mathrm{d}t\,\mathrm{d}\zeta \\
&\leq \int_0^\tau \int_{\mathbb{R}} \zeta^2 \mathrm{e}^{-t} \mathrm{e}^{-\zeta^2\varepsilon t} |\hat{f}(\zeta)|^2\,\mathrm{d}t\,\mathrm{d}\zeta \\
&\quad + \sum_{n=1}^{\infty} \int_0^\tau \int_{\mathbb{R}} \mathrm{e}^{-t} \frac{t^n}{n!} \frac{4n^2}{t^2} \frac{(1 - \exp\{-(\zeta^2/2)\varepsilon t\})^2}{\zeta^2\varepsilon^2}\,\mathrm{d}\zeta\,\mathrm{d}t.
\end{aligned}$$

Setting $C = \int_{\mathbb{R}}((1-\mathrm{e}^{-x^2/2})/x)^2\,\mathrm{d}x$, the change of variable $x = \zeta\sqrt{\varepsilon t}$ yields

$$2\pi \int_0^\tau \int_{\mathbb{R}} \left|\frac{\partial p}{\partial y}(t,y)\right|^2 \mathrm{d}t\,\mathrm{d}y \le \int_{\mathbb{R}} \int_0^\tau \zeta^2 \mathrm{e}^{-\zeta^2 \varepsilon t}\,\mathrm{d}t\,|\hat{f}(\zeta)|^2\,\mathrm{d}\zeta + \sum_{n=1}^{\infty} \frac{4Cn}{\varepsilon^{3/2}(n-1)!} \int_0^\tau \mathrm{e}^{-t} t^{n-3/2}\,\mathrm{d}t.$$

Using the fact that $\int_0^\tau \mathrm{e}^{-t} t^{n-3/2}\,\mathrm{d}t \le \tau^{n-1/2}/(n-1/2)$, we conclude that $p \in L_t^2([0,\tau], H_x^1)$.

We are now ready to prove Proposition 1.

Proof of Proposition 1. To obtain this result we proceed by inductive reasoning. The idea is to build a positive, increasing sequence $0 \le t_1 \le \cdots \le t_K = T$ such that, for $k \in \{1,\dots,K\}$, we are able to prove the following property: for all $t \in [0,t_k]$, the marginal distribution P_t has a probability density $p(t,\cdot)$, and $(p(t,\cdot))_{0\le t\le t_k}$ belongs to $L_t^\infty([0,t_k], L_x^1 \cap L_x^2) \cap L_t^2([0,t_k], H_x^1)$. Since, by Lemma 1, p is a weak solution to (1), by the uniqueness result in Theorem 1, $(p(t,\cdot))_{0\le t\le t_k}$ can then be identified with the restriction of ρ to the time interval $[0,t_k]$.

Let α be such that the conclusion of Corollary 1 holds, and let $K \in \mathbb{N}^*$ be such that $T/K \le ((\alpha-1)/2\|b\|_{L^2})^2$. We set $t_k = kT/K$, $k \in \{1,\dots,K\}$.

- As a first step, we use the fact that if Y_0 is distributed according to the density ρ_0, then, by Lemma 2(i), P is the distribution of the solution to the stochastic differential equation

$$Y_t = Y_0 + \int_0^t \sigma\sqrt{P_s([-1,1]^{\mathrm{c}})}\,\mathrm{d}W_s + \int_0^t b(s)\,\mathrm{d}s - \int_0^t Y_{s^-}\mathbf{1}_{\{|Y_{s^-}|>1\}}\,\mathrm{d}N_s.$$

  Let $t \in [0,t_1]$. Since $t_1 \le ((\alpha-1)/2\|b\|_{L^2})^2$, we have $\int_0^t |b(s)|\,\mathrm{d}s \le \|b\|_{L^2}\sqrt{t} \le (\alpha-1)/2$. Therefore,

$$\begin{aligned} &P_t([-1,1]^{\mathrm{c}}) \\ &\quad\ge \mathrm{P}\left(|Y_0|>\alpha,\ N_t=0,\ \left|Y_0+\sigma\int_0^t \sqrt{P_s([-1,1]^{\mathrm{c}})}\,\mathrm{d}W_s + \int_0^t b(s)\,\mathrm{d}s\right|>1\right) \\ &\quad\ge \mathrm{e}^{-t}\int_{|x|>\alpha}\rho_0(x)\,\mathrm{d}x\,\mathrm{P}\left(\left|\sigma\int_0^t\sqrt{P_s([-1,1]^{\mathrm{c}})}\,\mathrm{d}W_s\right| < \alpha-1-\int_0^t|b(s)|\,\mathrm{d}s\right) \\ &\quad\ge \frac{\nu\mathrm{e}^{-t}}{\sqrt{2\pi}}\int_0^{(\alpha-1)/2\sigma\sqrt{t}}\mathrm{e}^{-x^2/2}\,\mathrm{d}x, \end{aligned} \tag{7}$$

  by Corollary 1.

  Therefore, the diffusion coefficient, $a(t) = (\sigma^2/2)P_t([-1,1]^{\mathrm{c}})$, of the martingale problem satisfied by P is bounded from below by a positive constant on the time interval $[0,t_1]$. From Lemma 2(ii), we deduce that, for all $t \in [0,t_1]$, P_t has a density $p(t,\cdot)$ with respect to the Lebesgue measure on $\mathbb{R}$ and that the function p belongs to $L_t^\infty([0,t_1], L_x^1 \cap L_x^2) \cap L_t^2([0,t_1], H_x^1)$. On the other hand, by Lemma 1, p is a weak solution to (1). From Theorem 1 we deduce that, for $t \in [0,t_1]$, $p(t,\cdot) = \rho(t,\cdot)$.

- Now we assume that the inductive assumption is true at order $k-1$, $k \in \{2,\dots,K\}$, and show that this property remains true at order k. The image, $\tilde{P}$, of P under the mapping

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$x \in D([0,T],\mathbb{R}) \mapsto (x_{t+t_{k-1}})_{t\in[0,t_k-t_{k-1}]}$ solves the nonlinear martingale problem on the time interval $[0, t_k - t_{k-1}]$ with the initial probability distribution $\tilde{P}_0 = P_{t_{k-1}}$. Now,

$$\tilde{P}_0([-1,1]^{\mathrm{c}}) \geq \int_{|x|>\alpha} p(t_{k-1},x)\,\mathrm{d}x = \int_{|x|>\alpha} \rho(t_{k-1},x)\,\mathrm{d}x \geq \frac{\nu}{2},$$

by Corollary 1. From computations similar to the ones made in the first step, we find that, for $t \in [0, t_k - t_{k-1}]$, $\tilde{P}_t([-1,1]^{\mathrm{c}})$ is greater than the right-hand side of (7). Again we deduce from Lemma 2(ii) that, for $t \in [0, t_k - t_{k-1}]$, $\tilde{P}_t$ has a density $\tilde{p}(t,\cdot)$ and the function $\tilde{p}$ belongs to $L_t^\infty([0,t_k-t_{k-1}], L_x^1 \cap L_x^2) \cap L_t^2([0,t_k-t_{k-1}], H_x^1)$. By putting all this material together, we conclude that, for all $t \in [0,t_k]$, P_t has a density $p(t,\cdot)$, and $(p(t,\cdot))_{0\leq t\leq t_k}$ belongs to $L_t^\infty([0,t_k], L_x^1 \cap L_x^2) \cap L_t^2([0,t_k], H_x^1)$. Moreover, $(p(t,\cdot))_{0\leq t\leq t_k}$ can be identified with the restriction of ρ to the interval $[0,t_k]$.

This concludes the proof.

3. Propagation of chaos

We define a system of n interacting particles using the following stochastic differential equation:

$$Y_t^{i,n} = Y_0^i + \sigma \int_0^t \sqrt{\frac{1}{n}\sum_{j=1}^n \mathbf{1}_{\{|Y_s^{j,n}|>1\}} \vee \frac{1}{n}}\,\mathrm{d}W_s^i + \int_0^t b(s)\,\mathrm{d}s - \int_0^t Y_{s^-}^{i,n}\mathbf{1}_{\{|Y_{s^-}^{i,n}|>1\}}\,\mathrm{d}N_s^i, \quad 1 \leq i \leq n. \tag{8}$$

Here $(W^i)_{1\leq i\leq n}$ are independent Brownian motions, $(N^i)_{1\leq i\leq n}$ are independent Poisson processes with (common) intensity 1, and $(Y_0^i)_{1\leq i\leq n}$ are independent random variables distributed according to $\rho_0(x)\,\mathrm{d}x$. We assume that $(W^i)_{1\leq i\leq n}$, $(N^i)_{1\leq i\leq n}$, and $(Y_0^i)_{1\leq i\leq n}$ are independent. Between the jump times of the Poisson processes, $(Y^{1,n},\dots,Y^{n,n})$ evolves as an n-dimensional diffusion process with a piecewise-constant (in the n-dimensional spatial variable) and non-degenerate diffusion matrix. Hence, according to [1] and [9, Exercise 7.3.2], existence and uniqueness in law hold for (8).

Let $\mu^n = (1/n)\sum_{i=1}^n \delta_{Y^{i,n}}$ denote the empirical measure of the particle system. We are going to prove the following law of large numbers.

Theorem 3. *Assume that ρ_0 satisfies condition (H). As n tends to ∞, the $\mathcal{P}(D([0,T],\mathbb{R}))$-valued random variables μ^n converge in probability to P, the unique solution to the nonlinear martingale problem (MP).*

Since the particles $Y^{i,n}$, $1 \leq i \leq n$, are exchangeable, according to [10, Proposition 2.2], this result is equivalent to the propagation of chaos: for any fixed $k \in \mathbb{N}^*$, as n goes to ∞, the joint distribution of the processes $(Y_t^{1,n},\dots,Y_t^{k,n})_{t\in[0,T]}$ converges to $P^{\otimes k}$. In order to establish the theorem we need to control the possibility of the diffusion coefficient vanishing. This is why, for $\varepsilon > 0$, we introduce the stopping time

$$\tau_n^\varepsilon := \inf\left\{t > 0 \colon \frac{1}{n}\sum_{j=1}^n \mathbf{1}_{\{|Y_t^{j,n}|>1\}} < \varepsilon\right\}.$$

Let π^n denote the probability distribution of the empirical measure μ^n. We will denote by Q the canonical variable on $\mathcal{P}(D([0,T],\mathbb{R}))$. The next lemma implies that if $\mathrm{P}(\tau_n^\varepsilon \leq t)$ converges to 0 as n tends to ∞ then any weak limit π^∞ of the sequence $(\pi^n)_n$ has the following regularity property, which is desirable when taking the limit in the martingale problem formulation: $\pi^\infty(\mathrm{d}Q)$-almost everywhere, $\mathrm{d}r$-almost everywhere on $[0, t]$, Q_r does not weight the set of discontinuity points, $\{-1, 1\}$, of the characteristic function $x \mapsto \mathbf{1}_{\{|x|>1\}}$ which appears in the nonlinear diffusion coefficient, $(\sigma^2/2) P_s([-1, 1]^c) = (\sigma^2/2)\,\mathrm{E}^P(\mathbf{1}_{\{|X_s|>1\}})$, in problem (MP).

Lemma 3. *There is a constant $C > 0$ such that, for all $t \in [0, T]$ and all bounded functions f in $L^2(\mathbb{R})$,*

$$\left|\mathrm{E}^{\pi^n}\left(\int_0^t \langle Q_s, f\rangle\,\mathrm{d}s\right)\right| \leq t\|f\|_\infty \mathrm{P}(\tau_n^\varepsilon \leq t) + C\|f\|_{L^2},$$

where $\langle\cdot,\cdot\rangle$ denotes the duality bracket between a measure and a function.

The second technical lemma prepares an inductive argument as to why $\mathrm{P}(\tau_n^\varepsilon \leq T)$ tends to 0 as n tends to ∞.

Lemma 4. *For all $\alpha > 1$ and all $\kappa > 0$, there exist $\varepsilon > 0$ and $K \in \mathbb{N}^*$ such that*

$$\limsup_{n\to\infty} \mathrm{P}\left(\tau_n^\varepsilon \leq k\frac{T}{K}\right) \leq \sum_{\ell=0}^{k-1} \limsup_{n\to\infty} \mathrm{P}(\mu^n_{\ell T/K}([-\alpha, \alpha]^c) \leq \kappa) \quad \textit{for all } k \in \{1, \dots, K\}. \tag{9}$$

For the reader's convenience, the proofs of the above technical lemmas are postponed until after the proof of the theorem.

Proof of Theorem 3. By exchangeability of the particles, the tightness of the sequence $(\pi^n)_{n\geq 1}$ is equivalent to the tightness of the laws of the random variables $(Y^{1,n})_{n\geq 1}$ (again see [10, Proposition 2.2]). As the diffusion coefficient and the drift coefficient are uniformly bounded in n and the intensity of jumps remains smaller than 1, the tightness of the sequence $(Y^{1,n})_{n\geq 1}$ holds (using the Aldous criterion, for instance).

Let π^∞ be the limit of a convergent subsequence that we still index with n for notational simplicity. We are going to check that Q π^∞-almost surely solves the martingale problem (MP). To do so, for $p \in \mathbb{N}^*$, $\phi \in C_b^2(\mathbb{R})$, g is a continuous and bounded function on $\mathbb{R}^p$, and $T \geq S \geq t \geq s \geq s_1 \geq \cdots \geq s_p \geq 0$ we associate

$$\begin{aligned} F(Q) = \Bigg\langle Q, \Bigg(\phi(X_t) - \phi(X_s) - \int_s^t \left(b(r)\phi'(X_r) + \frac{\sigma^2}{2} Q_r([-1, 1]^c)\phi''(X_r)\right)\mathrm{d}r \\ - \int_s^t (\phi(0) - \phi(X_r))\,\mathbf{1}_{\{|X_r|>1\}}\,\mathrm{d}r\Bigg) g(X_{s_1}, \dots, X_{s_p})\Bigg\rangle \end{aligned} \tag{10}$$

with any $Q \in \mathcal{P}(D([0, S], \mathbb{R}))$. We want to prove that $\mathrm{E}^{\pi^\infty}(|F(Q)|) = 0$. By computing $F(\mu^n)$ using Itô's formula and then using the independence of the Brownian motions and Poisson processes, we can easily check that $\mathrm{E}(F^2(\mu^n)) \leq C/n$. Therefore,

$$\mathrm{E}^{\pi^n}(|F(Q)|) = \mathrm{E}(|F(\mu^n)|) \leq \sqrt{\mathrm{E}(F(\mu^n)^2)} \leq \frac{C}{\sqrt{n}}, \tag{11}$$

where the constant C does not depend on n. Hence, $\mathrm{E}^{\pi^n}(|F(Q)|)$ converges to 0 as n tends to ∞. Unfortunately, the mapping F is not continuous on $\mathcal{P}(D([0, T], \mathbb{R}))$ and we cannot deduce that $\mathrm{E}^{\pi^\infty}(|F(Q)|) = 0$. Nevertheless, F is continuous at any Q such that $Q_r(\{-1, 1\}) = 0$ $\mathrm{d}r$-almost surely. Thus, we should first prove that π^∞ gives full weight to such probability measures. To do so, we need to bound the diffusion coefficient of the particle system from below. We are only able to obtain such control on a small time interval. For this reason we first consider the limit on such a time interval. Then, to iterate our reasoning, we take advantage of the bound

$$P_s([-\alpha,\alpha]^{\mathrm{c}}) \geq \frac{\nu}{2}, \qquad s \in [0,T], \tag{12}$$

which holds for some $\alpha > 1$ according to Corollary 1 and Theorem 2. Applying Lemma 4 with this α and $\kappa = \nu/4$, we deduce that we can choose $\varepsilon > 0$ and $K \in \mathbb{N}^*$ such that

$$\limsup_{n\to\infty} \mathrm{P}\biggl(\tau_n^\varepsilon \leq k\frac{T}{K}\biggr) \leq \sum_{\ell=0}^{k-1} \limsup_{n\to\infty} \mathrm{P}\biggl(\mu^n_{\ell T/K}([-\alpha,\alpha]^{\mathrm{c}}) \leq \frac{\nu}{4}\biggr). \tag{13}$$

Let $\pi^{\infty,k}$ be the law of the image of Q under the restriction mapping $(Y_s)_{s\leq T} \in D([0,T],\mathbb{R}) \mapsto (Y_s)_{s\leq kT/K} \in D([0,kT/K],\mathbb{R})$ under π^∞, and let P^k be the image of P under this mapping. We are going to prove, by induction on $k \in \{0,\dots,K\}$, that $\pi^{\infty,k} = \delta_{P^k}$. Since the initial variables Y_0^i are independent and identically distributed according to $\rho_0(x)\,\mathrm{d}x$, the inductive property holds for $k=0$. We then assume that it holds at order $k-1$ and show that it remains true at order k.

From the recurrence assumption at order $k-1$, since under P the canonical process is quasi-left continuous, we can deduce that, for all $s \in [0,(k-1)T/K]$, μ_s^n converges weakly to P_s (see [7, Lemma 4.8]). Let $(m_n)_{n\geq 1}$ and m be probability measures on $\mathbb{R}$. It is well known that the weak convergence of $(m_n)_{n\geq1}$ to m implies that $\liminf_{n\to\infty} m_n(O) \geq m(O)$ for all open sets O of $\mathbb{R}$. This proves that $\{m \in \mathcal{P}(\mathbb{R})\colon m([-\alpha,\alpha]^{\mathrm{c}}) > \nu/4\}$ is an open set for the topology of weak convergence. Thus, by (12),

$$\begin{aligned}\liminf_{n\to\infty} \mathrm{P}\biggl(\mu^n_{\ell T/K}([-\alpha,\alpha]^{\mathrm{c}}) > \frac{\nu}{4}\biggr) \\ \geq \mathrm{P}\biggl(P_{\ell T/K}([-\alpha,\alpha]^{\mathrm{c}}) > \frac{\nu}{4}\biggr) = 1 \quad \text{for all } \ell \in \{0,\dots,k-1\}.\end{aligned}$$

Then, by (13), $\limsup_{n\to\infty} \mathrm{P}(\tau_n^\varepsilon \leq kT/K) = 0$. From Lemma 3, we deduce that, for any continuous, bounded function $f \in L^2(\mathbb{R})$,

$$\biggl|\mathrm{E}^{\pi^{\infty,k}}\biggl(\int_0^{kT/K} \langle Q_r, f\rangle\,\mathrm{d}r\biggr)\biggr| \leq C\|f\|_{L^2}.$$

Now let $f_\eta(x) := \max(0, 1-|1-|x||/\eta)$ for $0<\eta<1$. As $\|f_\eta\|_{L^2} = \sqrt{4\eta/3}$, if we replace f by f_η in the equation above and we let η go to 0, we deduce that, $\pi^{\infty,k}$-almost surely and $\mathrm{d}r$-almost everywhere, $Q_r(\{-1,1\}) = 0$.

Let the parameter t in (10) be smaller than kT/K. Since F, considered as a function on $\mathcal{P}(D([0,kT/K],\mathbb{R}))$, is continuous at all points Q $\mathrm{d}r$-almost everywhere satisfying $Q_r(\{-1,1\}) = 0$, we deduce from (11) that $\mathrm{E}^{\pi^{\infty,k}}(|F(Q)|) = \lim_{n\to\infty} \mathrm{E}^{\pi^n}(|F(Q)|) = 0$. Hence, $\pi^{\infty,k} = \delta_{P^k}$, which concludes the proof.

Let us now prove Lemma 3.

Proof of Lemma 3. Let f be a nonnegative or nonpositive bounded function on the real line, and let $t \in [0,T]$. Then

$$\biggl|\mathrm{E}^{\pi^n}\biggl(\int_0^t \langle Q_s, f\rangle\,\mathrm{d}s\biggr)\biggr| \leq t\|f\|_\infty \mathrm{P}(\tau_n^\varepsilon \leq t) + \biggl|\frac{1}{n}\sum_{i=1}^n \mathrm{E}\biggl(\mathbf{1}_{\{\tau_n^\varepsilon > t\}} \int_0^t f(Y_s^{i,n})\,\mathrm{d}s\biggr)\biggr|. \tag{14}$$

Setting

$$\sigma_t^{n,\varepsilon} := \mathbf{1}_{\{\tau_n^\varepsilon > t\}}\, \sigma \sqrt{\frac{1}{n}\sum_{j=1}^{n} \mathbf{1}_{\{|Y_t^{j,n}|>1\}} \vee \frac{1}{n}} + \mathbf{1}_{\{\tau_n^\varepsilon \le t\}}\, \sigma\sqrt{\varepsilon},$$

we introduce the stochastic differential equation

$$\begin{aligned} Y_0^{i,n,\varepsilon} &= Y_0^i, \\ \mathrm{d}Y_t^{i,n,\varepsilon} &= \sigma_t^{n,\varepsilon}\,\mathrm{d}W_t^i + b(t)\,\mathrm{d}t - Y_{t-}^{i,n,\varepsilon}\,\mathbf{1}_{\{|Y_{t-}^{i,n,\varepsilon}|>1\}}\,\mathrm{d}N_t^i, \qquad 1 \le i \le n. \end{aligned}$$

Up to time τ_n^ε, the processes $(Y_t^{i,n,\varepsilon},\ 1 \le i \le n)$ and $(Y_t^{i,n},\ 1 \le i \le n)$ coincide. This result, combined with the exchangeability of $(Y^{i,n,\varepsilon})_{1\le i\le n}$, enables us to replace $Y_t^{i,n}$ by $Y_t^{1,n,\varepsilon}$ in (14). We obtain

$$\left|\mathrm{E}^{\pi^n}\left(\int_0^t \langle Q_s, f\rangle\,\mathrm{d}s\right)\right| \le t\|f\|_\infty\,\mathrm{P}(\tau_n^\varepsilon \le t) + \left|\mathrm{E}\left(\int_0^t f(Y_s^{1,n,\varepsilon})\,\mathrm{d}s\right)\right|. \tag{15}$$

Now we are ready to apply the following estimation, which is a consequence of [5, Theorem 2].

Lemma 5. *Let $t \le T$, let $(\xi_s)_{s\ge 0}$ be an $(\mathcal{F}_s)$-standard real Brownian motion, and let*

$$x_s = x + \int_0^s \sigma_r\,\mathrm{d}\xi_r + \int_0^s \beta(r)\,\mathrm{d}r, \qquad s \in [0,t],$$

where $x \in \mathbb{R}$, β is a deterministic function integrable on $[0,t]$, and σ_r is an $\mathcal{F}_r$-adapted process. Let us assume that there exist constants $\underline{\sigma}$ and $\bar{\sigma}$ such that $0 < \underline{\sigma} \le \bar{\sigma}$ and $\underline{\sigma} \le \sigma_r \le \bar{\sigma}$ for all $r \in [0,t]$. Then, for all $f \in L^2(\mathbb{R})$,

$$\left|\mathrm{E}\left(\int_0^t f(x_s)\,\mathrm{d}s\right)\right| \le C\|f\|_{L^2},$$

where the constant C depends only on $\underline{\sigma}$, $\bar{\sigma}$, and T.

Coming back to our process $(Y_s^{1,n,\varepsilon})_{0\le s\le t}$, a simple decomposition of $Y_s^{1,n,\varepsilon}$ on the subsets $\{N_s^1 = k\}$, $k \in \mathbb{N}$, with the use of the conditional distribution of the jump times of N^1 given $\{N_s^1 = k\}$ yields

$$\begin{aligned} \mathrm{E}\left(\int_0^t f(Y_s^{1,n,\varepsilon})\,\mathrm{d}s\right) &= \mathrm{E}\left(\int_0^t \mathrm{e}^{-s} f(x_s^{n,\varepsilon,0})\,\mathrm{d}s\right) \\ &\quad + \sum_{k=1}^{\infty} \int_{0<s_1<\cdots<s_k<t} \mathrm{E}\left(\int_{s_k}^t \mathrm{e}^{-s} f(x_s^{n,\varepsilon,k})\,\mathrm{d}s\right)\mathrm{d}s_1\cdots\mathrm{d}s_k, \end{aligned}$$

where

$$x_s^{n,\varepsilon,k} = Y_{s_k}^{1,n,\varepsilon} + \int_{s_k}^s \sigma_r^{n,\varepsilon}\,\mathrm{d}W_r^1 + \int_{s_k}^s b(r)\,\mathrm{d}r$$

with the convention that $s_0 = 0$.

Noticing that $\sigma\sqrt{\varepsilon} \le \sigma_r^{n,\varepsilon} \le \sigma(1+\varepsilon)$, and applying Lemma 5, we deduce that, for all $f \in L^2(\mathbb{R})$,

$$\left|\mathrm{E}\left(\int_0^t f(Y_s^{1,n,\varepsilon})\,\mathrm{d}s\right)\right| \le C\mathrm{e}^t\|f\|_{L^2}. \tag{16}$$

Equations (15) and (16) together conclude the proof.

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Let us now prove Lemma 4.

Proof of Lemma 4. Let $\alpha > 1$ and $\kappa > 0$. As in the proof of Proposition 1, we introduce a $K \in \mathbb{N}^*$ such that $T/K \leq ((\alpha-1)/2\|b\|_{L^2})^2$ and set $t_1 = T/K$. Let $\varepsilon = \kappa\beta(t_1)/2$, where

$$\beta(t_1) = \mathrm{P}\bigg(\sup_{s\leq t_1} |W_s^i| \leq \frac{\alpha-1}{2\sigma},\ N_{t_1}^i = 0\bigg).$$

Let I denote the set of indexes $\{i \leq n \colon |Y_0^i| > \alpha\}$. If we decompose the event $\{\tau_n^\varepsilon \leq t_1\}$ on the event $\{\mathrm{card}(I) < \kappa n\}$ and its complement, we obtain

$$\mathrm{P}(\tau_n^\varepsilon \leq t_1) \leq \mathrm{P}(\mu_0^n([-\alpha,\alpha]^{\mathrm{c}}) < \kappa) + \mathrm{P}(\mathrm{card}(I) \geq \kappa n,\ \tau_n^\varepsilon \leq t_1). \tag{17}$$

We are going to prove that the limit, as $n \to \infty$, of the second term on the right-hand side of (17) is 0. Since $\int_0^{t_1} |b(r)|\,\mathrm{d}r \leq \|b\|_{L^2}\sqrt{t_1} = (\alpha-1)/2$, for $j \in I$ the existence of an $s \in [0, t_1]$ such that $|Y_s^{j,n}| \leq 1$ implies that either $N_{t_1}^j \neq 0$ or $\sup_{s\leq t_1} |\int_0^s \sigma_r^n\,\mathrm{d}W_r^j| > (\alpha-1)/2$, where

$$\sigma_r^n := \sigma\sqrt{\frac{1}{n}\sum_{j=1}^n \mathbf{1}_{\{|Y_s^{j,n}|>1\}} \vee \frac{1}{n}}.$$

Therefore, the second term on the right-hand side of (17) is bounded from above by

$$\mathrm{P}\bigg(\mathrm{card}(I) \geq \kappa n,\ \sum_{j\in I} \mathbf{1}_{\{N_{t_1}^j\neq 0 \text{ or } \sup_{s\leq t_1} |\int_0^s \sigma_r^n\,\mathrm{d}W_r^j|>(\alpha-1)/2\}} > \mathrm{card}(I) - n\varepsilon\bigg).$$

On the other hand, considering the filtration

$$\mathcal{G}_t := \sigma(Y_0^i, (N_s^i)_{s\leq T,\,1\leq i\leq n}, (W_s = (W_s^1,\dots,W_s^n))_{s\leq t}), \qquad t \in [0,T]$$

and the $\mathcal{G}_t$-martingale $M_t := \int_0^t \sigma_r^n\,\mathrm{d}W_r$, with $A_t := \int_0^t (\sigma_r^n)^2\,\mathrm{d}r$ and $\tau_t := \inf\{s \colon A_s \geq t\}$, by the Dambis and Dubins–Schwarz theorem [8, Theorem 1.6, p. 170] $B_t := M_{\tau_t} = \int_0^{\tau_t} \sigma_r^n\,\mathrm{d}W_r$ is an $\mathbb{R}^n$-valued $\mathcal{G}_{\tau_t}$-Brownian motion and $\int_0^t \sigma_r^n\,\mathrm{d}W_r = B_{A_t}$. This implies that $\mathrm{P}(\mathrm{card}(I) \geq \kappa n,\ \tau_n^\varepsilon \leq t_1)$ is smaller than

$$\mathrm{P}\bigg(\mathrm{card}(I) \geq \kappa n,\ \sum_{j\in I} \mathbf{1}_{\{N_{t_1}^j\neq 0 \text{ or } \sup_{s\leq t_1} |B_{A_s}^j|>(\alpha-1)/2\}} > \mathrm{card}(I) - n\varepsilon\bigg).$$

Noting that $A_s \leq \sigma^2 s$, and by using the definition of ε, we can replace the last upper bound by

$$\mathrm{P}\bigg(\mathrm{card}(I) \geq \kappa n,\ \frac{1}{\mathrm{card}(I)}\sum_{j\in I} \mathbf{1}_{\{N_{t_1}^j=0,\ \sup_{s\leq\sigma^2 t_1} |B_s^j|\leq(\alpha-1)/2\}} \leq \frac{\beta(t_1)}{2}\bigg).$$

Now, as $\sigma(Y_0^i,\ (N_s^i)_{s\leq T},\ 1 \leq i \leq n) = \mathcal{G}_0 \subset \mathcal{G}_{\tau_t}$, we deduce that $(N_s^i,\ s \leq T,\ 1 \leq i \leq n)$, $(B_s^i,\ s \leq T,\ 1 \leq i \leq n)$, and $(Y_0^i,\ 1 \leq i \leq n)$ are independent. With $\mathcal{F}_0 := \sigma(Y_0^i,\ 1 \leq i \leq n)$, this probability reads

$$\mathrm{E}\bigg(\mathbf{1}_{\{\mathrm{card}(I)\geq\kappa n\}}\,\mathrm{P}\bigg(\frac{1}{\mathrm{card}(I)}\sum_{j\in I} \mathbf{1}_{\{N_{t_1}^j=0,\ \sup_{s\leq t_1} |B_s^j|\leq(\alpha-1)/2\sigma\}} \leq \frac{\beta(t_1)}{2}\ \bigg|\ \mathcal{F}_0\bigg)\bigg).$$

Using the Bienaymé–Chebyshev inequality, we obtain

$$\mathrm{P}\bigg(\frac{1}{\mathrm{card}(I)}\sum_{j\in I}\mathbf{1}_{\{N^j_{t_1}=0,\ \sup_{s\le t_1}|B^j_s|\le(\alpha-1)/2\sigma\}}\le\frac{\beta(t_1)}{2}\ \bigg|\ \mathcal{F}_0\bigg)\le\frac{4}{\beta(t_1)\,\mathrm{card}(I)}.$$

Finally, the second term on the right-hand side of (17) is smaller than $4/\kappa\beta(t_1)n$ and converges to 0.

Next we use induction on $k\in\{1,\dots,N\}$ to establish (9). Since

$$\begin{aligned}\mathrm{P}(\tau^\varepsilon_n\le kt_1)\le{}&\mathrm{P}(\tau^\varepsilon_n\le(k-1)t_1)+\mathrm{P}(\mu^n_{(k-1)t_1}([-\alpha,\alpha]^{\mathrm{c}})\le\kappa)\\&+\mathrm{P}(\mu^n_{(k-1)t_1}([-\alpha,\alpha]^{\mathrm{c}})>\kappa,\ (k-1)t_1<\tau^\varepsilon_n\le kt_1),\end{aligned}$$

assuming that (9) holds at order $k-1$ we have

$$\begin{aligned}\limsup_{n\to\infty}\mathrm{P}(\tau^\varepsilon_n\le kt_1)\le{}&\sum_{\ell=0}^{k-1}\limsup_{n\to\infty}\mathrm{P}(\mu^n_{\ell t_1}([-\alpha,\alpha]^{\mathrm{c}})\le\kappa)\\&+\limsup_{n\to\infty}\mathrm{P}(\mu^n_{(k-1)t_1}([-\alpha,\alpha]^{\mathrm{c}})>\kappa,\ (k-1)t_1<\tau^\varepsilon_n\le kt_1).\end{aligned}$$

With $\tilde I=\{i\le n\colon |Y^i_{(k-1)t_1}|>\alpha\}$, by reasoning similar to that above for the time interval $[0,t_1]$, we obtain

$$\begin{aligned}\mathrm{P}(\mu^n_{(k-1)t_1}([-\alpha,\alpha]^{\mathrm{c}})>\kappa,\ (k-1)t_1<\tau^\varepsilon_n\le kt_1)&\le\mathrm{E}\bigg(\mathbf{1}_{\{\mathrm{card}(\tilde I)\ge\kappa n\}}\frac{4}{\beta(t_1)\,\mathrm{card}(\tilde I)}\bigg)\\&\le\frac{4}{\kappa\beta(t_1)n},\end{aligned}$$

which vanishes as n goes to ∞.

From a physical point of view, the average stress, $\int_{\mathbb{R}}x\rho(t,x)\,\mathrm{d}x$, is of particular interest. From Theorem 3 we can deduce the convergence of the particle approximation $(1/n)\sum_{i=1}^n Y^{i,n}_t$ to this quantity as n tends to ∞.

Corollary 2. *Assume that ρ_0 satisfies condition (H). Then*

$$\lim_{n\to\infty}\mathrm{E}\bigg|\frac{1}{n}\sum_{i=1}^n Y^{i,n}_t-\int_{\mathbb{R}}x\rho(t,x)\,\mathrm{d}x\bigg|=0.$$

Proof. From Theorem 3, since under P the canonical process is quasi-left continuous, for any $t\in[0,T]$, μ^n_t converges in probability to $P_t=\rho(t,x)\,\mathrm{d}x$ as n tends to ∞. We have

$$|Y^{1,n}_t|\le|Y^1_0|+\int_0^T|b(s)|\,\mathrm{d}s+2\sigma\sup_{s\le T}\bigg|\int_0^s\sqrt{\frac{1}{n}\sum_{j=1}^n\mathbf{1}_{\{|Y^{j,n}_s|>1\}}\vee\frac{1}{n}}\,\mathrm{d}W^1_s\bigg|.$$

Since the diffusion coefficient is bounded by 1 and the random variable $|Y^1_0|+\int_0^T|b(s)|\,\mathrm{d}s$ is integrable, the random variables $(|Y^{1,n}_t|)_{n\ge1}$ are uniformly integrable. Combining this property with (4) and the convergence in probability of μ^n_t to $P_t=\rho(t,x)\,\mathrm{d}x$, we easily obtain the result.

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4. Numerical results

To check the validity of the results obtained in the previous section with computer simulations, we consider the example of steady states given in [2]. According to [2, Proposition 5.1], if the function $b(t) = b$ is constant, then (1) admits a unique stationary solution in the following two cases.

- If $b = 0$ and $\sigma^2 > 1$, then

$$p(x) = \frac{1 - |x| + \sqrt{D}}{\sigma^2} \mathbf{1}_{\{x \in [-1,1]\}} + \frac{\sqrt{D}}{\sigma^2} \exp\left\{\frac{1 - |x|}{\sqrt{D}}\right\} \mathbf{1}_{\{x \notin [-1,1]\}},$$

with $D = D(p) > 0$ given by $D + \sqrt{D} = (\sigma^2 - 1)/2$.

- If $b \neq 0$ and $\sigma^2 \neq 0$, then

$$\begin{aligned} p(x) &= a_1 \exp\{\beta_{\operatorname{sgn}(x)} x\} \mathbf{1}_{\{x \notin [-1,1]\}} \\ &\quad + \left(a_2\left(1 + \exp\left\{\frac{b}{D} x\right\}\right) - \frac{2D}{b\sigma^2} \exp\left\{\frac{b}{D} x^+\right\}\right) \mathbf{1}_{\{x \in [-1,1]\}}, \end{aligned}$$

where $\operatorname{sgn}(x)$ denotes the sign of x, with $\beta_\pm = b/2D \mp \frac{1}{2}\sqrt{(b^2 + 4D)/D^2}$, $x^+ = \sup(0, x)$, and

$$a_1 = \frac{2 \exp\{\frac{1}{2}\sqrt{(b^2 + 4D/D^2)}\}}{\sigma^2(\beta_- \exp\{b/2D\} - \beta_+ \exp\{-b/2D\})},$$

$$a_2 = \frac{2D\beta_- \exp\{b/2D\}}{\sigma^2 b(\beta_- \exp\{b/2D\} - \beta_+ \exp\{-b/2D\})}.$$

This function always satisfies $D = D(p) > 0$, and the normalization condition

$$\int_{\mathbb{R}} p(x)\,\mathrm{d}x = 1$$

reads

$$\frac{D}{b} \frac{(1 + \beta_-) + (\beta_+ - 1)\exp\{-b/D\}}{\beta_- - \beta_+ \exp\{-b/D\}} + D = \frac{\sigma^2}{2}.$$

For fixed n, we want to simulate n interacting particles described by the stochastic differential equation (8). In order to discretize time, we assign n particle positions $(\hat{Y}^{i,n}_{kT/K})_{1 \leq i \leq n}$ to each time $k(T/K)$, $0 \leq k \leq K$, where K is a given integer. Let $\{G^i_k,\ 1 \leq i \leq n,\ 1 \leq k \leq K\}$ and $\{U^i_k,\ 1 \leq i \leq n,\ 1 \leq k \leq K\}$ be two independent sequences of independent and identically distributed random variables respectively distributed according the normal law and the uniform law on $[0, 1]$. At $k = 0$ we simulate n independent particles with initial density $\rho_0(x)$. For $k \in \{1, \ldots, K\}$, the discretized particles evolve as follows: for all $i \in \{1, \ldots, n\}$,

$$\hat{Y}^{i,n}_{kT/K} = \begin{cases} 0 & \text{if } |\hat{Y}^{i,n}_{(k-1)T/K}| > 1 \text{ and } U^i_k \leq T/K, \\ \hat{Y}^{i,n}_{(k-1)T/K} + \sigma D_{(k-1)T/K} \sqrt{\dfrac{T}{K}} G^i_k + b\dfrac{T}{K} & \text{otherwise,} \end{cases}$$

with

$$D_{(k-1)T/K} = \sqrt{\frac{1}{n} \sum_{i=1}^{n} \mathbf{1}_{\{|\hat{Y}^{i,n}_{(k-1)T/K}| > 1\}}}.$$

TABLE 1: Convergence of ε_n with respect to n.

n	ε_n $k = 100$	$k = 1000$
1 000	0.0360	0.0369
5 000	0.0158	0.0169
10 000	0.0116	0.0115
20 000	0.0080	0.0081
40 000	0.0058	0.0060
60 000	0.0046	0.0047
80 000	0.0040	0.0044
100 000	0.0036	0.0037

TABLE 2: Convergence of $n\mathrm{var}(\tau_1^n)$ with respect to n.

n	$n\,\mathrm{var}(\tau_1^n)$
1 000	0.5022943
5 000	0.4662847
10 000	0.4844257
20 000	0.4435595
40 000	0.4628567
60 000	0.4513587
80 000	0.4543330
100 000	0.4840270

The average stress in the physical model is given by $\tau(t) = \int_{\mathbb{R}} xp(t,x)\,\mathrm{d}x$ and it is approximated at the points kT/K, $k \in \{0,\dots,K\}$, by the empirical mean $\tau_{kT/K}^n = (1/n)\sum_{i=1}^n \hat{Y}_{kT/K}^{i,n}$. The simulation of $\tau_{kT/K}^n$ for $k \in \{0,\dots,K\}$ must therefore confirm the convergence toward $\int_{\mathbb{R}} xp(x)\,\mathrm{d}x$ as K and n tend to ∞.

4.1. Convergence with respect to n, the number of particles

Here we are interested in an example of the second type, namely steady states with $b = 1$ and $D = 0.5$. We start from equilibrium, i.e. we choose $\rho_0 = p$. We have

$$\int_{\mathbb{R}} xp(x)\,\mathrm{d}x = 1.126\,734\,8.$$

We take $T = 1$ first with $K = 100$ then with $K = 1000$. We simulate $M = 1000$ independent realizations, $(\tau_1^{j,n})_{1\le j\le M}$, of the random variable τ_1^n, with different values of n. We consider the empirical mean,

$$\varepsilon_n = \frac{1}{M}\sum_{j=1}^{M}\left|\tau_1^{j,n} - \int_{\mathbb{R}} xp(x)\,\mathrm{d}x\right|,$$

of the absolute value of the difference between the stress tensor $\int_{\mathbb{R}} xp(x)\,\mathrm{d}x$ and its particle approximation. In Table 1 we display results showing the convergence of the approximation ε_n of $\mathrm{E}(|\tau_1^n - \int_{\mathbb{R}} xp(x)\,\mathrm{d}x|)$ to 0 as n tends to ∞. As is easily seen by comparing ε_{5000}, $\varepsilon_{20\,000}$, and $\varepsilon_{100\,000}$, the error decreases like $C/\sqrt{n}$. Therefore, it is natural to try to check experimentally if the central limit theorem is satisfied in n, the number of particles. To do so, we choose $b = 0$

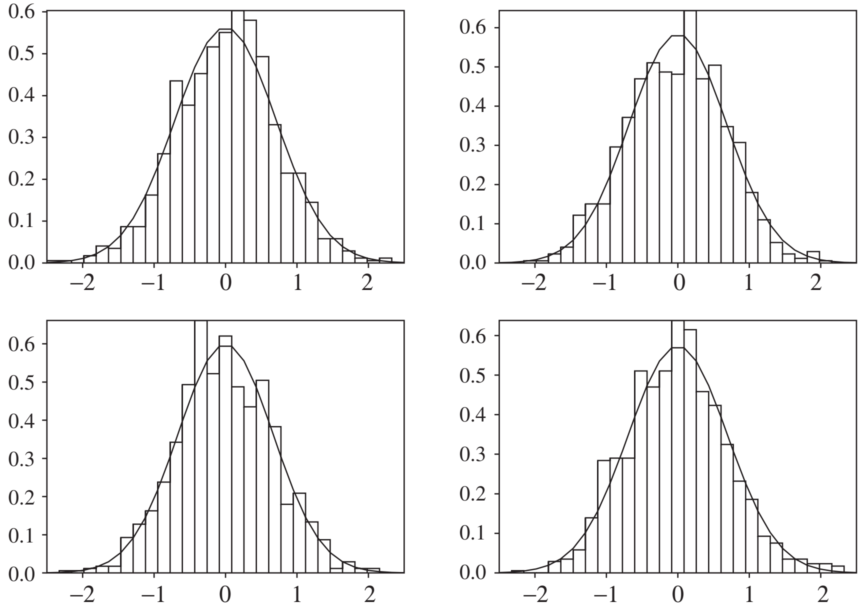

FIGURE 1: Convergence in distribution of the stress $\sqrt{n}\bar{\tau}_1^n$ with respect to n. From left to right, and top to bottom we have $n = 1000$, $n = 5000$, $n = 20\,000$, and $n = 100\,000$.

and $D = 1 - \sqrt{2}/2$, and initialize the particles with the first example of a steady distribution, which is such that

$$\int_{\mathbb{R}} x p(x)\, \mathrm{d}x = 0.$$

For $K = 100$ and different values of n, in Figure 1 we plot the histogram of $\sqrt{n}\tau_1^{j,n}$, $1 \le j \le M$, on the interval $[-2.5S_n, 2.5S_n]$, where

$$S_n^2 = \left[\frac{n}{(M-1)}\right]\sum_{j=1}^{M}(\tau_1^{j,n} - \bar{\tau}_1^n)^2,$$

with $\bar{\tau}_1^n = (1/M)\sum_{j=1}^{M} \tau_1^{j,n}$, is an estimator of $n\operatorname{var}(\tau_1^n)$. We compare this histogram with the centered Gaussian density with variance S_n^2. We have $n\operatorname{var}(\tau_1^n) = (1/n)\operatorname{var}(\sum_{i=1}^{n} \hat{Y}_1^{i,n})$, and Table 2 shows numerical convergence of this quantity as $n \to \infty$, despite the lack of theoretical proof.

The graphical representation in Figure 1 illustrates the convergence in law of the sequence $\sqrt{n}\tau_1^n$ towards the Gaussian distribution.

4.2. Convergence with respect to K, the number of time-steps

To investigate the influence of K, we choose $b = 0$ and $D = 1 - \sqrt{2}/2$, and initialize the particles with the nonequilibrium density of $2|G_1| - 3|G_2|$, where G_1 and G_2 are independent, normal variables. For fixed $n = 1000$ and for $T = 1$, we approximate $\mathrm{E}(\tau_1^n)$ by the Monte Carlo method over $M = 100\,000$ independent trajectories for different values of K. In Table 3 we display results showing the convergence of the approximation $\bar{\tau}_1^n = (1/M)\sum_{j=1}^{M} \tau_1^{j,n}$ of $\mathrm{E}(\tau_1^n)$ with K.

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TABLE 3: Convergence of $\bar{\tau}_1^n$ with K.

Number of time steps (K)	Stress	Variance	Confidence intervals at 95%
2	−0.2062	0.0018	[−0.2065, −0.2060]
4	−0.2582	0.0022	[−0.2585, −0.2579]
8	−0.2801	0.0023	[−0.2804, −0.2798]
16	−0.2898	0.0024	[−0.2901, −0.2895]
100	−0.2981	40.0025	[−0.2984, −0.2978]

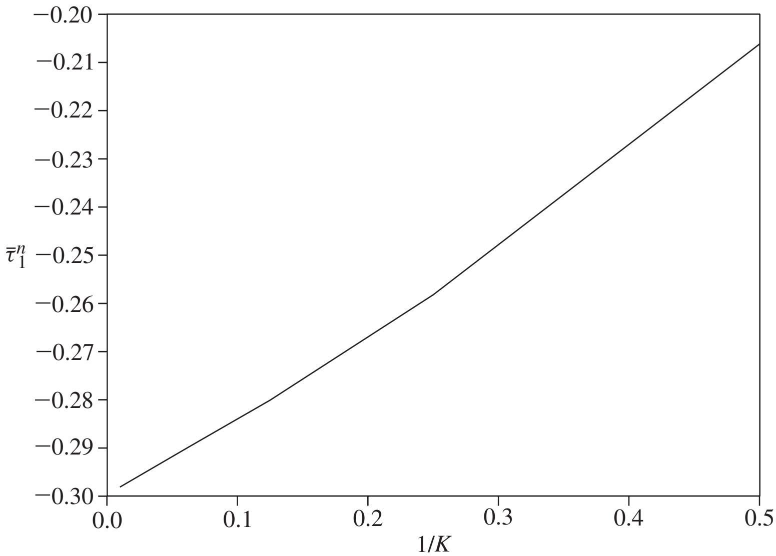

FIGURE 2: Convergence of the stress with K.

The graphical representation in Figure 2 shows that, despite the lack of theoretical study of the weak convergence of the discretization scheme, $\bar{\tau}_1^n$ converges like C/K with K.

5. Conclusion

The propagation-of-chaos theorem proved in the present paper provides a theoretical basis for the practical simulation of the average stress, which is of interest in physics. Some first numerical tests are completely conclusive with respect to the convergence and seem promising with respect to the rate of convergence. From a theoretical point of view, the next question is now to investigate the latter subject.

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