GROUP THEORETIC PROPERTIES INHERITED BY LOWER CENTRAL FACTORS

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Several properties are known to be inherited from the derived factor group of a group G by other factors $\Gamma_i = \gamma_i(G)/\gamma_{i+1}(G)$ of the lower central series. Derek Robinson proved in [**1**] that the defining property of any class $\mathfrak{X}$ of groups which is closed under the forming of homomorphic images of tensor products is so inherited. Possibilities for $\mathfrak{X}$ here include the following classes.

(1) Finite groups, groups which satisfy the maximal (resp. minimal) condition, minimax groups, min-by-max groups, groups with finite rank, groups with finite p-ranks for all $p = 0$ or a prime.

The property of being a periodic π-group (resp. a group with finite exponent dividing some fixed integer e) is easily seen to be inherited by Γ_{i+1} from Γ_i for any integer $i \geqslant 1$. In Lemma 2.2 of [**3**] J. P. Williams showed that the property of having finite rank passes from Γ_2 to $\Gamma_i (i > 2)$, while the lemma in [**2**] states that if Γ_n is periodic with finite p-rank for all p then so is Γ_m for all $m > n$. Here we shall prove the following (where $r_p(X)$ denotes the p-rank and $r_0(X)$ the torsion-free rank of an abelian group X).

THEOREM. (a) *If* $r_0(\Gamma_n) = r < \infty$, *then* $r_0(\Gamma_{n+1}) \leqslant r^2 n$.
(b) *If* $r_p(\Gamma_n) = s < \infty$ *and* $r_0(\Gamma_n) = r < \infty$, *then* $r_p(\Gamma_{n+1}) \leqslant (r+s)^2 n$.

COROLLARY. *If G is a group such that Γ_n belongs to one of the classes $\mathfrak{X}$ listed in* (1) *above then Γ_m belongs to $\mathfrak{X}$ for all $m \geqslant n$.*

The classes (1) indicate some of the finiteness conditions most commonly referred to. A general result corresponding to that of Robinson on tensor products would be more satisfactory, but Γ_2 may be cyclic while Γ_3 is non-cyclic and so there is no obvious generalisation here.

A rather trivial example points to further obstacles to extending the above theorem.

EXAMPLE. Let H be the group defined by taking the free nil-3 group on generators a and b and adding the relations $a^{25} = 1 = b^{25}$. For $i = 1, 2, 3$, let Γ_i denote $\gamma_i(H)/\gamma_{i+1}(H)$ and let x be the automorphism of H which maps a to a^2 and b to b^3. Then $[a, b]^{25} = 1$, $[a, b, a]^{25} = 1 = [a, b, b]^{25}$ and Γ_2 is cyclic of order 25 and Γ_3 is non-cyclic of order 625. It is routine to check that $[a, b, x, x] = [a, b, b]^7[a, b, a]^{-2}$ and hence that $[\Gamma_2, \langle x \rangle, \langle x \rangle] = 1$. But $[a, b, a, x] = [a, b, a]^{11}$ and $[a, b, b, x] = [a, b, b]^{17}$ generate $\gamma_3(H)$ and so $[\Gamma_3, \langle x \rangle] = \Gamma_3$. Thus Γ_2 is a "polytrivial" $\langle x \rangle$-module but Γ_3 is not.

Proof of the theorem. Suppose G satisfies the hypothesis of part (a) and assume $\gamma_{n+2}(G) = 1$. As in [**2**] we define certain commutators as follows. Let $\sigma = [x_1, \ldots, x_n]$ for

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some $x_i \in G$ and write $\sigma_1 = x_1$, $\sigma_i = [x_1, \ldots, x_i]$ for $i = 1, 2, \ldots, n$. Let $g \in G$ and set $\alpha_j = [g, x_n, \ldots, x_j]$, $j = 1, \ldots, n$, and $\alpha_{n+1} = g$. We can use the Jacobi identity to deduce that

$$[\sigma, g] \in \langle \alpha_1, [\sigma_{n-j}, \alpha_{n-j+2}, x_{n-j+1}], \qquad j = 1, \ldots, n-1 \rangle. \tag{2}$$

Now suppose that $\sigma(1), \ldots, \sigma(r)$ are left-normed commutators of weight n which generate a free abelian subgroup of rank r of γ_n modulo γ_{n+1}. Writing $S = \langle \sigma(1), \ldots, \sigma(r) \rangle$, we have that $[S, G]$ is generated by elements $[\sigma(k), g]$, and each of these lies in an n-generated subgroup $U_k(g)$ of the type described in (2). Since $\gamma_n / S\gamma_{n+1}$ is periodic, each of the elements α_2, $[\sigma_{n-j}, \alpha_{n-j+2}]$, $j = 1, \ldots, n-1$ (for each k, with the obvious notation) has some power in $S\gamma_{n+1}$ and so there are integers $v_k(g)$ such that

$$(U_k(g))^{v_k(g)} \leq \langle [S, x_{k1}], \ldots, [S, x_{kn}] \rangle = T_k, \text{ say},$$

where $x_{k1}, \ldots, x_{kn}$ are the entries of the commutator $\sigma(k)$.

Let $T = T_1 \ldots T_r$. Then $[S, G]/T$ is periodic and T is at most $r^2 n$-generated. It follows that γ_{n+1} has torsion-free rank at most $r^2 n$, thus proving (a).

Now let T^* be a free abelian subgroup of maximal rank in T. Then the image of γ_{n+1} in G/T^* is periodic, and in order to prove part (b) we may assume that γ_{n+1} is a p-group. Suppose $\sigma \in \gamma_n$ and q is a prime different from p such that $\sigma^q \in S\gamma_{n+1}$. Then, for $g \in G$, $\langle [\sigma, g] \rangle = \langle [s, g] \rangle$ for some $s \in S$. This is because $[\sigma, g]^q = [\sigma^q, g]$ and γ_{n+1} is q-divisible and central. It follows that if F is a finitely generated subgroup of G then there is a subgroup R of $\gamma_n(F)$ which is generated by at most r_p commutators of weight n such that $\gamma_{n+1}(F) \leq [\langle R, S \rangle, F]$. We can now proceed as in the proof of the lemma in [**2**] to deduce that $\gamma_{n+1}(F)$ is generated by at most $(r_p + r)^2 n$ commutators, as required.

Proof of the corollary. Note first that the finite rank case is an immediate consequence of the theorem, while an abelian group satisfies min if and only if it is periodic of finite rank with $r_p = 0$ for almost all p. For the maximal condition, if $\gamma_n(G) = R\gamma_{n+1}(G)$ for some finitely generated subgroup R, then $\gamma_{n+1}(G) = [R, G]$ is also finitely generated, by the argument used in the above proof. If Γ_n is minimax, we choose a finitely generated subgroup S of γ_n which generates a torsion-free subgroup of maximal rank modulo γ_{n+1} and note that the integers $v_k(g)$ may be chosen to be π-numbers, where π is a finite set of primes such that $\gamma_n / S\gamma_{n+1}$ is a π-group. Then, with the same notation, $[S, G]/T$ is a π-group of finite rank and so $[S, G]$ is minimax. Factoring, we may assume S is central. It is clear that γ_{n+1} is then a π-group of finite rank and so minimax. The case where Γ_n is min-by-max is dealt with similarly, thus concluding the proof.

REFERENCES

1. D. J. S. Robinson, A property of the lower central series of a group, *Math. Z.* **107** (1968), 225–231.

https://doi.org/10.1017/S0017089500006698 Published online by Cambridge University Press

2. H. Smith, Hypercentral groups with all subgroups subnormal II, *Bull. London Math. Soc.* **18** (1986) 343–348.

3. J. P. Williams, The join of several subnormal subgroups, *Math. Proc. Cambridge Philos. Soc.* **92** (1982), 391–399.

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