THE CLIFFORD ALGEBRA AND THE GROUP OF SIMILITUDES

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Let $C(M, Q)$ be the Clifford algebra of an even dimensional vector space M relative to a quadratic form Q. When Q is non-degenerate, it is well known that there exists an isomorphism of the orthogonal group $O(Q)$ onto the group of those automorphisms of $C(M, Q)$ which leave invariant the space $M \subset C(M, Q)$. These automorphisms are inner and the group of invertible elements of $C(M, Q)$ which define such inner automorphisms is called the Clifford group.

If instead of the group $O(Q)$ we take the group of similitudes $\gamma(Q)$ or even the group of semi-similitudes $\Gamma\gamma(Q)$, it is possible to associate in a natural way with any element of these groups an automorphism or semi-automorphism, respectively, of the subalgebra of even elements $C^+(M, Q) \subset C(M, Q)$. Each one of the automorphisms of $C^+(M, Q)$ so defined can be extended, as it is shown here (Theorem 2), to an inner automorphism of $C(M, Q)$, although the extension is not unique. A semi-automorphism of $C^+(M, Q)$ associated to a semi-similitude of Q can be extended to all of $C(M, Q)$ if and only if the ratio of the semi-similitude satisfies the conditions given in Theorem 3. Although the extension is not unique, Theorem 3 gives all the possible extensions. We do not know if there exist semi-similitudes whose ratios do not satisfy the conditions of Theorem 3. In particular, Theorem 2 asserts that the conditions do hold for the similitudes. This gives a new proof of a result of Dieudonné (cf. **(6)** and the corollary of Theorem 3).

In the case that the characteristic of the ground field K is $\neq 2$, we show that $C(M, Q)$ can be expressed as a direct sum of certain subspaces. This permits us to consider $C(M, Q)$ as a graded space. In II, we use this gradation to characterize the automorphisms of $C(M, Q)$ associated with the similitudes and to characterize the semi-automorphisms of $C^+(M, Q)$ associated to the semi-similitudes (Theorem 4).

In III we use our characterization of such automorphisms to define what we call *the extended Clifford group*. Then we apply to the elements of this group the usual definition of spin-norm. In this way we obtain a mapping of the extended Clifford group into the centre of the algebra $C^+(M, Q)$.

If S is a similitude of ratio ρ, $\rho^{-1}S^2$ is an orthogonal transformation. We determine the spin-norm of this orthogonal transformation using the mapping

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mentioned above (Proposition 1). In the same way we find the spin-norm of the commutator of elements of the group of similitudes (Proposition 2). These results can be used when the Witt index of the quadratic form Q is > 0, to prove that the commutator group of the group of similitudes coincides with the group generated by the elements of the form $\rho^{-1}S^2$ (Theorem 5). Theorem 5 also gives necessary and sufficient conditions for this to hold true for the group of proper similitudes.

I

The well-known results on the Clifford algebras, (**3**; **7**; **9**), and the exterior algebras, (**1**; **2**; **4**), will be used, and computations with the elements of these algebras will be made assuming that the reader is familiar with them. However, it has been considered convenient to start with the general definitions of these algebras.

The Clifford algebra. Let M be a finite dimension vector space over a field K of characteristic $\neq 2$, Q a quadratic form on M. The Clifford algebra $C(M, Q)$ relative to Q is the factor algebra of the free algebra

$$F(M) = K \oplus M \oplus M_2 \oplus \ldots \oplus M_i \oplus \ldots$$

where $M_i = M \otimes {}_K M \otimes_K \ldots \otimes_K M$ i times, by the ideal I generated by the elements $x \otimes x - Q(x)$. That is $C(M, Q) = F(M)/I$ and it is an algebra over K.

If $\dim M = n$, the algebra $C(M, Q)$ has dimension 2^n and given any basis $x_1, x_2, \ldots, x_n$ of M the coset $x_1^{\epsilon_1} x_2^{\epsilon_2} \ldots x_n^{\epsilon_n}$, $\epsilon_i = 0, 1$, of the elements $x_1^{\epsilon_1} \otimes x_2^{\epsilon_2} \otimes \ldots \otimes x_n^{\epsilon_n} \in F(M)$ form a basis for $C(M, Q)$.

The free algebra $F(M)$ has a main antiautomorphism of order 2 which carries the element

$$x_{i_1} \otimes x_{i_2} \otimes \ldots \otimes x_{i_n} \text{ into } x_{i_n} \otimes \ldots \otimes x_{i_1}.$$

Since this antiautomorphism leaves invariant the generators of I, I is invariant under it (as an ideal), and therefore it induces an antiautomorphism in the Clifford algebra which we denote by * and call the main antiautomorphism of $C(M, Q)$.

$F(M)$ can be made into a graded algebra if we define the degree of an element $a \in M_h$ to be h. It also can be made into a semi-graded algebra, that is, an algebra with a gradation in which the set of indices is the group with two elements $\{1, -1\}$ (cf. **4**, chapter I). The elements of degree 1 or positive elements form the subalgebra $F^+(M) = K(= M_0) \oplus M_2 \oplus \ldots \oplus M_{2i} \oplus \ldots$ and the elements of degree -1 or negative elements form the subspace $F^-(M) = M_1 \oplus M_3 \oplus \ldots \oplus M_{2i+1} \oplus \ldots$.

The ideal I is generated by elements of $F^+(M)$, therefore I is homogeneous under the semi-gradation and can be written as $I = I^+ \oplus I^-$, where

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$I^+ = I \cap F^+(M)$, $I^- = I \cap F^-(M)$. It follows that the Clifford algebra has a structure of semi-graded algebra, namely,

$$C(M, Q) = F(M)/I \approx F^+(M)/I^+ \oplus F^-(M)/I^- = C^+(M, Q) \oplus C^-(M, Q).$$

The exterior algebra. When the quadratic form Q is identically 0, the ideal I, generated now by the homogeneous elements $x \otimes x$, is homogeneous under the gradation of $F(M)$ defined before. It follows that the factor algebra $E(M) = F(M)/I$ is a graded algebra with the same set of indices; however, in this case the subspaces of degree $s > n = \dim M$ are 0. Therefore $E(M)$, the exterior algebra of M, is the direct sum of its subspaces of degree $0, 1, \ldots, n$; that is, $E(M) = K \oplus E_1 \oplus E_2 \oplus \ldots \oplus E_n$, where E_r is the subspace of elements of degree r, and $K = E_0$ that of elements of degree 0.

A linear isomorphism of E(M) onto C(M,Q)(1). If $x_1, x_2, \ldots, x_r \in M$ we define the element $[x_1 \ldots x_r] \in C(M, Q)$ inductively by

$$\begin{aligned}
&[x] = x \\
&[x_1 \ldots x_{2k-1}x_{2k}] = [[x_1 \ldots x_{2k-1}], x_{2k}] \qquad ([a, b] = ab - ba)\\
&[x_1 \ldots x_{2k}x_{2k+1}] = \{[x_1 \ldots x_{2k}], x_{2k+1}\} \qquad (\{a, b\} = ab + ba).
\end{aligned}$$

LEMMA 1. *The function $[x_1 \ldots x_r]$ vanishes when any two of its arguments are equal.*

Proof. Since the function is multilinear it suffices to show that $[x_1 \ldots x_{r-2}xx] = 0$. This follows from the following two calculations

$$\begin{aligned}
[\{a, x\}, x] &= (ax + xa)x - x(ax + xa) = ax^2 - x^2a = Q(x)a - Q(x)a = 0,\\
\{[a, x], x\} &= (ax - xa)x + x(ax - xa) = ax^2 - x^2a = 0.
\end{aligned}$$

LEMMA 2. *The space spanned by* 1 *and the* $[x_1 \ldots x_r]$ *is the whole of* $C(M, Q)$.

Proof. Let $y_1, y_2, \ldots, y_n$ be an orthogonal basis for M with respect to the bilinear form associated to Q. Then one proves by induction that

$$[y_{i_1}y_{i_2} \ldots y_{i_r}] = 2^{r-1}y_{i_1}y_{i_2} \ldots y_{i_r}$$

if the i_j are different. This implies the lemma.

THEOREM 1. *There is a linear isomorphism of the exterior algebra $E(M)$ onto $C(M, Q)$ sending* 1 *into* 1, *$x_1 \wedge \ldots \wedge x_r$ into $[x_1 \ldots x_r]$. If $M_{[r]}$ denotes the subspace of elements $[x_1, \ldots x_r]$ in $C(M, Q)$, image of the subspace E_r, then $C(M, Q) = K \oplus M_{[1]} \oplus M_{[2]} \oplus \ldots \oplus M_{[n]}$.*

Proof. Since $[x_1 \ldots x_r]$ is multilinear we have a linear mapping of the free algebra $F(M)$ into $C(M, Q)$ sending 1 into 1, $x_1 \otimes \ldots \otimes x_r$ into $[x_1 \ldots x_r]$. By Lemma 1 this induces a linear mapping of the exterior algebra $E(M)$

(1) I am indebted to Professor Jacobson for this definition.

into $C(M, Q)$. By Lemma 2 this mapping is onto $C(M, Q)$. Comparison of dimensionalities shows that this is an isomorphism which gives

$$C(M, Q) = K \oplus M_{[1]} \oplus M_{[2]} \oplus \ldots \oplus M_{[n]}.$$

Since $E_1 = M$ and under the linear isomorphism E is isomorphically mapped onto $M_{[1]}$, this space is identified with M.

DEFINITION 1. *If an element of $C(M, Q)$ belongs to $M_{[r]}$ it will be said that it has degree r. The field K is considered as $M_{[0]}$.*

Under this gradation $C(M, Q)$ is not a graded algebra but a graded vector space over K, and $C^+(M, Q) = K \oplus M_{[2]} \oplus \ldots \oplus M_{[2r]}$ if dim $M = 2r$ or $2r + 1$.

Let $y_1, y_2, \ldots, y_r \in M$ span the subspace $Y \subset M$, and $z_1, \ldots, z_r$ the subspace $Z \subset M$. Then $y_1 \wedge \ldots \wedge y_r = \alpha z_1 \wedge \ldots \wedge z_r$, $\alpha \in K$, if and only if $Y = Z$. Therefore $[y_1 \ldots y_r] = \alpha[z_1 \ldots z_r]$ if and only if $Y = Z$, which implies

LEMMA 3. *If $y_1, y_2, \ldots, y_r$ and $z_1, \ldots, z_r$ are orthogonal bases for Y and Z, respectively, $y_1y_2 \ldots y_r = \alpha z_1 \ldots z_r$ if and only if $Y = Z$.*

II

DEFINITION 2. *A transformation Σ of an algebra onto itself will be called a semi-automorphism relative to σ if Σ is an automorphism of the algebra considered as a ring, that is*

$$(a + b)^\Sigma = a^\Sigma + b^\Sigma, \qquad (ab)^\Sigma = a^\Sigma b^\Sigma,$$

and it is a semi-linear mapping relative to the automorphism σ of K with respect to multiplication by elements of K, that is, $(\alpha a)^\Sigma = \alpha^\sigma a^\Sigma$. In particular if the algebra has an identity $\alpha^\Sigma = (\alpha 1)^\Sigma = \alpha^\sigma 1 = \alpha^\sigma$.

If the automorphism σ is the identity, Σ will be called an automorphism.

If (S, σ) is a semi-similitude of Q of ratio ρ, that is, a semi-linear transformation of M onto M, relative to the automorphism σ of K, such that $Q(xS) = \rho Q(x)^\sigma$ (cf. **1**, chapter I), there exists a semi-automorphism Σ of $F^+(M)$ relative to σ, associated to (S, σ) and defined in the following way

$$(y_1 \otimes y_2 \otimes \ldots \otimes y_{2t})^\Sigma = \rho^{-t}(y_1S) \otimes y_2S \otimes \ldots \otimes y_{2t}S.$$

Under this automorphism I^+ is changed into itself. To prove this it is sufficient to take elements of the form

$$d = y_1 \otimes y_2 \otimes \ldots \otimes y_r \otimes u \otimes z_1 \otimes \ldots \otimes z_s$$

where $r + s = 2t$ and $u = x \otimes x - Q(x)$. Then

$$d^\Sigma = \rho^{-(t+1)}(y_1S) \otimes \ldots \otimes y_rS \otimes xS \otimes xS \otimes z_1S \otimes \ldots \otimes z_sS$$
$$- \rho^{-t}Q(x)^\sigma(y_1S) \otimes \ldots \otimes y_rS \otimes z_1S \otimes \ldots \otimes z_sS.$$

Since $\rho^{-1}(xS) \otimes xS - Q(x)^\sigma = \rho^{-1}(xS \otimes xS - Q(xS)) = \rho^{-1}v \in I^+$

$$d^\Sigma = \rho^{-(t+1)}(y_1S) \otimes \ldots \otimes y_rS \otimes v \otimes z_1S \otimes \ldots \otimes z_sS \in I^+.$$

Therefore Σ induces a semiautomorphism in $C^+(M, Q)$ which will be called the semi-automorphism associated to (S, σ). This is a semi-automorphism relative to σ. The image under this semi-automorphism of the element $x_1x_2 \ldots x_{2h}$ is $\rho^{-h}(x_1S)(x_2S) \ldots (x_{2h}S)$. If $x_1, \ldots, x_{2h}$ are orthogonal vectors so are the vectors $x_1S, \ldots, x_{2h}S$. This implies that Σ takes $M_{[2h]}$ into itself and therefore Σ is homogeneous of degree 0. From now on we are going to assume that Q is non-degenerate.

If $\dim M = 2r$, $M_{[2r]}$ is one dimensional. Let $0 \neq e \in M_{[2r]}$, then $e^\Sigma = \alpha e$. Since $e^2 = \delta \in K$, $\delta^\sigma = (e^2)^\Sigma = (e^\Sigma)^2 = \alpha^2\delta$. When σ is the identity, (S, σ) is called a similitude, Σ is an automorphism and one must have $\alpha^2 = 1$, $e^\Sigma = \pm e$. A similitude is proper (improper) if $e^\Sigma = e (e^\Sigma = - e)$. The proper similitudes form a subgroup γ^+ of index 2 of the group γ of similitudes; the improper similitudes γ^- form the other coset.

THEOREM 2. *Any automorphism Σ of $C^+(M, Q)$ associated to a similitude S of Q can be extended to an inner automorphism of $C(M, Q)$.*

Proof. It follows from their definitions that, if $xS = \alpha(xS_1)$, the automorphisms of $C^+(M, Q)$ associated to S and S_1 are equal. Therefore when M is odd dimensional the automorphism Σ is equal to the automophism associated to a rotation U (cf. **7**, chapter II, § 13). The rotation U defines an inner automorphism of $C(M, Q)$ which induces on $C^+(M, Q)$ the automorphism Σ (**3**, 2.3).

If $\dim M = 2r$ and S is proper the automorphism Σ leaves invariant the centre of $C^+(M, Q)$, $Z = K + Ke$, therefore it is an inner automorphism of $C^+(M, Q)$ and can be extended to an automorphism of $C(M, Q)$.

If S is improper and $x_1, x_2, \ldots, x_{2r}$ is an orthogonal basis for M, let us take the proper similitude S' defined as follows

$$x_iS' = x_iS \text{ for } i = 1, 2, \ldots, 2r - 1, \text{ and } x_{2r}S' = - x_{2r}S.$$

The automorphism Σ' associated to S' is inner and let $u \in C^+$ be such that $c^{\Sigma'} = u^{-1}cu$ and take $v = x_1x_2 \ldots x_{2r-1} \in C^-$. Then vu defines an inner automorphism of C which induces on C^+ the automorphism Σ.

THEOREM 3. *Let Σ be a semi-automorphism of the algebra $C^+(M, Q)$ associated to the semi-similitude (S, σ) of ratio ρ. Then*

(i) *if* $\dim M = 2r + 1$, *Σ can be extended to a semi-automorphism of $C(M, Q)$ if and only if $\rho = \mu^2$. Then if $x \in M$, $x^\Sigma = \mu^{-1}(xS)$ or $- \mu^{-1}(xS)$.*

(ii) *if* $\dim M = 2r$, *Σ can be extended to $C(M, Q)$ if and only if $\rho = N(\alpha + \beta e) = (\alpha + \beta e)(\alpha - \beta e)$. Then, if $x \in M, x^\Sigma = \rho^{-1}(xS)(\alpha + \beta e)$.*

Proof. Let $x_1, x_2, \ldots, x_n$ be an orthogonal basis of M, $e = x_1 \ldots x_n$ and $\delta = Q(x_1) \ldots Q(x_n)$. Since the algebra $C(M, Q)$ is generated by x_1 and $x_1x_2, x_1x_3, \ldots, x_1x_n$, given a semi-automorphism of C^+ it would be possible to extend it to C if and only if we can find an element $c \notin C^+$ which anticommutes with $(x_1x_i)^\Sigma$ $i = 2, 3, \ldots, n$ and whose square is equal to $(x_1^2)^\Sigma = Q(x_1)^\sigma$. For, if the extension exists x_1^Σ has all these properties and

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conversely if there exists a c with these properties the map taking $u = u^+ + u^- = u^+ + x_1v^+$, where $u^+, v^+ \in C^+$, $u^- \in C^-$, into $u^\Sigma = (u^+)^\Sigma + c(v^+)^\Sigma$ can easily be seen to give a ring endomorphism of $C(M, Q)$.

The image of C that we get under this endomorphism has dimension greater than dim $C^+ = \frac{1}{2}$ dim C. Since C is either simple or a direct sum of two simple algebras of equal dimension, the defined endomorphism is onto and hence a semi-automorphism of the algebra C.

The element x_1S certainly anticommutes with $(x_1x_i)^\Sigma = \rho^{-1}(x_1S)(x_iS)$, $i = 2, \ldots, n$. If another element c also has this property $c(x_1S)$ commutes with all the $(x_1x_i)^\Sigma$ and therefore with C^+. Hence $c(x_1S) = \alpha' + \beta' e$, $c = (x_1S)(\alpha + \beta e)$.

(i) If dim $M = 2r + 1$ and the extension exists $x_1^\Sigma = c = (x_1S)(\alpha + \beta e)$ and $(x_1^\Sigma)^2 = (x_1S)^2(\alpha + \beta e)^2 = \rho Q(x_1)^\sigma(\alpha^2 + 2\alpha\beta e + (-1)^r\beta^2\delta) = Q(x_1)^\sigma = (x_1^2)^\Sigma$ which implies that either $\alpha = 0$ or $\beta = 0$. But if $\alpha = 0$, $(x_1S)\beta e \in C^+$ and we do not get a semi-automorphism. Hence $\beta = 0$ and $\rho\alpha^2 = 1$, that is, $\rho = \mu^2$ and $x_1^\Sigma = \mu^{-1}(x_1S)$ or $x_1^\Sigma = -\mu^{-1}(x_1S)$.

Suppose $x_1^\Sigma = \mu^{-1}(x_1S)$, then if $x \in M$, $x^\Sigma = (Q(x_1)^{-1}x_1(x_1x))^\Sigma = Q(x_1)^{-\sigma}\mu^{-1}(x_1S)\rho^{-1}(x_1S)(xS) = \mu^{-1}(xS)$, that is, $x^\Sigma = \mu^{-1}(xS)$.

(ii) If dim $M = 2r$ and the extension exists $(x_1^\Sigma)^2 = ((x_1S)(\alpha + \beta e))^2 = (x_1S)^2(\alpha - \beta e)(\alpha + \beta e) = \rho Q(x_1)^\sigma(\alpha^2 - (-1)^r\beta^2\delta)$ must be equal to $(x_1^2)^\Sigma = Q(x_1)^\sigma$. Hence $\rho(\alpha^2 - (-1)^r\beta^2\delta) = 1$ and therefore $\rho = (\rho\alpha)^2 - (-1)^r(\rho\beta)^2\delta = (\alpha_1 + \beta_1 e)(\alpha_1 - \beta_1 e) = N(\alpha_1 + \beta_1 e)$, where $\alpha_1 = \alpha\rho$, $\beta_1 = \beta\rho$ and $N(\alpha_1 + \beta_1 e)$ means the norm of $\alpha_1 + \beta_1 e$.

Then if ρ is a norm, say $\rho = N(\alpha + \beta e)$, taking $x_1^\Sigma = \rho^{-1}(x_1S)(\alpha + \beta e)$ we get an extension of Σ and for any $x \in M$ we have $x^\Sigma = \rho^{-1}(xS)(\alpha + \beta e)$.

Combining the result (ii) with Theorem 2 we get the

COROLLARY. *When* dim $M = 2r$, *the ratio of a similitude is of the form* $\rho = N(\alpha + \beta e) = \alpha^2 + (-1)^{r-1}\beta^2\delta$ (cf. **6**).

Up to now it has been seen that to any semi-similitude (S, σ) can be associated a semi-automorphism of $C^+(M, Q)$ relative to σ which is homogeneous of degree 0. If σ is the identity S is a similitude and we have an automorphism.

Our purpose now is to prove that any semi-automorphism of $C^+(M, Q)$ which is homogeneous of degree 0 is a semi-automorphism associated to a semi-similitude. If the semi-automorphism is an automorphism it is associated to a similitude. In fact, it is sufficient to assume that the semi-automorphism takes $M_{[2]}$ into itself to deduce that it is associated to a semi-similitude.

The proof will be decomposed into steps which we present as lemmas.

LEMMA 4. *Any element* $c \in M_{[2]}$ *whose square is an element of* K *different from zero is the product of two non-isotropic orthogonal vectors of* M.

Proof. Let $x_1, x_2, \ldots, x_n$ be an orthogonal basis for M. Then $x_1x_2, x_1x_3, \ldots, x_1x_n, x_2x_3, \ldots, x_{n-1}x_n$ form a basis of $M_{[2]}$.

If $c \in M_{[2]}$, $c = \sum_{i<j}\alpha_{ij}x_ix_j$. Because we assume that $c^2 = \mu \neq 0$, $c^{-1} = \mu^{-1}c$. We are going to prove that c belongs to the Clifford group, that is, the inner automorphism defined by c leaves invariant the subspace M. It is sufficient to prove that for x_h, $h = 1, 2, \ldots, n$, $c^{-1}x_hc \in M$.

Since

$$x_hc = x_h \sum_{i<j} \alpha_{ij}x_ix_j = \left(\sum_{i,j\neq h} \alpha_{ij}x_ix_j - \sum_{j>h}^{j=n} \alpha_{hj}x_hx_j - \sum_{i=1}^{i<h} \alpha_{ih}x_ix_h\right)x_h$$

$$= \left(c - 2\sum_{j>h}^{j=n} \alpha_{hj}x_hx_j - 2\sum_{i=1}^{i<h} \alpha_{ih}x_ix_h\right)x_h,$$

$$c^{-1}x_hc = c^{-1}\left(c - 2\sum_{j>h}^{j=n} \alpha_{hj}x_hx_j - 2\sum_{i=1}^{i<h} \alpha_{ih}x_ix_h\right)x_h$$

$$= \mu^{-1}c\left(cx_h + 2Q(x_h)\sum_{j>h}^{j=n} \alpha_{hj}x_j - 2Q(x_h)\sum_{i=1}^{i<h} \alpha_{ih}x_i\right) = d_{[1]} + d_{[3]}$$

where $d_{[1]} \in M$ and $d_{[3]} \in M_{[3]}$.

Now from $(c^{-1}x_hc)^* = (\mu^{-1}cx_hc)^* = \mu^{-1}cx_hc = c^{-1}x_hc$ follows $(c^{-1}x_hc)^* = (d_{[1]} + d_{[3]})^* = d_{[1]} - d_{[3]} = d_{[1]} + d_{[3]}$ and $d_{[3]} = 0$. Therefore $c^{-1}x_hc \in M$ and defining $xG = c^{-1}xc$ for any $x \in M$, G is an orthogonal transformation. Moreover G is a proper orthogonal involution since $c \in M_{[2]} \subset C^+$ and $c^2 = \mu$.

If U is the minus space of G (cf. **8**, I.2, Lemma 2), it has even dimension, Let $y_1, y_2, \ldots, y_n$ be an orthogonal basis of M such that the $2r$ first vectors form a basis of U. When M is even dimensional any element of the Clifford group defining an inner automorphism which induces on M the rotation G has the form $\gamma y_1y_2 \ldots y_{2r}$, $\gamma \in K$ (**3**, II.3). Therefore $c = \gamma y_1y_2 \ldots y_{2r}$ and since $c \in M_{[2]}$, $2r = 2$ and $c = \gamma y_1y_2 = y_1'y_2$. When M is odd dimensional, let $e = y_1y_2 \ldots y_n$, then $c = (\alpha + \beta e)y_1y_2 \ldots y_{2r}$, but, since $c \in M_{[2]}$ and $ey_1y_2 \ldots y_{2r} = \delta y_{2r+1} \ldots y_n \in M_{[n-2r]} \neq M_{[2]}$ because $n - 2r$ is odd, $\beta = 0$, and $c = \alpha y_1y_2 = y_1'y_2$.

Lemma 5. *Let y_1 and y_2 be two non-zero vectors of M. Then $y_1y_2 \in K$ if and only if $y_2 = \alpha y_1$.*

Proof. The linear isomorphism of $E(M)$ onto $C(M, Q)$ defined in Part I, takes $y_1 \wedge y_2 \in E_2$ into $[y_1y_2] = y_1y_2 - y_2y_1 \in M_{[2]}$. On the other hand $y_1y_2 + y_2y_1 = (y_1, y_2) \in K$, where (y_1, y_2) is the bilinear form associated to Q. Then $2y_1y_2 = [y_1y_2] + (y_1, y_2) \in M_{[2]} + K$. Since we assume that $y_1y_2 \in K$, $[y_1y_2] = 0$; therefore $y_1 \wedge y_2 = 0$, which implies $y_2 = \alpha y_1$.

Definition 3. *The vectors of a set are called properly independent (p.i.) if they are non-isotropic and orthogonal to each other.*

Lemma 6. *Let x, y and u, v be two pairs of p.i. vectors. Let P_{xy} and P_{uv} be the two planes spanned by x and y, and u and v, respectively. Then xy and uv*

anticommute if and only if $P_{xy} \cap P_{uv} = Kz$, *and the space* Kz *is non-isotropic and its orthogonal complement in* P_{xy} *is orthogonal to* P_{uv}.

Proof. The plane P_{uv} is non-isotropic. Therefore $M = P_{uv} \oplus P_{uv}{}^{\perp}$ where $P_{uv\perp}$ is the orthogonal complement of P_{uv}. In this decomposition of M, $x = c_1 + d_1$, $y = c_2 + d_2$, $c_i \in P_{uv}$, $d_i \in P_{uv\perp}$, which implies that d_i commute with uv and c_i anticommute. Hence if xy and uv anticommute

$$xyuv = (c_1 + d_1)(c_2 + d_2)uv = uv(- c_1 + d_1)(- c_2 + d_2)$$
$$xyuv = - uvxy = - uv(c_1 + d_1)(c_2 + d_2).$$

Since uv has an inverse we have

$$(- c_1 + d_1)(- c_2 + d_2) = - (c_1 + d_1)(c_2 + d_2) \text{ or } 2(c_1c_2 + d_1d_2) = 0.$$

But c_1c_2 belongs to the subalgebra over K, A, generated by u and v; d_1d_2 belongs to the subalgebra over K, B, generated by the vectors in $P_{uv\perp}$, and $A \cap B = K$. Therefore $c_1c_2 + d_1d_2 = 0$ if and only if $c_1c_2 = \mu$, $d_1d_2 = - \mu$, $\mu \in K$.

Case I. Suppose c_1, c_2, d_1, d_2 are all different from zero. Then by Lemma 5 $c_2 = \alpha c_1$, $d_2 = \beta d_1$. Therefore $x = c_1 + d_1$, $y = \alpha c_1 + \beta d_1$ and since $(x, y) = 0$ we get $xy = (\beta - \alpha)c_1d_1$. This implies, by Lemma 3, $P_{xy} = P_{c_1d_1}$ and therefore $c_1 \in P_{xy}$, $P_{uv} \cap P_{xy} = Kc_1$ and the orthogonal complement of c_1 in P_{xy}, $d_1 \in P_{uv}{}^{\perp}$.

Case II. Suppose one of the vectors c_1, c_2, d_1, d_2, is zero, let us say $d_2 = 0$. Since $y \neq 0$, $y = c_2 \neq 0$, $Q(c_2) = Q(y) \neq 0$. But then $c_1c_2 + d_1d_2 = 0$ is reduced to $c_1c_2 = 0$. Because $Q(c_2) \neq 0$, c_2 has an inverse and $c_1c_2 = 0$ implies $c_1 = 0$. Therefore $x = d_1$, $y = c_2$, $P_{uv} \cap P_{xy} = Kc_2$ and the orthogonal complement of Kc_2 in P_{xy} is $d_1 \in P_{uv}{}^{\perp}$.

In the same way, if $c_2 = 0$, we get $d_2 \neq 0$, $Q(d_2) \neq 0$ and $c_1c_2 + d_1d_2 = 0$ reduces to $d_1d_2 = 0$ which implies $d_1 = 0$, $x = c_1$, $y = d_2$.

Conversely, if the space spanned by x and y contains two non-isotropic vectors x' and y' such that $x' \in P_{uv}$ and $y' \in P_{uv}{}^{\perp}$, x' anticommutes with uv and y' commutes with it. Therefore $xy = \alpha x'y'$ anticommutes with uv.

LEMMA 7. *Let* $x_1, x_2, \ldots, x_n$ *be an orthogonal basis of* M. *Then under any semi-automorphism* Σ *of* $C(M, Q)$ *or* $C^+(M, Q)$ *taking* $x_1x_2, x_1x_3, \ldots, x_1x_n$ *into elements of* $M_{[2]}$, *the images of these elements can be written in the form* $y_1'y_2$, $y_1'y_3, \ldots, y_1'y_n$, *where* $y_1', y_2, \ldots, y_n$ *are p.i. vectors.*

Proof. By hypothesis $(x_1x_i)^{\Sigma} = \Sigma_{h<j}\alpha_{hj}x_hx_j$, and since $0 \neq (x_1x_i)^2 \in K$, $(\Sigma\alpha_{hj}x_hx_j)^2 \in K$. Hence by Lemma 4 $(x_1x_i)^{\Sigma} = t_iz_i$ where t_i, z_i, $i = 2, 3, \ldots, n$ are p.i. vectors.

Since t_2z_2 and t_3z_3 anticommute, by Lemmas 6 and 3, $t_3z_3 = (\alpha t_2 + \beta z_2)y_3$ and t_2, z_2, y_3 are p.i. and $y_1' = (\alpha t_2 + \beta z_2)$ is determined up to a scalar factor. By Lemma 3 we know that $t_2z_2 = y_1'y_2$ where $y_2 = \alpha' t_2 + \beta' z_2$ and y_1', y_2, y_3 are p.i.

Let us consider $t_i z_i$ $i \neq 2, 3$. Since it anticommutes with $(x_1x_2)^{\Sigma} = y_1'y_2$ and $(x_1x_3)^{\Sigma} = y_1'y_3$ we have $t_i z_i = (\alpha_i y_1' + \beta_i y_2)u = (\alpha_i' y_1' + \beta_i' y_3)v$ where y_1', y_2, u are p.i. and so are y_1', y_3, v.

Applying again Lemma 3 we get

$$\mu(\alpha_i y_1' + \beta_i y_2) + \nu u = \alpha_i' y_1' + \beta_i' y_3.$$

If $\nu \neq 0$, $u = \delta_1 y_1' + \delta_2 y_2 + \delta_3 y_3$ and

$$\begin{aligned}(x_1x_i)^{\Sigma} &= t_i z_i = (\alpha_i y_1' + \beta_i y_2)u = \gamma_1 + \gamma_2 y_1' y_2 + \gamma_3 y_1' y_3 + \gamma_4 y_2 y_3 \\ &= (\gamma_1^{\sigma^{-1}} + \gamma_2^{\sigma^{-1}} x_1x_2 + \gamma_3^{\sigma^{-1}} x_1x_3 - Q(y_1)^{\sigma^{-1}} \gamma_4^{\sigma^{-1}} x_1x_2x_1x_3)^{\Sigma}\end{aligned}$$

where σ is the automorphism of K related to Σ. But this contradicts the fact that Σ is a semi-automorphism because $1, x_1x_2, x_1x_3, x_2x_3, x_1x_i$ are linearly independent.

Therefore $\nu = 0$ and $\beta_i = \beta_i' = 0$ for y_1', y_2, y_3 are p.i. If we write $\alpha_i u = y_i$ we have $(x_1x_i)^{\Sigma} = y_1'y_i$ and y_1', y_2, y_3, y_i are p.i.

To prove that $y_1', y_2, \ldots, y_n$ are p.i. it is sufficient to show that they form a set of pairwise orthogonal vectors, for since they are non-isotropic this would be impossible if they were not independent.

We know that $y_1'y_i$ are p.i. and y_1', y_j are p.i. Besides if $i \neq j$ $y_1'y_i$ and $y_1'y_j$ must anticommute; then by Lemma 6 y_i and y_j are orthogonal.

THEOREM 4. *Any semi-automorphism Σ of $C^+(M, Q)$ mapping $M_{[2]}$ into $M_{[2]}$ is associated to a semi-similitude. Moreover, if Σ is an automorphism it is associated to a similitude. If* dim $M > 2$ *the semi-similitude is defined by Σ up to a scalar factor.*

Proof. The elements $1, x_1x_2, x_1x_3, \ldots, x_1x_n$, where $x_1, x_2, \ldots, x_n$ is an orthogonal basis of M, form a set of generators of $C^+(M, Q)$ over K. By Lemma 7 this set of generators is taken by Σ into $1, y_1'y_2, y_1'y_3, \ldots, y_1'y_n$ where $y_1', y_2, \ldots, y_n$ are p.i. and the y_1' and therefore the y_i's are defined up to a scalar factor, when dim $M > 2$.

Let $\rho = Q(y_1')^{-1}Q(x_1)^{\sigma}$, then

$$((x_1x_i)^2)^{\Sigma} = -Q(x_1)^{\sigma}Q(x_i)^{\sigma} = ((x_1x_i)^{\Sigma})^2 = (y_1'y_i)^2 = -Q(y_1')Q(y_i)$$

and hence

$$Q(y_i) = \rho Q(x_i)^{\sigma} \qquad i = 2, 3, \ldots, n.$$

Take $y_1 = \rho y_1'$, then $Q(y_1) = \rho^2 Q(y_1') = \rho Q(x_1)^{\sigma}$ and define the semi-similitude (S, σ): $\Sigma \alpha_i x_i \rightarrow \Sigma \alpha_i^{\sigma} y_i$.

The semi-automorphism of $C^+(M, Q)$ associated to (S, σ) takes x_1x_i into $\rho^{-1}y_1y_i = y_1'y_i$ and since it coincides with Σ on a set of generators over K both semi-automorphisms are equal.

COROLLARY 1. *When* dim $M > 2$, *there exists an isomorphism between the projective group of semi-similitudes (similitudes) and the group of semi-automorphisms (automorphisms) of $C^+(M, Q)$ leaving invariant the subspace $M_{[2]}$.*

This isomorphism takes the projective group of proper similitudes onto the group of inner automorphisms of $C^+(M, Q)$ *leaving invariant* $M_{[2]}$.

The last part of this corollary is a consequence of the proof of Theorem 2.

COROLLARY 2. *If* $\dim M > 2$, *there exists a homomorphism between the group of inner automorphisms of* $C(M, Q)$ *leaving invariant the subspace* $M_{[2]}$ *and the projective group of similitudes* $P\gamma(M, Q)$. *The kernel consists of the inner automorphisms defined by elements of the centre of* $C^+(M, Q)$.

Proof. Any inner automorphism Σ of $C(M, Q)$ taking $M_{[2]}$ into itself induces an automorphism on $C^+(M, Q)$ uniquely associated to the coset of a similitude S in $P\gamma(M, Q)$. The mapping of Σ into the coset of S is a homomorphism of the group of inner automorphisms of $C(M, Q)$ onto $P\gamma(M, Q)$ by Theorem 2. The inner automorphisms which are mapped into the coset of the identity are defined by the invertible elements of the centralizer of $C^+(M, Q)$ in $C(M, Q)$. This centralizer is the centre of $C^+(M, Q)$, if $\dim M = 2r$, and if $\dim M = 2r+1$ it is the centre of C which defines the same inner automorphisms that the centre K of C^+, namely, only the identity.

Remark. With the exception of the case of the algebra $C^+(M, Q)$ when $\dim M = 4$, any automorphism of $C(M, Q)$ or $C^+(M, Q)$ considered as rings, taking $M_{[2]}$ into itself must take K into itself and therefore it is a semi-automorphism of $C(M, Q)$ or $C^+(M, Q)$ considered as algebras.

III

The Clifford group Γ of $C(M, Q)$ is the group of invertible elements which define inner automorphism of $C(M, Q)$ leaving invariant the space M. The transformations induced on M for such automorphisms are orthogonal transformations with respect to Q (cf. **3**, 2.3). It is clear then that the elements of Γ define inner automorphisms of $C(M, Q)$ which are homogeneous of degree 0, with respect to the gradation of $C(M, Q)$ defined in part I.

DEFINITION 3. *The extended Clifford group* Θ *of* $C(M, Q)$ *is the group of invertible elements* s *of* $C(M, Q)$ *such that the inner automorphism* $s^{-1}cs$ *leaves invariant the space* $M_{[2]}$.

We have seen in part II that the elements of Θ defined inner automorphisms which induce on C^+ automorphisms associated to similitudes and therefore these automorphisms induced on C^+ are homogeneous of degree 0.

It is clear that $\Theta \supset \Gamma$ and Theorem 3 (i) proves that when M is odd dimensional $\Theta = \Gamma$. Therefore the only interesting case is when $\dim M = 2r$ and Θ properly contains Γ.

From now on it is assumed that $\dim M = 2r$.

Any automorphism associated to a proper (improper) similitude takes any element $e \in M_{[2r]}$ into $e(-e)$. Hence the inner autmorphism defined by

$s \in \Theta$ is associated to a proper (improper) similitude if and only if s commutes (anticommutes) with e, that is, $s \in \Theta^+ = \Theta \cap C^+$ ($\Theta^- = \Theta \cap C^-$).

Therefore if $s \in \Theta$ either $s \in \Theta^+$ or $s \in \Theta^-$.

Let us recall that there exists a homomorphism λ of Γ into the multiplicative group of invertible elements of the centre of $C(M, Q)$. If $s \in \Gamma$, the value $\lambda(s) = ss^*$ is called the spin-norm of s and also the spin-norm of the orthogonal transformation of M taking $x \in M$ into $xG = s^{-1}xs$.

Now we are going to study the values of $\lambda(s) = ss^*$ for $s \in \Theta$.

LEMMA 8. (i) *If* $\dim M = 4r + 2$, $\lambda(.)$ *is a homomorphism of* Θ *into* K.

(ii) *If* $\dim M = 4r$, $\lambda(.)$ *maps the elements of* Θ *into the centre* Z *of* $C^+(M, Q)$. *The restriction of* $\lambda(.)$ *to* Θ^+ *is a homomorphism of* Θ^+ *into this centre. Moreover,* $\lambda(s) \in K$ *if and only if* $s \in \Gamma$.

Proof. When $\dim M = 4r + 2$, $e^* = -e$ for $e \in M_{[4r+2]}$. Let $s \in \Theta$ and $x \in M$; the inner automorphism defined by s is an extension to $C(M, Q)$ of an automorphshim of C^+ associated to a similitude. Therefore by Theorem 3 (ii) $s^{-1}xs = y(\alpha + \beta e)$, $y \in M$ and $(s^{-1}xs)^* = s^*x(s^*)^{-1} = (y(\alpha + \beta e))^* = y(\alpha + \beta e) = s^{-1}xs$, that is, $s^*x(s^*)^{-1} = s^{-1}xs$, $xss^* = ss^*x$ for every $x \in M$. Hence $\lambda(s) = ss^* \in K$, the centre of C, and $\lambda(s_1s_2) = s_1s_2s^*{}_2s^*{}_1 = \lambda(s_1)\lambda(s_2)$.

(ii) When $\dim M = 4r$, $e^* = e$, $e \in M_{[4r]}$. Let $x_1, x_2, \ldots, x_{4r}$ be an orthogonal basis of M. Then $s^{-1}x_ix_js = \rho^{-1}y_iy_j$ where $y_h = x_hS$, $h = 1, \ldots, 4r$ and S is a similitude of ratio ρ. Hence

$$(s^{-1}x_ix_js)^* = (\rho^{-1}y_iy_j)^* = -\rho^{-1}y_iy_j = -(s^{-1}x_ix_js);$$

$$(s^{-1}x_ix_js)^* = -s^*x_ix_j(s^{-1})^* = -s^{-1}x_ix_js;\ ss^*x_ix_j = x_ix_jss^*,$$

that is, $\lambda(s) = ss^* \in Z$ and if $s_1, s_2 \in \Theta^+$ $\lambda(s_1s_2) = s_1s_2s^*{}_2s^*{}_1 = \lambda(s_1)\lambda(s_2)$.

Assume now that $\lambda(s) = \delta \in K$. Then, for any $x \in M$, $ss^*x = xss^*$; $s^{-1}xs = s^*x(s^*)^{-1} = (s^{-1}xs)^*$. But $s^{-1}xs = y(\alpha + \beta e)$ and $(y(\alpha + \beta e))^* = y(\alpha + \beta e)$ if and only if $\beta = 0$ and $s \in \Gamma$.

Consider a similitude of ratio ρ. We know by the corollary of Theorem 3 that $\rho = N(\alpha + \beta e)$ and by the theorem that the automorphism of C^+ associated to S can be extended to an automorphism Σ of C such that $x^\Sigma = \rho^{-1}(xS)(\alpha + \beta e)$. This is an automorphism of the simple algebra $C(M, Q)$ which leaves the centre invariant. Therefore there exists an element $s \in \Theta$ defined up to a scalar such that $s^{-1}xs = \rho^{-1}(xS)(\alpha + \beta e) = (xS)(\alpha' + \beta' e)$ where $\alpha' = \rho^{-1}\alpha$, $\beta' = \rho^{-1}\beta$ and $N(\alpha' + \beta' e) = \rho^{-2}N(\alpha + \beta e) = \rho^{-1}$.

DEFINITION 4. *An element* $s \in \Theta$ *is said to be associated to the similitude* S *with the factor* $\alpha + \beta e$ *if* $s^{-1}xs = (xS)(\alpha + \beta e)$. *When the factor is* 1, *we simply say that* $s \in \Gamma$ *is associated to the orthogonal transformation* S.

LEMMA 9. *Let* $s \in \Theta$ *be associated to* S *with the factor* $\alpha + \beta e$. *Then* s^{-1} *is associated to* S^{-1} *with the factor* $(\alpha + \beta e)^{-1}[(\alpha - \beta e)^{-1}]$ *if* $s \in \Theta^+[s \in \Theta^-]$.

Proof. (a) Let $s \in \Theta^+$, $x \in M$. Then $s^{-1} \in \Theta^+$ and $xS^{-1} = ss^{-1}(xS^{-1})ss^{-1} = sx(\alpha + \beta e)s^{-1} = sxs^{-1}(\alpha + \beta e)$; $sxs^{-1} = (xS^{-1})(\alpha + \beta e)^{-1}$.

(b) Let $s \in \Theta^-$, $x \in M$. Then $s^{-1} \in \Theta^-$, $xS^{-1} = ss^{-1}(xS^{-1})ss^{-1} = sx(\alpha+\beta e)s^{-1} = sxs^{-1}(\alpha - \beta e)$; $sxs^{-1} = (xS^{-1})(\alpha - \beta e)^{-1}$.

LEMMA 10. *If* $\dim M = 4r$ *and* $s \in \Theta$ *is associated to* S *with the factor* $\alpha+\beta e$, *then*

(i) $\lambda(s) = \mu(\alpha + \beta e)$ *and* $\lambda(s^{-1}) = \mu^{-1}(\alpha + \beta e)^{-1}$ *if* $s \in \Theta^+$.

(ii) $\lambda(s) = \mu(\alpha - \beta e)$ *and* $\lambda(s^{-1}) = \mu^{-1}(\alpha + \beta e)^{-1}$ *if* $s \in \Theta^-$.

Proof. (i) When $s \in \Theta^+$, $\lambda(s) = ss^* = \alpha' + \beta' e$ implies $s^{-1} = (\alpha' + \beta' e)^{-1}s^*$. Since $(\alpha' + \beta' e)^{-1}s^*xs = (xS)(\alpha + \beta e)$, $s^*xs = (s^*xs)^* = (xS)(\alpha' - \beta' e)(\alpha + \beta e)$, that is, $((xS)(\alpha' - \beta' e)(\alpha + \beta e))^* = (\alpha + \beta e)(\alpha' - \beta' e)(xS) = (xS)(\alpha' + \beta' e)(\alpha - \beta e) = (xS)(\alpha' - \beta' e)(\alpha + \beta e)$, $(\alpha' + \beta' e)(\alpha - \beta e) = \overline{(\alpha' - \beta' e)(\alpha + \beta e)}$. This shows that if we define the automorphism — of $K + Ke$ as $\overline{\alpha + \beta e} = \alpha - \beta e$, $(\alpha' + \beta' e)(\alpha - \beta e)$ is invariant under this automorphism and therefore is an element of K, hence $\alpha' + \beta' e = \mu(\alpha + \beta e)$. As for s^{-1} we have $\lambda(s)\lambda(s^{-1}) = \lambda(ss^{-1}) = 1$ so that $\lambda(s^{-1}) = \mu^{-1}(\alpha + \beta e)^{-1}$.

(ii) When $s \in \Theta^-$, $\lambda(s) = ss^* = \alpha' + \beta' e$ implies $s^{-1} = (\alpha' - \beta' e)^{-1}s^*$, therefore $(\alpha' - \beta' e)^{-1}s^*xs = (xS)(\alpha + \beta e)$, $s^*xs = (s^*xs)^* = (xS)(\alpha' + \beta' e)(\alpha + \beta e)$ which implies $(\alpha' + \beta' e)(\alpha + \beta e) = (\alpha' - \beta' e)(\alpha - \beta e) \in K$ and $\alpha' + \beta' e = \mu(\alpha - \beta e)$.

Now $1 = \lambda(ss^{-1}) = ss^{-1}(s^{-1})^*s^* = \overline{\lambda(s^{-1})}\lambda(s)$, therefore $\lambda(s^{-1}) = \mu^{-1}(\alpha + \beta e)^{-1}$.

Let K' be the multiplicative group of non-zero elements of K and K'^2 the subgroup of squares of elements of K'. If an element $s \in \Gamma$ has spin-norm $\lambda(s) \in K'^2$, say $\lambda(s) = \mu^2$, $\lambda(\mu^{-1}s) = \mu^{-2}\lambda(s) = 1$. The elements s and $\mu^{-1}s$ define the same inner automorphism and *a fortiori* the same orthogonal transformation on M. Conversely, if s and s' define the same inner automorphism on $C(M, Q)$, $s' = \mu s$ and $\lambda(s') \equiv \lambda(s)(K'^2)$.

The group of elements of $\Gamma^+ = \Gamma \cap C^+$ of spin-norm 1 is a subgroup denoted by Γ_0^+ and called the reduced Clifford group. When the index of $Q > 0$ the commutator group Ω of the orthogonal group coincides with the group of orthogonal rotations associated to elements of Γ_0^+. Therefore an orthogonal transformation associated to an element $s \in \Gamma^+$ belongs to Ω if and only if $\lambda(s) \equiv 1(K'^2)$.

PROPOSITION 1. *Let* S_ρ *be a similitude of ratio* ρ *and* $g \in \Gamma^+$ *be associated to the rotation* G *defined by* $xG = \rho^{-1}(xS_\rho^2)$, $x \in M$. *Then*

(i) *If* $\dim M = 4r + 2$, $\lambda(g) \equiv \rho(K'^2)$ *if* $S_\rho \in \gamma^+$ *and* $\lambda(g) \equiv 1(K'^2)$ *if* $S_\rho \in \gamma^-$. *When index of* $Q > 0$, $G \in \Omega$ *if and only if either* $S_\rho \in \gamma^-$ *or* $\rho \equiv 1(K'^2)$.

(ii) *If* $\dim M = 4r$, $\lambda(g) \equiv 1(K'^2)$ *if* $S_\rho \in \gamma^+$ *and* $\lambda(g) \equiv \rho(K'^2)$ *if* $S_\rho \in \gamma^-$. *When index of* $Q > 0$, $G \in \Omega$ *if and only if either* $S_\rho \in \gamma^+$ *or* $\rho \equiv 1(K'^2)$.

Proof. (i) Let $s \in \Theta^+$ be associated to S_ρ with the factor $\alpha + \beta e$, hence $N(\alpha + \beta e) = \rho^{-1}$. Then $(s^2)^{-1}xs^2 = s^{-1}(xS_\rho)(\alpha + \beta e)s = (xS_\rho^2)(\alpha + \beta e)^2$. This implies that the element $g(\alpha+\beta e)$ defines the same inner automorphism that s^2, for, since $N(\alpha+\beta e)=\rho^{-1}$, $(\alpha+\beta e)^{-1}g^{-1}xg(\alpha+\beta e)=\rho(\alpha-\beta e)g^{-1}xg$ $(\alpha+\beta e)=\rho(\alpha-\beta e)\rho^{-1}(xS_\rho^2)(\alpha+\beta e)=xS_\rho^2(\alpha+\beta e)^2$.

Hence $s^2=\gamma g(\alpha+\beta e)$ and $\lambda(s^2)=(\lambda(s))^2=\nu^2=\lambda(\gamma g(\alpha+\beta e))=\gamma^2$ $\rho^{-1}\lambda(g)$. Therefore $\lambda(g)\equiv\rho(K'^2)$ and if $\rho\not\equiv 1(K'^2)$, $G\notin\Omega$.

If $s\in\Theta^-$ is associated to S_ρ with the factor $\alpha+\beta e$, $N(\alpha+\beta e)=\rho^{-1}$ and $(s^2)^{-1}xs^2=\rho^{-1}(xS_\rho^2)$. Therefore $s^2=\gamma g$ and $\lambda(s^2)=\gamma^2\lambda(g)=\nu^2$ implies that $G\in\Omega$ if index of $Q>0$.

(ii) Let $s\in\Theta^+$ be associated to S_ρ with the factor $\alpha+\beta e$, $(s^2)^{-1}xs^2=s^{-1}$ $(xS_\rho)(\alpha+\beta e)s=(xS_\rho^2)(\alpha+\beta e)^2$ and $s^2=\gamma g(\alpha+\beta e)$. Then by Lemma 10 (i) $\lambda(s^2)=sss^*s^*=\mu^2(\alpha+\beta e)^2$, and on the other hand $\lambda(s^2)=\lambda(\gamma g(\alpha+\beta e))=\gamma^2gg^*(\alpha+\beta e)(\alpha+\beta e)^*=\gamma^2\lambda(g)(\alpha+\beta e)^2$. Therefore $\mu^2(\alpha+\beta e)^2=\gamma^2\lambda(g)(\alpha+\beta e)^2$, $\lambda(g)\equiv 1(K'^2)$, which implies that $G\in\Omega$ if index of $Q>0$.

If $s\in\Theta^-$, $s^{-2}xs^2=\rho^{-1}(xS_\rho^2)$ and therefore $s^2=\gamma g$. Hence $\lambda(s^2)=\lambda(\gamma g)=\gamma^2\lambda(g)$ and by Lemma 10 (ii) $\lambda(s^2)=sss^*s^*=\mu^2N(\alpha+\beta e)=\mu^2\rho^{-1}$. which proves that $\lambda(g)\equiv\rho(K'^2)$.

PROPOSITION 2. *Let S_{ρ_1}, S_{ρ_2} be similitudes of ratio ρ_1, ρ_2 respectively. Let $\delta g\in\Gamma^+$ be an element associated to the rotation $S_{\rho_1}\,S_{\rho_2}\,S_{\rho_1}^{-1}\,S_{\rho_2}^{-1}$. Then*

$$\lambda(g)\equiv\rho_1^{\epsilon_2}\rho_2^{\epsilon_1}(K'^2)\quad\text{where}\quad \epsilon_i=\begin{cases}0 \text{ if } S_{\rho i}\in\gamma^+\\ 1 \text{ if } S_{\rho i}\in\gamma^-\end{cases}\qquad i=1,2.$$

Proof. Let s_1, s_2 be elements of Θ associated to S_{ρ_1}, S_{ρ_2} with the factors $\alpha_1+\beta_1 e$, $\alpha_2+\beta_2 e$, respectively. We consider two different cases.

Case 1. dim $M=4r+2$. Then $\lambda(s_1s_2s_1^{-1}s_2^{-1})=1$. Suppose

(a) $s_1,s_2\in\Theta^+$, $(s_1s_2s_1^{-1}s_2^{-1})^{-1}xs_1s_2s_1^{-1}s_2^{-1}=(s_2s_1^{-1}s_2^{-1})^{-1}(xS_{\rho_1})s_2s_1^{-1}s_2^{-1}(\alpha_1+\beta_1 e)=xS_{\rho_1}S_{\rho_2}S_{\rho_1}^{-1}S_{\rho_2}^{-1}$ by Lemma 9. Therefore $s_1s_2s_1^{-1}s_2^{-1}=g\in\Gamma^+$ and $\lambda(g)=1$; g is associated to $S_{\rho_1}S_{\rho_2}S_{\rho_1}^{-1}S_{\rho_2}^{-1}$.

(b) $s_1\in\Theta^+$, $s_2\in\Theta^-$, $(s_1s_2s_1^{-1}s_2^{-1})^{-1}xs_1s_2s_1^{-1}s_2^{-1}=(xS_{\rho_1}S_{\rho_2}S_{\rho_1}^{-1}S_{\rho_2}^{-1})(\alpha_1+\beta_1 e)(\alpha_1-\beta_1 e)^{-1}$. Therefore $s_1s_2s_1^{-1}s_2^{-1}=g(\alpha_1+\beta_1 e)$ and $\lambda(g)\equiv\rho_1(K'^2)$.

(c) $s_1\in\Theta^-$, $s_2\in\Theta^+$, $(s_1s_2s_1^{-1}s_2^{-1})^{-1}xs_1s_2s_1^{-1}s_2^{-1}=(xS_{\rho_1}S_{\rho_2}S_{\rho_1}^{-1}S_{\rho_2}^{-1})(\alpha_2-\beta_2 e)(\alpha_2+\beta_2 e)^{-1}$. Hence $s_1s_2s_1^{-1}s_2^{-1}=g(\alpha_2-\beta_2 e)$ and $\lambda(g)\equiv\rho_2(K'^2)$.

(d) $s_1,s_2\in\Theta^-$, $(s_1s_2s_1^{-1}s_2^{-1})^{-1}xs_1s_2s_1^{-1}s_2^{-1}=(xS_{\rho_1}S_{\rho_2}S_{\rho_1}^{-1}S_{\rho_2}^{-1})(\alpha_1-\beta_1 e)$ $(\alpha_2+\beta_2 e)(\alpha_1+\beta_1 e)^{-1}(\alpha_2-\beta_2 e)^{-1}$. Hence $s_1s_2s_1^{-1}s_2^{-1}=g(\alpha_1-\beta_1 e)(\alpha_2+\beta_2 e)$ $\lambda(g)\equiv\rho_1\rho_2(K'^2)$.

Case 2. dim $M=4r$. In this case we apply Lemma 9 in the same way it was done before, but we also need Lemma 10 to compute $\lambda(s_1s_2s_1^{-1}s_2^{-1})$.

(a) $s_1,s_2\in\Theta^+$. Since $\lambda(.)$ is a homomorphism of Θ^+ into a commutative group $\lambda(s_1s_2s_1^{-1}s_2^{-1})=1$ and the result is the same as that for the case 1 (a).

(b) $s_1\in\Theta^+$, $s_2\in\Theta^-$. As before, $s_1s_2s_1^{-1}s_2^{-1}=g(\alpha_1+\beta_1 e)$.

Now $\lambda(s_1s_2s_1^{-1}s_2^{-1})=s_1s_2s_1^{-1}\mu_2^{-1}(\alpha_2+\beta_2 e)^{-1}s_1^{-*}s_2^*s_1^*=\mu_2^{-1}(\alpha_2-\beta_2 e)^{-1}$ $\mu_1^{-1}(\alpha_1-\beta_1 e)^{-1}\mu_2(\alpha_2-\beta_2 e)\mu_1(\alpha_1+\beta_1 e)$ by Lemma 10. Hence

$$\lambda(g(\alpha_1 + \beta_1 e)) = \lambda(g)(\alpha_1 + \beta_1 e)^2 = \frac{(\alpha_1 + \beta_1 e)^2}{N(\alpha_1 + \beta_1 e)}$$

and $\lambda(g) = \rho_1$.

(c) $s_1 \in \Theta^-$, $s_2 \in \Theta^+$. We know $s_1 s_2 s_1^{-1} s_2^{-1} = g(\alpha_2 - \beta_2 e)$ and by Lemma 10 $\lambda(s_1 s_2 s_1^{-1} s_2^{-1}) = (\alpha_2 + \beta_2 e)^{-1}(\alpha_2 - \beta_2 e) = \lambda(g)(\alpha_2 - \beta_2 e)^2$. Therefore $\lambda(g) = \rho_2$

(d) $s_1 \in \Theta^-$, $s_2 \in \Theta^-$. Then

$$\lambda(s_1 s_2 s_1^{-1} s_2^{-1}) = \frac{(\alpha_1 - \beta_1 e)^2}{N(\alpha_1 + \beta_1 e)} \cdot \frac{(\alpha_2 + \beta_2 e)^2}{N(\alpha_2 + \beta_2 e)} = \lambda(g)(\alpha_1 - \beta_1 e)^2(\alpha_2 + \beta_2 e)^2$$

and $\lambda(g) = \rho_1\rho_2$.

COROLLARY 1. *The centralizer of an improper similitude S_ρ of ratio ρ in the group of similitudes is generated by the scalar multiples of S_ρ and of a subgroup of O^+ (the rotation group).*

Proof. If T belongs to the centralizer of S_ρ in γ, $S_\rho T S_\rho^{-1} T^{-1} = I$ is the identity, $1 \in \Gamma$ is associated to I and $\lambda(1) = 1$. If T is a proper similitude, applying the proposition, we get that the ratio of T must be a square, say, α^2. Then $xT = \alpha(xG)$ where G is a rotation.

If T is an improper similitude its ratio, by the proposition, must be of the form $\alpha^2\rho$. Therefore the ratio of TS_ρ^{-1} is α^2 which implies that $xT = \alpha(xGS_\rho)$ where G is a rotation.

COROLLARY 2. *No element of γ^- belongs to the centralizer in the group of similitudes γ of a proper similitude S_ρ of ratio ρ if $\rho \not\equiv 1(K'^2)$.*

COROLLARY 3. *If the index of Q is greater than 0, the first commutator group γ' of the group of similitudes γ consists of the transformations of O^+ associated to elements $g \in \Gamma^+$ such that $\lambda(g) \equiv \rho(K'^2)$, where ρ is the ratio of some similitude of γ. The second commutator group $\gamma'' = \Omega$ if $\dim M > 4$.*

Proof. Since the group $\Omega \subset \gamma'$ contains all the rotations defined by elements of Γ_0^+, any rotation associated to an element $g \in \Gamma^+$ with spin-norm $\lambda(g) \equiv \rho(K'^2)$ belongs to γ' if γ' contains one with such a spin-norm. Proposition 2 shows that if there exists a proper similitude S_ρ of ratio ρ, $S_\rho U S_\rho^{-1} U^{-1}$ is a rotation with such spin-norm if $U \in \gamma^-$.

Conversely, any element of γ' is a product of elements of the form $S_{\rho_1} S_{\rho_2} S_{\rho_1}^{-1} S_{\rho_2}^{-1}$ associated to a $g \in \Gamma^+$ and by the proposition $\lambda(g) \equiv \rho_1^{\epsilon_2}\rho_2^{\epsilon_1}(K'^2), \epsilon_i = 0,1$. Hence the product of elements of such form is associated to a $g' \in \Gamma^+$ with spin-norm $\lambda(g') = \rho_1\rho_2 \ldots \rho_i\ (K'^2)$, where the ρ_i's are ratios of similitudes and therefore $\rho_1\rho_2 \ldots \rho_i$ is also the ratio of a similitude.

The last part of the corollary is a consequence of the well-known fact that if $\dim M > 4$, index of $Q > 0$ and F is any subgroup of the orthogonal group $O(M, Q)$ such that $\Omega \subset F \subset O(M, Q)$ the commutator of F is Ω (cf. **5**; **9**).

In the proof of the next theorem we are going to use another known result of the theory of orthogonal groups, namely: the commutator group Ω of the orthogonal group $O(M, Q)$ is generated by the squares of the elements of $O(M, Q)$. If dim $M > 2$, the commutator group of the group of rotations is also Ω, which is generated by the squares of the rotations.

THEOREM 5. *Let γ be the group of similitudes of M with respect to Q, index of $Q > 0$,* dim $M > 2$. *Then the commutator group γ' of γ is the group generated by the rotations G of the form $xG = \rho^{-1}(xS_\rho^{\,2})$, where ρ is the ratio of S_ρ. The commutator $(\gamma^+)'$ of the group of proper similitudes γ^+ is equal to Ω.*

The rotations of the form G, $xG = \rho^{-1}(xS_\rho^{\,2})$, $S_\rho \in \gamma^+$, generate

(i) γ' *if* dim $M = 4r + 2$.

(ii) Ω *if* dim $M = 4r$.

$\gamma' = \Omega$ *if and only if the ratio of any similitude is a square in K'.*

Proof. Since the squares of the elements of O^+, that is, the squares of the rotations, generate Ω and we assume index of $Q > 0$, all the rotations associated to elements of Γ_0^+ belong to the group generated by the G's. But then any rotation associated to a $g \in \Gamma^+$ with $\lambda(g) \equiv \rho(K'^2)$, ρ the ratio of some similitude, also belongs to this group by Proposition 1. Conversely, any element of the group generated by the G's is a rotation associated to a $g \in \Gamma^+$ with $\lambda(g) \equiv \rho(K'^2)$, where ρ is the ratio of some similitude. Therefore this group is γ' by the preceding corollary.

The same argument proves that if we take only the G's defined by proper similitudes they generate γ' if dim $M = 4r + 2$ (Proposition 1 (i)), and Ω if dim $M = 4r$ (Proposition 1 (ii)). Proposition 2 proves that, when index of $Q > 0$, $(\gamma^+)' = \Omega$. The last statement is proved by Corollary 3 of Proposition 2.

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