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Reynolds stress models of convection in convective cores

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Abstract. We investigate the properties of non-local Reynolds stress models of turbulent convection in a spherical geometry. Regularity at the centre $r = 0$ places constraints on the behaviour of 3^{rd} order moments. Some of the down-gradient and algebraic closure models have inconsistent behaviour at $r = 0$. A combination of down-gradient and algebraic closures gives a consistent prescription that can be used to model convection in stellar cores.

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1. Reynolds Stress Models

With no mean flow the equations for the second order correlations in a general coordinate system $\{x^j\}$ with metric $ds^2 = g_{ij}dx^i dx^j$ are (cf. Canuto 1992)

$$\frac{\partial}{\partial t}(\overline{\theta^2}) + (\overline{u^j\theta^2})_{;j} = 2(\overline{u^j\theta})\beta_j + \chi g^{jk}(\overline{\theta^2})_{;jk} - 2\epsilon_\theta \tag{1}$$

$$\frac{\partial}{\partial t}(\overline{u_i\theta}) + (\overline{\theta u_i u^j})_{;j} = \overline{u_i u^j}\beta_j + g_i\alpha\overline{\theta^2} - \Pi_i^\theta + \eta_i \tag{2}$$

$$\frac{\partial}{\partial t}(\overline{u_i u^j}) + (\overline{u_i u^j u^k})_{;k} = \alpha g_i(\overline{u^j\theta}) + \alpha g^j(\overline{u_i\theta}) - \frac{2}{3}\delta_i^j(\overline{pu^k})_{;k} + \nu g^{mk}(\overline{u_i u^j})_{;mk} - \Pi_i^j - \epsilon_i^j \tag{3}$$

where $\{;j\}$ denotes the covariant derivatives with respect to x^j. Here (u_1, u_2, u_3) is the covariant velocity, $T = T_o + \theta$ is the temperature with T_o the mean and θ the fluctuating component, ν, χ the kinematic viscosity and thermometric conductivity, $\alpha = 1/T_o$ is the coefficient of thermal expansion, (g_1, g_2, g_3) is the covariant acceleration due to gravity. The covariant superadiabatic temperature gradient $(\beta_1, \beta_2, \beta_3)$ and ϵ_θ are defined as

$$\beta_i = \left(\frac{\partial T_o}{\partial x^i}\right)_{\rm ad} - \frac{\partial T_o}{\partial x_i}, \quad \epsilon_\theta = \chi g^{jk}\overline{\theta_{;j}\theta_{;k}} \tag{4}$$

For simplicity we here neglect the pressure correlations and make the approximations

$$\eta_i = \frac{(\nu+\chi)}{2} g^{jk}(\overline{u_i\theta})_{;jk} \quad \epsilon_i^j = \frac{2}{3}\delta_i^j\epsilon \tag{5}$$

In covariant spherical polar coordinates $(x^1, x^2, x^3) = (r, \theta, \phi)$, the line element and covariant and contravariant velocities are

$$ds^2 = g_{ij}dx^i dx^j = dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2$$

$$\{u_i\} = \{v_r, rv_\theta, r\sin\theta v_\phi\}, \quad \{u^i\} = \left\{v_r, \frac{v_\theta}{r}, \frac{v_\phi}{r\sin\theta}\right\} \tag{6}$$

where v_r, v_θ, v_ϕ are the physical components of velocity along the coordinate directions.

With spherical symmetry the acceleration due to gravity $\{g_i\} = (g, 0, 0)$, the superadiabatic temperature gradient $\{\beta_i\} = (\beta, 0, 0)$, and the non vanishing correlations are

$$\overline{\theta^2}, \quad \overline{v_r^2}, \quad \overline{v_\theta^2} = \overline{v_\phi^2}, \quad \overline{v_r\theta}, \quad \overline{v_r^2\theta}, \quad \overline{v_\theta^2\theta} = \overline{v_\phi^2\theta}, \quad \overline{v_r\theta^2}, \quad \overline{v_r^3}, \quad \overline{v_r v_\theta^2} = \overline{v_r v_\phi^2} \tag{7}$$

Equations (1-3) in spherical polar coordinates then reduce to the 4 independent equations

$$\frac{\partial \overline{\theta^2}}{\partial t} + \frac{1}{r^2}\frac{d}{dr}(r^2\,\overline{v_r\theta^2}) = 2\beta\,\overline{v_r\theta} + \chi\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\,\overline{\theta^2}}{dr}\right) - 2\epsilon_\theta \tag{8}$$

$$\frac{\partial\,\overline{v_r\theta}}{\partial t} + \frac{d\,\overline{v_r^2\theta}}{dr} + 2\,\frac{\overline{v_r^2\theta} - \overline{v_\theta^2\theta}}{r} = \beta\,\overline{v_r^2} + g\alpha\,\overline{\theta^2} + \left(\frac{\nu+\chi}{2}\right)\frac{d}{dr}\left(\frac{1}{r^2}\frac{d}{dr}(r^2\,\overline{v_r\theta})\right) \tag{9}$$

$$\frac{\partial\,\overline{v_r^2}}{\partial t} + \frac{d\,\overline{v_r^3}}{dr} + 2\,\frac{\overline{v_r^3}}{r} - 4\,\frac{\overline{v_r v_\theta^2}}{r} = 2\alpha g\,\overline{v_r\theta} + \nu\left[\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\,\overline{v_r^2}}{dr}\right) - 4\,\frac{\overline{v_r^2} - \overline{v_\theta^2}}{r^2}\right] - \frac{2}{3}\epsilon \tag{10}$$

$$\frac{\partial\,\overline{v_\theta^2}}{\partial t} + \frac{d\,\overline{v_r v_\theta^2}}{dr} + 4\,\frac{\overline{v_r v_\theta^2}}{r} = \nu\left[\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\,\overline{v_\theta^2}}{dr}\right) + 2\,\frac{\overline{v_r^2} - \overline{v_\theta^2}}{r^2}\right] - \frac{2}{3}\epsilon \tag{11}$$

2. Requirements on the 3rd order moments for regularity at $r = 0$

As $r \to 0$, $g, \beta \propto r$ and $\overline{\theta^2}$, $\overline{v_r^2}$, $\overline{v_\theta^2}$ have even expansions (with $a_{10} = a_{20}$ from (11))

$$\overline{\theta^2} = a_{00} + a_{02}r^2 + \ldots, \quad \overline{v_r^2} = a_{10} + a_{12}r^2 + \ldots, \quad \overline{v_\theta^2} = a_{20} + a_{22}r^2 + \ldots \tag{12}$$

With regularity at $r = 0$ it follows from equations (8), (9), (10), and (11) that

$$\overline{v_r\theta} = a_{31}r + a_{33}r^3 + \ldots, \quad \overline{v_r\theta^2} = a_{41}r + a_{43}r^3 + \ldots, \tag{13}$$

$$\overline{v_r^2\theta} = a_{52}r^2 + a_{54}r^4 + \ldots, \quad \overline{v_\theta^2\theta} = a_{62}r^2 + a_{64}r^4 + \ldots, \tag{14}$$

$$\overline{v_r^3} = a_{71}r + a_{73}r^3 + \ldots, \quad \overline{v_r v_\theta^2} = a_{81}r + a_{83}r^3 + \ldots \tag{15}$$

3. Down-gradient closure approximations for third order moments

The down-gradient closures for the third order moments $\overline{u_i\theta^2}$, $\overline{u_iu_j\theta}$, $\overline{u_iu_ju_k}$ need to be expressed in tensorial form. For plane symmetry we take these as

$$\overline{w\theta^2} = -\chi_t\frac{d\,\overline{w^2}}{dz}, \quad \overline{w^2\theta} = -\frac{(\nu_t+\chi_t)}{2}\frac{d\,\overline{w\theta}}{dz}, \quad \overline{w^3} = -\nu_t\frac{d\,\overline{w^2}}{dz} \tag{16}$$

where w is the fluctuating vertical velocity and ν_t, χ_t are eddy transport coefficients.

In a general coordinate system with velocity $\{u_i\}$ these can be expressed as

$$\overline{u_i\theta^2} = -\chi_t(\overline{\theta^2})_{;i}, \quad \overline{u_iu_j\theta} = -\frac{(\nu_t+\chi_t)}{2}\left(\frac{\overline{u_i\theta}_{;j} + \overline{u_j\theta}_{;i}}{2}\right)$$

$$\overline{u_iu_ju_k} = -\nu_t\left(\frac{\overline{u_iu_j}_{;k} + \overline{u_iu_k}_{;j} + \overline{u_ju_k}_{;i}}{3}\right) \tag{17}$$

which are invariant under a re-ordering of indices and reduce to (16) for plane symmetry.

In spherical polar coordinates, with spherical symmetry ($\overline{v_\theta^2} = \overline{v_\phi^2}$), these reduce to

$$\overline{v_r\theta^2} = -\chi_t\frac{d}{dr}\left(\overline{\theta^2}\right), \quad \overline{v_r^2\theta} = -\frac{(\nu_t+\chi_t)}{2}\frac{d}{dr}\left(\overline{v_r\theta}\right), \quad \overline{v_\theta^2\theta} = -\frac{(\nu_t+\chi_t)}{2}\left(\frac{\overline{v_r\theta}}{r}\right) \tag{18}$$

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$$\overline{v_r^3} = -\nu_t \frac{d}{dr}\left(\overline{v_r^2}\right), \quad \overline{v_r v_\theta^2} = -\frac{\nu_t}{3}\left[\frac{d}{dr}(\overline{v_\theta^2}) + \frac{2}{r}(\overline{v_r^2} - \overline{v_\theta^2})\right] \tag{19}$$

We note that the down-gradient closure approximations for $\overline{v_r^2\theta}$ and $\overline{v_\theta^2\theta}$ are incompatible with the behaviour as $r \to 0$ deduced from regularity of the governing equations, unless the flux $\overline{v_r\theta} \propto r^3$ as $r \to 0$. As both the radiative flux and the total flux $\propto r$ as $r \to 0$, this would require the core to be convectively neutral at $r = 0$ (cf. Xiong, 1979).

4. Algebraic closures for 3rd order moments

An alternative closure procedure is to use algebraic relations as deduced by Gryanik and Hartmann (2002) which for plane symmetry are

$$\overline{w^2\theta} = \frac{\overline{w^3}\ \overline{w\theta}}{\overline{w^2}}, \quad \overline{w\theta^2} = \frac{\overline{\theta^3}\ \overline{w\theta}}{\overline{\theta^2}} \tag{20}$$

where w is the fluctuating vertical velocity. In tensorial form with velocity $\{u_i\}$ this generalises to

$$\overline{u_i u_j \theta} = \frac{1}{3}\left(\frac{\overline{u_i u_j u^k}\ \overline{u_k\theta}}{\overline{V^2}} + \frac{\overline{u_i u_k u^k}\ \overline{u_j\theta}}{\overline{V^2}} + \frac{\overline{u_j u_k u^k}\ \overline{u_i\theta}}{\overline{V^2}}\right), \quad \overline{u_i\theta^2} = \frac{\overline{\theta^3}\ \overline{u_i\theta}}{\overline{\theta^2}} \tag{21}$$

where $\overline{V^2} = \overline{u_i u^i}$. For spherical symmetry this reduces to

$$\overline{v_r^2\theta} = \frac{1}{3}\left(\frac{\overline{v_r^3} + 2\ \overline{v_r V^2}}{\overline{V^2}}\right)\overline{v_r\theta}, \quad \overline{v_\theta^2\theta} = \frac{1}{3}\frac{\overline{v_r v_\theta^2}}{\overline{V^2}}\ \overline{v_r\theta}, \quad \overline{v_r\theta^2} = \frac{\overline{\theta^3}}{\overline{\theta^2}}\ \overline{v_r\theta} \tag{22}$$

We note that the algebraic closures give the correct behaviour ($\propto r^2$) for $\overline{v_r^2\theta}$, $\overline{v_\theta^2\theta}$ as $r \to 0$, with the flux $\overline{v_r\theta} \propto r$ as $r \to 0$. However, they do not give the correct behaviour of $\overline{v_r\theta^2}$ since by symmetry $\overline{\theta^3} \to 0$ as $r \to 0$ so that $\overline{v_r\theta^2}$ cannot be $\propto r$. These results are independent of the form of $\overline{u_i u_j \theta}$ and the values of a_{31} and a_{33} in (13).

5. A combined down-gradient and algebraic closure model

By combining the above results we can obtain a closure approximation that at least has the correct behaviour as $r \to 0$, namely:

$$\overline{v_r\theta^2} = -\chi_t \frac{d}{dr}\left(\overline{\theta^2}\right), \quad \overline{v_r^3} = -\nu_t \frac{d}{dr}\left(\overline{v_r^2}\right), \quad \overline{v_r v_\theta^2} = -\frac{\nu_t}{3}\left[\frac{d}{dr}(\overline{v_\theta^2}) + \frac{2}{r}(\overline{v_r^2} - \overline{v_\theta^2})\right] \tag{23}$$

$$\overline{v_r^2\theta} = \frac{1}{3}\left(\frac{\overline{v_r^3} + 2\ \overline{v_r V^2}}{\overline{V^2}}\right)\overline{v_r\theta}, \quad \overline{v_\theta^2\theta} = \frac{1}{3}\frac{\overline{v_r v_\theta^2}}{\overline{V^2}}\ \overline{v_r\theta} \tag{24}$$

where $\overline{V^2} = \overline{v_r^2} + \overline{v_\theta^2} + \overline{v_\phi^2}$. This could be used in models of convection in stellar cores.

Here we have only considered closure approximations for third order moments. It would be desirable to develop the equations for third order moments in spherical geometry and to use algebraic closures on the fourth order moments. This is work in progress.

References

Canuto, V. M., 1992, *ApJ* 392, 218
Gryanik, V. M. & Hartmann, J., 2002, *J. Atmos. Sci.* 59, 2729
Xiong, D. R., 1979, *Acta. Astr. Sinica* 20, 238