ANZIAM J. 47(2005), 203–235

A COMPARISON PRINCIPLE AND STABILITY FOR LARGE-SCALE IMPULSIVE DELAY DIFFERENTIAL SYSTEMS

XINZHI LIU[✉ 1], XUEMIN SHEN[2] and YI ZHANG[13]

(Received 6 October, 2004; revised 16 August, 2005)

Abstract

This paper studies the stability of large-scale impulsive delay differential systems and impulsive neutral systems. By developing some impulsive delay differential inequalities and a comparison principle, sufficient conditions are derived for the stability of both linear and nonlinear large-scale impulsive delay differential systems and impulsive neutral systems. Examples are given to illustrate the main results.

1. Introduction

Many evolution processes exhibit abrupt changes of their states at certain moments in time, such as threshold phenomena in biology, bursting rhythm models in medicine, optimal control models in economics, circuit networks and frequency modulated systems, *etc*. These abrupt changes are of short-term duration and may be described by impulsive differential equations. The theory of impulsive differential equations has been significantly developed in the past two decades, see [2, 5, 9, 11, 12, 15, 16, 20] and references therein. However, the corresponding theory for impulsive delay differential equations is less developed due to some theoretical and technical difficulties. Some existence and uniqueness results have been developed recently in [3] for general impulsive delay differential equations and some special classes were considered in [1, 4, 8]. Some exponential stability results for linear delay impulsive differential equations are obtained in [1, 4] utilising fundamental matrices. Weakly exponential stability is studied in [19]. Two criteria on asymptotic behaviour are given for a

[1]Department of Applied Mathematics, University of Waterloo, Waterloo, Ontario N2L 3G1, Canada; e-mail: xzliu@math.uwaterloo.ca.
[2]Department of Electric and Computer Engineering, University of Waterloo, Waterloo, Ontario N2L 3G1, Canada.
[3]China University of Petroleum, Beijing 102249, China.

https://doi.org/10.1017/S1446181100009998 Published online by Cambridge University Press

nonlinear neutral differential equation with an impulse in [14]. Impulsive integro-differential questions are studied in a Banach space in [7]. The method of Lyapunov is used to study the stability problem of impulsive delay differential equations in [6, 17, 18]. However, how to construct a suitable Lyapunov function or functional for a large-scale complex system remains a challenging issue. Recently, a new approach has been proposed in [13] for studying the stability problem of large-scale dynamic systems, where the study of a complicated large-scale system is converted to that of a lower order linear system by using a comparison principle.

In this paper, we shall study stability problems for both linear and nonlinear large-scale impulsive differential equations with a time delay in the spirit of [13]. We shall first establish the comparison principle and some inequalities for impulsive delay differential equations, and then derive some sufficient conditions to guarantee stability of the nonlinear impulsive large-scale differential equations. These conditions are simple and easy to verify. Examples are given to illustrate the main results. The remainder of this paper is organised as follows. In Section 2, the comparison principle and several inequalities for linear impulsive differential equations with delay are proved, which are useful in studying the stability of large-scale impulsive delay differential equations. Some stability criteria for both linear and nonlinear large-scale impulsive delay differential equations are established in Section 3. In Section 4, the stability problem for large-scale impulsive neutral differential systems is investigated. Finally, the conclusion is given in Section 5.

2. Preliminaries

Let R be the set of real numbers, R^n be the space of n-dimensional column vectors $x = \mathrm{col}(x_1, \dots, x_n)$ with the norm $\|x\| = \sum_{i=1}^n |x_i|$ and let $\|A\| = \max_{1\le j\le m} \sum_{i=1}^n |a_{ij}|$ denote the norm of an $n \times m$ matrix $A = (a_{ij})$.

Let $I = \{t_k \mid t_1 < t_2 < \cdots,\ t_i - t_{i-1} > \alpha > 0,\ i = 1, 2, \dots\}$, $J = \{t \mid t \ge t_0,\ t_0 \in R\}$, $J_k = \{t \mid t_{k-1} \le t < t_k\}$ and $\Delta(t) = \{k \mid t_0 \le t_k < t\}$. Without loss of generality, let $t_0 \le t_1$, where t_0 is the initial time of the IVP (initial value problem, see Sections 3–4) and t_1 is the first instant of I.

For $a, b \in R$, $a < b$, define

$$\begin{aligned} PC[[a,b], R^n] = \{\phi : [a,b] \to R^n \mid\ & \phi(t+0) = \phi(t), \text{ for all } t \in [a,b); \\ & \phi(t-0) \text{ exists in } R^n, \text{ for all } t \in (a,b] \text{ and } \phi(t-0) = \phi(t) \\ & \text{for all but at most a finite number of points } t \in (a,b]\}; \\ PC[[a,\infty), R^n] = \{\phi : [a,\infty) \to R^n \mid\ & \text{for all } b > a, \phi \in PC[[a,b], R^n]\}. \end{aligned}$$

Let $\|\phi_t\| = \sup_{t-\tau\le\theta\le t} \|\phi(\theta)\|$ denote the norm of functions $\phi \in PC[[t-\tau, t], R^{n\times m}]$, where $\tau > 0$ is a constant.

Let $A = (a_{ij})$, $B = (b_{ij})$ be $n \times m$ matrices, denote $A \leq B$ if $a_{ij} \leq b_{ij}$ for all $i = 1, \dots, n$; $j = 1, \dots, m$. Denote by Θ the zero matrix, that is, all of the entries of Θ are 0.

LEMMA 2.1. *Assume that $H \in C[J, R^{r \times r}]$, $H = (h_{ij}(t))_{r \times r}$, $h_{ij}(t) \geq 0$, $i \neq j$, $i, j = 1, 2, \dots, r$, $f \in C[J, R^r]$, $D_k \in R^{r \times r}$ and $D_k \geq \Theta$. Let $x, y \in C^1[J \backslash I, R^r]$ be such that*

$$\begin{cases} dx(t)/dt \leq H(t)x(t) + f(t), & t \in J \backslash I, \\ x(t_k) \leq D_k x(t_k - 0), & t_k \in I, \\ x(t_0) \leq x_0 \end{cases}$$

and

$$\begin{cases} dy(t)/dt = H(t)y(t) + f(t), & t \in J \backslash I, \\ y(t_k) = D_k y(t_k - 0), & t_k \in I, \\ y(t_0) = y_0. \end{cases}$$

Then $x_0 \leq y_0$ implies $x(t) \leq y(t)$ for $t \in J$.

PROOF. It follows from the comparison principle in [10] that $x(t) \leq y(t)$ for $t \in J_1$. Since $D_1 \geq \Theta$, we get $x(t_1) \leq y(t_1)$. Let $x(t) \leq y(t)$, $t \in [t_0, t_k)$, then $x(t_k) \leq y(t_k)$ since $D_k \geq \Theta$. With [10], $x(t) \leq y(t)$ for $t \in [t_k, t_{k+1})$ and so $x(t) \leq y(t)$ for $t \in [t_0, t_{k+1})$. By induction, $x(t) \leq y(t)$, $t \in J$. The proof is complete. □

LEMMA 2.2 (Comparison principle). *Assume that $H, G \in C[J, R^{r \times r}]$, $H(t) = (h_{ij}(t))$, $h_{ij}(t) \geq 0$, $i \neq j$, $G(t) = (g_{ij}(t))$, $G \geq \Theta$, $f \in C[J, R^r]$ and that $D_k \geq \Theta$ are $r \times r$ matrices. Let x, y be the solutions of the following systems*:

$$\begin{cases} dx(t)/dt \leq H(t)x(t) + G(t)x(t - \tau) + f(t), & t \in J \backslash I, \\ x(t_k) \leq D_k x(t_k - 0), & t_k \in I, \\ x(\theta) \leq \phi(\theta), & t_0 - \tau \leq \theta \leq t_0 \end{cases}$$

and

$$\begin{cases} dy(t)/dt = H(t)y(t) + G(t)y(t - \tau) + f(t), & t \in J \backslash I, \\ y(t_k) = D_k y(t_k - 0), & t_k \in I, \\ y(\theta) = \psi(\theta), & t_0 - \tau \leq \theta \leq t_0, \end{cases}$$

respectively, where $\phi, \psi \in PC[[-\tau, 0], R^r]$. Then $\phi(\theta) \leq \psi(\theta)$ implies $x(t) \leq y(t)$.

PROOF. We first prove that $x(t) \leq y(t)$ for $t \in [t_0 - \tau, t_1)$.

Consider the system

$$\begin{cases} dY(t)/dt = H(t)Y(t) + G(t)Y(t-\tau) + f(t) + \epsilon, & t \in [t_0, t_1), \\ Y(\theta) = \psi(\theta) + \epsilon, & t_0 - \tau \le \theta \le t_0. \end{cases}$$

We claim that $\psi(\theta) \ge \phi(\theta)$ implies that $Y(t) > x(t)$ for $t \in [t_0 - \tau, t_1)$.

In fact, if this is not true, then there exists a $t_0 < t^* < t_1$ and some i such that

$$\begin{aligned} &x_i(t) < Y_i(t), \quad t \in [t_0 - \tau, t^*), \\ &x_i(t^*) = Y_i(t^*) \quad \text{and} \\ &x_j(t) \le Y_j(t), \quad t \in [t_0 - \tau, t^*], \ j \ne i. \end{aligned}$$

Thus $Y_i'(t^*) \le x_i'(t^*)$. On the other hand,

$$\begin{aligned} Y_i'(t^*) &= \sum_{j=1}^{r} h_{ij}(t^*)Y_j(t^*) + \sum_{j=1}^{r} g_{ij}(t^*)Y_j(t^* - \tau) + f_i(t^*) + \epsilon \\ &\ge \sum_{j=1}^{r} h_{ij}(t^*)x_j(t^*) + \sum_{j=1}^{r} g_{ij}(t^*)x_j(t^* - \tau) + f_i(t^*) + \epsilon \\ &> \sum_{j=1}^{r} h_{ij}(t^*)x_j(t^*) + \sum_{j=1}^{r} g_{ij}(t^*)x_j(t^* - \tau) + f_i(t^*) = x_i'(t^*). \end{aligned}$$

This contradiction indicates that $Y(t) > x(t)$ for $t_1 > t \ge t_0 - \tau$. Let $\epsilon \to 0$, then $y(t) \to Y(t)$ and hence $y(t) \ge x(t)$ for $t_1 > t \ge t_0 - \tau$.

Since $D_1 \ge 0$, $x(t_1) = D_1 x(t_1 - 0) \le D_1 y(t_1 - 0) = y(t_1)$. Let $x(t) \le y(t)$ for $t \in [t_0 - \tau, t_k)$, then $x(t_k) = D_k x(t_k - 0) \le D_k y(t_k - 0) = y(t_k)$. Similar to the previous process, we have $x(t) \le y(t)$ when $t \in [t_0 - \tau, t_{k+1})$. By induction, it follows that $x(t) \le y(t)$, $t \in [t_0 - \tau, \infty)$. The proof is complete. □

LEMMA 2.3. *Let $A, B \in C[[t_0, \infty), R^{n\times n}]$, $\Phi(t, t_0)$ be the fundamental matrix of $dx/dt = A(t)x$ and $x(t) = x(t, t_0, \phi)$ be the solution of the system*

$$\begin{cases} dx/dt = A(t)x(t) + B(t)x(t-\tau) \\ x(\theta) = \phi(\theta), & \theta \in [t_0 - \tau, t_0], \end{cases} \tag{2.1}$$

where $\phi \in PC[[t_0 - \tau, t_0], R^n]$. Assume that there exist positive numbers $\gamma > 0$ and $M > 0$ such that $\|\Phi(t, t_0)\| \le Me^{-\gamma(t-t_0)}$ and $\gamma > M \sup_{t\ge t_0} \|B(t)\|$. Then there exists an $\alpha > 0$ such that $\|x(t)\| \le M\|x_{t_0}\|e^{-\alpha(t-t_0)}$.

PROOF. Since $\gamma > M \sup_{t\ge t_0} \|B(t)\|$, there exists an $\alpha > 0$ such that

$$\gamma - \alpha - M \sup_{t\ge t_0} \|B(t)\| e^{\alpha\tau} > 0.$$

We claim that, for this α, $\|x(t)\| \le M\|x_{t_0}\|e^{-\alpha(t-t_0)}$. In fact, by the method of variation of parameters, the solution of (2.1) is given by

$$x(t) = \Phi(t, t_0)x(t_0) + \int_{t_0}^{t} \Phi(t, s)B(s)x(s-\tau)\,ds,$$

and

$$\begin{aligned}\|x(t)\| &\le \|\Phi(t, t_0)\|\|x(t_0)\| + \int_{t_0}^{t} \|\Phi(t, s)\|\|B(s)\|\|x(s-\tau)\|\,ds \\ &\le Me^{-\gamma(t-t_0)}\|x_{t_0}\| + \int_{t_0}^{t} Me^{-\gamma(t-s)}\|B(s)\|\|x(s-\tau)\|\,ds.\end{aligned}$$

Let $Q(t) = e^{\alpha(t-t_0)}\big[Me^{-\gamma(t-t_0)}\|x_{t_0}\| + \int_{t_0}^{t} Me^{-\gamma(t-s)}\|B(s)\|\|x(s-\tau)\|\,ds\big]$, $t \ge t_0$ and $Q(t) = \|x_{t_0}\|$, $t \in [t_0 - \tau, t_0]$. Then

$$Q(t) \ge \|x(t)\|e^{\alpha(t-t_0)}, \quad t \ge t_0 - \tau \tag{2.2}$$

and

$$\begin{aligned}Q'(t) &= \alpha e^{\alpha(t-t_0)}\left[Me^{-\gamma(t-t_0)}\|x_{t_0}\| + s\int_{t_0}^{t} Me^{-\gamma(t-s)}\|B(s)\|\|x(s-\tau)\|\,ds\right] \\ &\quad + e^{\alpha(t-t_0)}\left[-\gamma Me^{-\gamma(t-t_0)}\|x_{t_0}\| - \gamma\int_{t_0}^{t} Me^{-\gamma(t-s)}\|B(s)\|\|x(s-\tau)\|\,ds\right] \\ &\quad + Me^{\alpha(t-t_0)}\|B(t)\|\|x(t-\tau)\| \\ &= \alpha Q(t) - \gamma Q(t) + Me^{\alpha(t-t_0)}\|B(t)\|\|x(t-\tau)\| \\ &= (\alpha-\gamma)Q(t) + Me^{\alpha(t-t_0)}\|B(t)\|\|x(t-\tau)\| \\ &\le (\alpha-\gamma)Q(t) + M\|B(t)\|\|Q(t-\tau)\|e^{\alpha\tau}\end{aligned}$$

in view of (2.2).

For any $K > 1$, we claim that $Q(t) < K\|Q_{t_0}\| =: L$, $L > 0$, $t \ge t_0 - \tau$. If this is not true, then there exists $t^* > t_0$ such that $Q(t^*) = L$, $Q(t) < L$, $t_0 - \tau \le t < t^*$ and $Q'(t^*) \ge 0$. On the other hand,

$$\begin{aligned}Q'(t^*) &\le (\alpha-\gamma)Q(t^*) + M\|B(t^*)\|\|Q(t^*-\tau)\|e^{\alpha\tau} \\ &\le (\alpha-\gamma)L + M\|B(t^*)\|Le^{\alpha\tau} \\ &\le (\alpha-\gamma+M\|B(t^*)\|e^{\alpha\tau})L < 0.\end{aligned}$$

This contradiction implies $Q(t) < L$. Let $K \to 1$, then $Q(t) \le \|Q_{t_0}\|$ and

$$\|x(t)\| \le Q(t)e^{-\alpha(t-t_0)} \le \|Q_{t_0}\|e^{-\alpha(t-t_0)} = M\|x_{t_0}\|e^{-\alpha(t-t_0)}, \quad t \ge t_0 - \tau. \tag{2.3}$$

The proof is complete. □

LEMMA 2.4. *Let* $A, B \in C[[t_0, \infty), R^{r\times r}]$ *and* $\Phi(t, t_0)$ *be the fundamental matrix of* $dx/dt = A(t)x(t)$. *Assume that*

(1) *there exist positive numbers* $\gamma > 0$ *and* $M > 0$ *such that* $\|\Phi(t, t_0)\| \le Me^{-\gamma(t-t_0)}$ *and* $\gamma > M \sup_{t\ge t_0} \|B(t)\|$;

(2) $x(t, t_0, \phi)$ *is the solution of the IVP*

$$\begin{cases} dx(t)/dt = A(t)x(t) + B(t)x(t-\tau), & t \in J\backslash I, \\ x(t_k) = D_k x(t_k - 0), & t_k \in I, \\ x(\theta) = \phi(\theta), & t_0 - \tau \le \theta \le t_0, \end{cases} \tag{2.4}$$

where $\phi \in PC[[t_0 - \tau, t_0], R^n]$ *and* $t_{k+1} - t_k \ge \tau$.

Then there exists an $\alpha > 0$ *such that*

$$\|x(t)\| \le \|x_{t_0}\| \prod_{j\in\Delta(t)} M^{k+1} \max\left\{\|D_j\|, e^{\alpha\tau}\right\} e^{-\alpha(t-t_0)}, \quad t_k \le t < t_{k+1}.$$

PROOF. From Lemma 2.3, it follows that for any k, we have

$$\|x(t)\| \le M\|x_{t_k}\|e^{-\alpha(t-t_k)}, \quad t \in [t_k, t_{k+1}).$$

Since $x(t_k) = D_k x(t_k - 0)$, it follows that

$$\begin{aligned} \|x_{t_k}\| &= \sup_{t_k-\tau\le t\le t_k} \|x(t)\| \\ &\le \max\left\{\sup_{t_k-\tau\le t<t_k} \|x(t)\|, \|x(t_k)\|\right\} \\ &\le \max\left\{\sup_{t_k-\tau\le t<t_k} \|x(t)\|, \|D_k\|\|x(t_k - 0)\|\right\} \\ &\le \max\left\{\sup_{t_k-\tau\le t<t_k} M\|x_{t_{k-1}}\|e^{-\alpha(t-t_{k-1})}, M\|D_k\|\|x_{t_{k-1}}\|e^{-\alpha(t_k-t_{k-1})}\right\} \\ &\le M\|x_{t_{k-1}}\|e^{-\alpha(t_k-t_{k-1})} \max\{e^{\alpha\tau}, \|D_k\|\}. \end{aligned}$$

Using a similar argument, we have $\|x_{t_{k-1}}\| \le M\|x_{t_{k-2}}\|e^{-\alpha(t_{k-1}-t_{k-2})} \max\{e^{\alpha\tau}, \|D_{k-1}\|\}$ and so on. Thus we get

$$\begin{aligned} \|x(t)\| &\le M\|x_{t_k}\|e^{-\alpha(t-t_k)} \\ &\le M^2\|x_{t_{k-1}}\|e^{-\alpha(t-t_{k-1})} \max\{e^{\alpha\tau}, \|D_k\|\} \\ &\le \cdots \le \|x_{t_0}\| \prod_{j\in\Delta(t)} M^{k+1} \max\{\|D_j\|, e^{\alpha\tau}\} e^{-\alpha(t-t_0)}, \quad t_k \le t < t_{k+1}. \end{aligned}$$

The proof is complete. □

https://doi.org/10.1017/S1446181100009998 Published online by Cambridge University Press

3. Stability of large-scale impulsive delay systems

Consider the large-scale impulsive delay differential equations

$$\begin{cases} \dfrac{dx_i(t)}{dt} = A_{ii}(t)x(t) + \sum\limits_{j\neq i} A_{ij}(t)x_j(t) + \sum\limits_{j=1}^{r} B_{ij}(t)x_j(t-\tau), & t \in J\backslash I, \\ x_i(t_k) = \sum\limits_{j=1}^{r} D_{ijk}x_j(t_k - 0), & t_k \in I, \end{cases} \tag{3.1}$$

where $x_i = (x_1^i, \dots, x_{n_i}^i)^T \in R^{n_i}$, $A_{ij}, B_{ij} \in C[J, R^{n_i\times n_j}]$, $D_{ijk} \in R^{n_i\times n_j}$, $i, j = 1, \dots, r$, $k = 1, 2, \dots$ and $\sum_{i=1}^{r} n_i = n$.

In this section and in the following section, we always assume that $t_k - t_{k-1} \geq \tau$, $k = 1, 2, \dots$.

Assume that there exist positive numbers $\alpha_i > 0$, $c_i > 0$, $i = 1, \dots, r$ such that the fundamental solution matrix $R_i(t, t_0)$ of the isolated subsystem

$$dx_i(t)/dt = A_{ii}(t)x_i(t)$$

satisfies $\|R_i(t, t_0)\| \leq c_i e^{-\alpha_i(t-t_0)}$ and

$$\begin{aligned} \|A_{ij}(t)\| &\leq a_{ij}(t) < +\infty, && i, j = 1, 2, \dots, r,\ i \neq j, \\ \|B_{ij}(t)\| &\leq b_{ij}(t) < +\infty, && i, j = 1, 2, \dots, r \quad \text{and} \\ \|D_{ijk}\| &\leq d_{ijk}, && i, j = 1, 2, \dots, r,\ k = 1, 2, \dots. \end{aligned}$$

Denote by $\tilde{A}(t) = ((1 - \delta_{ij})c_i a_{ij}(t))_{r\times r}$, $B(t) = (c_i b_{ij}(t))_{r\times r}$ and $\tilde{D}_k = (d_{ijk})_{r\times r}$, where

$$\delta_{ij} = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$$

We are ready to state and prove our first result.

THEOREM 3.1. *Assume that* $-\alpha_j + \sum_{i\neq j} c_i a_{ij}(t) < -\gamma < 0$, $j = 1, \dots, r$, $-\gamma + \sup_{t\geq t_0} \|\tilde{B}(t)\| < 0$ *and* $\alpha > 0$ *is the solution of* $\gamma - \sup_{t\geq t_0} \|\tilde{B}(t)\| e^{\alpha\tau} - \alpha > 0$. *Then*

(1) $\limsup_{t\to\infty} \left(\prod_{j\in\Delta(t)} \max\{\|\tilde{D}_j\|, e^{\alpha\tau}\}\right)/e^{\alpha(t-t_0)} < \infty$ *implies system* (3.1) *is stable*;

(2) $\limsup_{t\to\infty} \left(\prod_{j\in\Delta(t)} \max\{\|\tilde{D}_j\|, e^{\alpha\tau}\}\right)/e^{\alpha(t-t_0)} = 0$ *implies system* (3.1) *is asymptotically stable*;

(3) *if there exists a positive number β, such that $\beta < \alpha$ and*

$$\limsup_{t\to\infty} \frac{\prod_{j\in\Delta(t)} \max\{\|\tilde{D}_j\|, e^{\alpha\tau}\}}{e^{\beta(t-t_0)}} < \infty,$$

then system (3.1) *is exponentially stable.*

PROOF. From (3.1), we have, for $t \in J_{k+1}$,

$$x_i(t) = R_i(t, t_k)x_i(t_k) + \int_{t_k}^{t} \left[R_i(t,s)\sum_{j\neq i} A_{ij}(s)x_j(s) + R_i(t,s)\sum_{j=1}^{r} B_{ij}(s)x_j(s-\tau)\right] ds$$

and

$$\|x_i(t)\| \leq c_i e^{-\alpha_i(t-t_k)}\|x_i(t_k)\| + \int_{t_k}^{t} c_i e^{-\alpha_i(t-s)} \sum_{j\neq i} a_{ij}(s)\|x_j(s)\|\, ds + \int_{t_k}^{t} c_i e^{-\alpha_i(t-s)} \sum_{j=1}^{r} b_{ij}(s)\|x_j(s-\tau)\|\, ds =: P_i(t).$$

Then $\|x_i(t)\| \leq P_i(t)$, $t \in J_{k+1}$, and

$$P_i'(t) \leq -\alpha_i P_i(t) + \sum_{j\neq i} c_i a_{ij}(t) P_j(t) + \sum_{j=1}^{r} c_i b_{ij}(t) P_j(t-\tau), \quad t \in J\backslash I.$$

Let $P = \mathrm{col}(P_1, \ldots, P_r)$, $P(t_k) = \tilde{D}_k P(t_k - 0)$, $t_k \in I$, then $\|x_i(t)| \leq P_i(t)$, $t \in J$.

Consider the comparison system

$$\begin{cases} P'(t) \leq \mathrm{diag}(-\alpha_1, \ldots, -\alpha_r)P(t) + \tilde{A}(t)P(t) + \tilde{B}(t)P(t-\tau), & t \in J\backslash I, \\ P(t_k) = \tilde{D}_k P(t_k - 0), & t_k \in I; \end{cases}$$

and

$$\begin{cases} \xi'(t) = \mathrm{diag}(-\alpha_1, \ldots, -\alpha_r)\xi(t) + \tilde{A}(t)\xi(t) + \tilde{B}(t)\xi(t-\tau), & t \in J\backslash I, \\ \xi(t_k) = \tilde{D}_k \xi(t_k - 0), & t_k \in I, \end{cases} \tag{3.2}$$

where $\xi(t) = \mathrm{col}(\xi_1(t), \ldots, \xi_r(t))$. Since $-\alpha_j + \sum_{i\neq j} c_i a_{ij}(t) < -\gamma < 0$, we claim that the solution $\eta(t)$ of the system

$$\eta'(t) = \mathrm{diag}(-\alpha_1, \ldots, -\alpha_r)\eta(t) + \tilde{A}(t)\eta(t) \tag{3.3}$$

satisfies $|\eta(t)| \leq \|\eta(t_0)\| e^{-\gamma(t-t_0)}$ and hence the fundamental matrix $\Phi(t, t_0)$ of (3.3) satisfies $\|\Phi(t, t_0)\| \leq e^{-\gamma(t-t_0)}$, where $\eta = \mathrm{col}(\eta_1, \ldots, \eta_r)$.

In fact, let $Q(t) = \sum_{i=1}^{n} |\eta_i(t)|$, $t \geq t_0$, then $Q(t_0) = \sum_{i=1}^{n} |\eta_i(t_0)|$ and

$$\begin{aligned} D^+Q(t) &= \sum_{i=1}^{n} D^+|\eta_i(t)| \leq \sum_{i=1}^{n}\left[-\alpha_i|\eta_i(t)| + \sum_{j\neq i}^{n} c_i a_{ij}(t)|\eta_j(t)|\right] \\ &= \sum_{j=1}^{n}(-\alpha_j|\eta_j(t)|) + \sum_{j=1}^{n}\left(\sum_{i\neq j}^{n} c_i a_{ij}(t)\right)|\eta_j(t)| \\ &= \sum_{j=1}^{n}\left(-\alpha_j + \sum_{i\neq j}^{n} c_i a_{ij}(t)\right)|\eta_j(t)| \leq -\gamma\sum_{j=1}^{n}|\eta_j(t)| = -\gamma Q(t). \end{aligned}$$

Thus

$$\|\eta(t)\| = \sum_{i=1}^{n}|\eta_i(t)| = Q(t) \leq Q(t_0)e^{-\gamma(t-t_0)} = \|\eta(t_0)\|e^{-\gamma(t-t_0)}$$

and so $\|\Phi(t, t_0)\| \leq e^{-\gamma(t-t_0)}$, $t \geq t_0$, since $\Phi(t, t_0)$ is the fundamental matrix.

From Lemma 2.4 and the condition $-\gamma + \sup_{t\geq t_0}\|\tilde{B}(t)\| < 0$, it follows that there exists an $\alpha > 0$ such that the solutions of system (3.2) satisfy

$$\|\xi(t, t_0, \xi_{t_0})\| \leq \|\xi_{t_0}\| \prod_{j\in\Delta(t)} \max\left\{\|\tilde{D}_j\|, e^{\alpha\tau}\right\} e^{-\alpha(t-t_0)}. \tag{3.4}$$

It follows from Lemma 2.2 that $\|x_i(t)\| \leq P_i(t) \leq \xi_i(t)$, $i = 1, \dots, r$, and from (3.4) that statements (1)–(3) of Theorem 3.1 are true. The proof is complete. □

COROLLARY 3.2. *Assume that the conditions of Theorem* 3.1 *hold and there exist positive numbers* $\eta > \tau$ *and* $M > 0$ *such that* $t_k - t_{k-1} = \eta$ *for all* $k = 1, 2, \dots$ *and* $\max\{\sup\{\|\tilde{D}_k\|\}, e^{\alpha\tau}\} \leq M$. *Then* $M < e^{\alpha\eta}$ *implies system* (3.1) *is exponentially stable.*

PROOF. In this case

$$\prod_{j\in\Delta(t)} \max\left\{\|\tilde{D}_j\|, e^{\alpha\tau}\right\} = e^{\sum_{j\in\Delta(t)} \ln\max\{\|\tilde{D}_j\|, e^{\alpha\tau}\}} \leq e^{k\ln M}, \quad t \in [t_k, t_{k+1})$$

and so

$$\frac{\prod_{j\in\Delta(t)} \max\left\{\|\tilde{D}_j\|, e^{\alpha\tau}\right\}}{e^{\alpha(t-t_0)}} \leq e^{k\ln M - \alpha(t-t_0)} \leq e^{k\ln M - k\alpha\eta},$$

which implies the required result. □

COROLLARY 3.3. *Assume that the conditions of Theorem* 3.1 *hold and there exist positive numbers* $M > 0$ *and* $p > 0$ *such that* $\max\{\sup\{\|\tilde{D}_k\|\}, e^{\alpha\tau}\} \le M$ *and*

$$\lim_{T\to\infty} \frac{\eta(t, t+T)}{T} = p,$$

where $\eta(t_0, t)$ *denotes the number of impulses in the time interval* $[t_0, t)$.

Then $M < e^{\alpha/p}$ *implies that system* (3.1) *is exponentially stable.*

PROOF. From the condition, it follows that for any $\epsilon > 0$, there exists $T > 0$ and $\tilde{M} = \tilde{M}(t_0) > 0$, such that when $t \ge t_0 + T$,

$$\prod_{j\in\Delta(t)} \max\{\|\tilde{D}_j\|, e^{\alpha\tau}\} = e^{\sum_{j\in\Delta(t)} \ln\max\{\|\tilde{D}_j\|, e^{\alpha\tau}\}} \le e^{\frac{\eta(t_0,t)}{t-t_0}(t-t_0)\ln M} \le \tilde{M} e^{(p+\epsilon)\ln M(t-t_0)}$$

and so

$$\frac{\prod_{j\in\Delta(t)} \max\{\|\tilde{D}_j\|, e^{\alpha\tau}\}}{e^{\alpha(t-t_0)}} \le \tilde{M} e^{[(p+\epsilon)\ln M-\alpha](t-t_0)}. \tag{3.5}$$

Since ϵ can be chosen arbitrarily small, (3.5) implies the required results. □

EXAMPLE 1. Consider the large-scale impulsive delay system

$$\begin{cases} dx_i(t)/dt = A_{i1}(t)x_1(t) + A_{i2}(t)x_2(t) \\ \qquad\qquad +B_{i1}(t)x_1(t-\tau) + B_{i2}(t)x_2(t-\tau), & i = 1, 2,\ t \in J\backslash I, \\ x_i(t_k) = D_{i1}(t_k)x_1(t_k - 0) + D_{i2}(t_k)x_2(t_k - 0), & i = 1, 2,\ t_k \in I, \end{cases} \tag{3.6}$$

where $x_1, x_2 \in R^2$, $A_{ij}, B_{ij}, D_{ij} \in R^{2\times 2}, i, j = 1, 2$.

Let $t_0 = 0$, $\tau = 1$,

$$A_{11}(t) = \begin{bmatrix} -4 & 0 \\ \frac{1}{2}\sin^2 t & -4 \end{bmatrix} \quad\text{and}\quad A_{22}(t) = \begin{bmatrix} -4 + \frac{3}{4}\sin t & 0 \\ 0 & -3 \end{bmatrix}.$$

Choose $A_{12}(t)$, $A_{21}(t)$ and $B(t) = (B_{ij}(t))$ such that $\|A_{12}(t)\| < 1 - (4e^2)^{-1}$, $\|A_{21}(t)\| < 1 - (4e^2)^{-1}$ and $\|B(t)\| \le e^{-1}$. Then $\alpha_i = -3$, $c_i = 1$, $i = 1, 2$. Choose $\gamma > 2$, $\alpha = 1$ and so $\gamma - \sup_{t\ge t_0} \|\tilde{B}(t)\|e^{\alpha\tau} - \alpha > 0$. Let

$$D_k = \begin{bmatrix} D_{11k} & D_{12k} \\ D_{21k} & D_{22k} \end{bmatrix} = \begin{bmatrix} e - 3/2 & -1/2 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 1/2 & 0 & 0 \end{bmatrix},$$

then $\|D_k\| = \|\tilde{D}_k\| = e - 1/2$ and if

(1) $t_k - t_{k-1} \geq \tau = 1$, then $\left(\prod_{j\in\Delta(t)} \max\{\|\tilde{D}_j\|, e^{\alpha\tau}\}\right)/e^t \leq 1$,

(2) $t_k - t_{k-1} \geq \tau + 1/k$, then $\lim_{t\to\infty}((\prod_{j\in\Delta(t)} \max\{\|\tilde{D}_j\|, e^{\alpha\tau}\})/e^{\alpha t} = 0$,

(3) $t_k - t_{k-1} \geq \tau + \eta$, $\eta > 0$, then $\left(\prod_{j\in\Delta(t)} \max\{\|\tilde{D}_j\|, e^{\alpha\tau}\}\right)/e^{t/(1+\eta)} \leq 1$.

Thus we can conclude that system (3.6) is: stable if $t_k - t_{k-1} \geq \tau = 1$; asymptotically stable if $t_k - t_{k-1} \geq \tau + 1/k$; and exponentially stable if $t_k - t_{k-1} \geq \tau + \eta$, $\eta > 0$.

REMARK. This example illustrates that Theorem 3.1 is simple and easily verified. It is very interesting to notice that, from the example, even if at every impulsive point t_k, $\|D_k\| > 1$, which implies that the norm of solutions is increased at impulsive points, that the system may still be stable or exponentially stable.

Consider the nonlinear impulsive delay differential system

$$\begin{cases} dx/dt = \mathrm{diag}(A_{11}(t), \dots, A_{rr}(t))x(t) + F(t, x(t), x(t-\tau)), & t \in J\backslash I, \\ x(t_k) = D_k x(t_k - 0), & t_k \in I, \end{cases} \tag{3.7}$$

where $F \in C[I \times R^n \times R^n, R^n]$, $F(t, 0, 0) \equiv 0$, $x = (x_1, \dots, x_r)^T \in R^n$, $x_i \in R^{n_i}$, $A_{ii}(t)$ are $n_i \times n_i$ matrices, $D_k = (D_{ijk})$ are $n \times n$ matrices, D_{ijk} are $n_i \times n_j$ matrices, $i, j = 1, 2, \dots, r$, $\sum_{i=1}^r n_i = n$, $k = 1, 2, \dots$. Rewriting (3.7) by components,

$$\begin{cases} dx_i/dt = A_{ii}(t)x_i(t) + F_i(t, x(t), x(t-\tau)), & t \in J\backslash I, \\ x_i(t_k) = \sum_{j=1}^r D_{ijk} x_j(t_k - 0), & t_k \in I,\ i = 1, 2, \dots, r. \end{cases} \tag{3.8}$$

THEOREM 3.4. *If*

(1) *there exist scalar functions $l_{ij}, k_{ij} \in C[J, R]$, such that*

$$\|F_i(t, x, y)\| \leq \sum_{j=1}^r l_{ij}(t)\|x_j\| + \sum_{j=1}^r k_{ij}(t)\|y_j\|, \quad i = 1, 2, \dots, r;$$

(2) *there exist constants c_i and scalar functions $\beta_i \in C[J, R]$, such that the fundamental solution matrix $R_{ii}(t, t_0)$ of the isolated subsystem $dx_i/dt = A_{ii}(t)x_i(t)$ satisfies $\|R_{ii}(t, t_0)\| \leq c_i e^{-\int_{t_0}^t \beta_i(s)\,ds}$, $i = 1, \dots, r$,*

then the stability, uniform stability, global asymptotic stability, global uniform asymptotic stability, global exponential stability, the Lagrange stability of all solutions, uniform Lagrange stability of the trivial solution of the lower dimension linear equations

$$\begin{cases} \dfrac{d\eta_i(t)}{dt} = -\beta_i(t)\eta_i(t) + \sum_{j=1}^r c_i l_{ij}(t)\eta_j(t) + \sum_{j=1}^r c_i k_{ij}(t)\eta_j(t-\tau), & t \in J\backslash I, \\ \eta_i(t_k) = \sum_{j=1}^r \|D_{ijk}\|\eta_j(t_k - 0), & t_k \in I, \end{cases} \tag{3.9}$$

where $i = 1, \dots, r$, imply the stability, uniform stability, global asymptotic stability, global uniform asymptotic stability, global exponential stability, the Lagrange stability of all solutions, uniform Lagrange stability of the high dimension nonlinear system (3.7), *respectively.*

PROOF. Let $x(t) = x(t, t_0, \phi)$ be the solution of (3.7) satisfying the initial condition $x(t) = \phi(t)$, $t_0 - \tau \le t \le t_0$. Then we have, for $i = 1, \dots, r$ and $t \in J_{k+1}$,

$$\begin{cases} x_i(t) = R_i(t, t_k)x_i(t_k) + \displaystyle\int_{t_k}^{t} R_i(t, s)F_i(s, x(s), x(s-\tau))\, ds, \\ x_i(t_k) = \displaystyle\sum_{j=1}^{r} D_{ijk}x_j(t_k - 0), \end{cases}$$

and

$$\begin{cases} \|x_i(t)\| \le c_i e^{-\int_{t_k}^{t} \beta_i(s)\,ds}\|x_i(t_k)\| \\ \qquad + \displaystyle\sum_{j=1}^{r}\int_{t_k}^{t} c_i e^{-\int_s^t \beta_i(v)dv}\big[l_{ij}(s)\|x_j(s)\| + k_{ij}(s)\|x_j(s-\tau)\|\big]\, ds, \\ \|x_i(t_k)\| \le \displaystyle\sum_{j=1}^{r} \|D_{ijk}\|\|x_j(t_k - 0)\|. \end{cases}$$

Let for $i = 1, \dots, r$ and $t \in J_{k+1}$,

$$\begin{cases} \xi_i(t) = c_i e^{-\int_{t_k}^{t} \beta_i(s)\,ds}\|x_i(t_k)\| \\ \qquad + \displaystyle\sum_{j=1}^{r}\int_{t_k}^{t} c_i e^{-\int_s^t \beta_i(v)dv}\big[l_{ij}(s)\|x_j(s)\| + k_{ij}(s)\|x_j(s-\tau)\|\big]\, ds, \\ \xi_i(t_k) = \displaystyle\sum_{j=1}^{r} \|D_{ijk}\|\xi_j(t_k - 0), \end{cases}$$

then $\|x_i(t)\| \le \xi_i(t)$, $i = 1, \dots, r$, and

$$\begin{cases} \dfrac{d\xi_i(t)}{dt} \le -\beta_i(t)\xi_i(t) + \displaystyle\sum_{j=1}^{r} c_i l_{ij}(t)\xi_j(t) + \sum_{j=1}^{r} c_i k_{ij}(t)\xi_j(t-\tau), \\ \xi_i(t_k) = \displaystyle\sum_{j=1}^{r} \|D_{ijk}\|\xi_j(t_k - 0). \end{cases}$$

With the comparison system for $i = 1, \dots, r$ and $t \in J_{k+1}$,

$$\begin{cases} \dfrac{d\eta_i}{dt} = -\beta_i(t)\eta_i(t) + \displaystyle\sum_{j=1}^{r} c_i l_{ij}(t)\eta_j(t) + \sum_{j=1}^{r} c_i k_{ij}(t)\eta_j(t-\tau), \\ \eta_i(t_k) = \displaystyle\sum_{j=1}^{r} \|D_{ijk}\|\eta_j(t_k - 0), \end{cases}$$

and from Lemma 2.2, we have $\|x_i(t)\| \le \xi_i(t) \le \eta_i(t)$, $t \ge t_0$, $i = 1, \dots, r$. The inequality implies the results of the theorem. The proof is complete. □

EXAMPLE 2. Consider the nonlinear impulsive delay differential system

$$\begin{cases} \dfrac{dx_1}{dt} = \begin{bmatrix} -5+e^{-t} & \ln(1+t^2) \\ -\ln(1+t^2) & -4 \end{bmatrix} x_1 + \dfrac{|\sin t|}{2\sqrt{2}(1+\|x_2\|)} x_2 \\ \qquad + \dfrac{(1+\sin t^2)\sin^2(x_1^T x_2)}{4\sqrt{2}e^2} x_1(t-1), \\ \dfrac{dx_2}{dt} = \begin{bmatrix} -6+\sin t & t \\ -t & -7+2\cos^2 t \end{bmatrix} x_2 + \dfrac{\sqrt{2}}{3(1+\ln(1+\|x_2\|^2))} x_1 \\ \qquad + \dfrac{\sqrt{2}}{4e^2(2-e^{-t\|x_1(t-1)\|^2})} x_2(t-1), \quad t \neq t_k, \\ x(t_k) = D_k x(t_k - 0), \quad k = 1, 2, \ldots, \end{cases} \tag{3.10}$$

where $x = (x_1^T, x_2^T)^T \in R^4$, $x_1, x_2 \in R^2$, $D_k = (D_{ijk}) \in R^{4\times 4}$ and $D_{ijk} \in R^{2\times 2}$, $i, j = 1, 2$.

Using the notation of Theorem 3.4, we have

$$A_{11} = \begin{bmatrix} -5+e^{-t} & \ln(1+t^2) \\ -\ln(1+t^2) & -4 \end{bmatrix}, \quad A_{22} = \begin{bmatrix} -6+\sin t & t \\ -t & -7+2\cos^2 t \end{bmatrix},$$

$$F_1(t, x, x(t-\tau)) = \frac{|\sin t|}{2\sqrt{2}(1+\|x_2\|)} x_2 + \frac{(1+\sin t^2)\sin^2(x_1^T x_2)}{4\sqrt{2}e^2} x_1(t-1) \quad \text{and}$$

$$F_2(t, x, x(t-\tau)) = \frac{\sqrt{2}}{3(1+ln(1+\|x_2\|^2))} x_1 + \frac{\sqrt{2}}{4e^2(2-e^{-t\|x_1(t-1)\|^2})} x_2(t-1).$$

By choosing $c_i = \sqrt{2}$, $\beta_i = 4$, $l_{11}(t) = l_{22}(t) = k_{12}(t) = k_{21}(t) = 0$, $l_{12}(t) = \sqrt{2}/4$, $l_{21}(t) = \sqrt{2}/3$, $k_{11} = (2\sqrt{2}e^2)^{-1}$ and $k_{22}(t) = (2\sqrt{2}e^2)^{-1}$, the comparison system is

$$\begin{cases} \dfrac{d\eta_1}{dt} = -4\eta_1(t) + \dfrac{1}{2}\eta_2(t) + \dfrac{1}{2e^2}\eta_1(t-1), \\ \dfrac{d\eta_2}{dt} = -4\eta_2(t) + \dfrac{2}{3}\eta_1(t) + \dfrac{1}{2e^2}\eta_2(t-1), \qquad t \neq t_k,\ k = 1, \ldots, \\ \eta_i(t_k) = \displaystyle\sum_{j=1}^{2} \|D_{ijk}\| \eta_j(t_k - 0), \qquad i = 1, 2,\ k = 1, \ldots. \end{cases} \tag{3.11}$$

For system (3.11), using the notation of Theorem 3.1,

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} -4 & 1/2 \\ 2/3 & -4 \end{bmatrix}, \quad B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} = \begin{bmatrix} (2e^2)^{-1} & 0 \\ 0 & (2e^2)^{-1} \end{bmatrix},$$

and

$$\tilde{A} = \begin{bmatrix} 0 & 1/2 \\ 2/3 & 0 \end{bmatrix}, \quad \tilde{B} = B = \begin{bmatrix} (2e^2)^{-1} & 0 \\ 0 & (2e^2)^{-1} \end{bmatrix}.$$

https://doi.org/10.1017/S1446181100009998 Published online by Cambridge University Press

By choosing $c_i = 1$, $\alpha_i = 4$, $i = 1, 2$, $\|B\| = (2e^2)^{-1}$, $\gamma = 3$, $\|\tilde{B}\| = (2e^2)^{-1}$, $\tau = 1$, $\alpha = 2$, then $\|R_i(t, t_0)\| \le c_i e^{-\alpha_i(t-t_0)}$, $i = 1, 2$, and $\gamma - \|\tilde{B}\|e^{\alpha\tau} - \alpha > 0$. Let

$$D_k = \begin{bmatrix} D_{11k} & D_{12k} \\ D_{21k} & D_{22k} \end{bmatrix} = \begin{bmatrix} e-3/2 & 0 & 1 & 1/2 \\ 1 & 0 & 0 & 1/2 \\ 0 & 1 & 1/2 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix},$$

then $\|D_k\| = \|\tilde{D}_k\| = e - 1/2$ and if

(1) $t_k - t_{k-1} \ge 2 > \tau$, then $\left(e^{-2t}\prod_{j\in\Delta(t)}\max\{\|\tilde{D}_j\|, e^{\alpha\tau}\}\right) \le 1$,

(2) $t_k - t_{k-1} \ge 2 + 1/k$, then $\lim_{t\to\infty}\left(e^{-2t}\prod_{j\in\Delta(t)}\max\{\|\tilde{D}_j\|, e^{\alpha\tau}\}\right) = 0$,

(3) $t_k - t_{k-1} \ge 2 + \eta$, $\eta > 0$, then $\left(e^{-t/(2+\eta)}\prod_{j\in\Delta(t)}\max\{\|\tilde{D}_j\|, e^{\alpha\tau}\}\right) \le 1$.

Thus we can conclude that system (3.10) is: stable in case (1), asymptotically stable in case (2), and exponentially stable in case (3).

4. A large-scale impulsive neutral system

In this section, we consider the large-scale impulsive neutral system

$$\begin{cases} \dfrac{dx_i(t)}{dt} = A_{ii}(t)x_i(t) + \displaystyle\sum_{j\neq i} A_{ij}(t)x_j(t) + \sum_{j=1}^{r} B_{ij}(t)x_j(t-\tau) \\ \qquad + \displaystyle\sum_{j=1}^{n} C_{ij}(t)x_j'(t-\tau), \quad t \in J\backslash I,\ i = 1, \dots, r, \\ x_i(t_k) = \displaystyle\sum_{j=1}^{r} D_{ijk}x_j(t_k - 0), \quad t_k \in I, \end{cases} \tag{4.1}$$

where $A_{ij}, B_{ij} \in C[J, R^{n_i\times n_j}]$, $D_{ijk} \in R^{n_i\times n_j}$, $x_i^T = (x_1^i, \dots, x_{n_i}^i) \in R^{n_i}$, $C_{ij} \in C^1[J, R^{n_i\times n_j}]$, $i, j = 1, \dots, r$, and $\sum_{i=1}^{r} n_i = n$.

If $n_i = n_j = 1$ and $D = (D_{ijk}) = E$ in the system (4.1), where E is an identity matrix, it means that the system does not have an impulse at the point τ_k. Then the system becomes

$$\frac{dx(t)}{dt} = A(t)x(t) + B(t)x(t-\tau) + C(t)x'(t-\tau). \tag{4.2}$$

Denote by $R(t, t_0)$ the fundamental matrix of $dx/dt = A(t)x$ and by $x(t) = x(t, t_0, \phi)$ the solution of (4.2) with initial condition $x(t) = \phi(t)$, $t \in [t_0 - \tau, t_0]$.

LEMMA 4.1. *Assume that*

(1) $\|C(t)\|$ *is a decreasing function;*
(2) *there exist* $\gamma \in C[R, R^+]$, $M > 0$ *such that* $\|R(t, t_0)\| \le Me^{-\int_{t_0}^t \gamma(s)\,ds}$;
(3) *there exists* $l > 0$ *such that* $e^{\int_{t-\tau}^t \gamma(s)\,ds} \le l$.

Then the solution $x(t) = x(t, t_0, \phi)$ *of* (4.2) *satisfies*

$$\|x(t)\| \le \left[M(1 + \|C(t_0)\|) + \|C(t_0 - \tau)\|l\right] \times \|x_{t_0}\| \exp\left(\int_{t_0}^t \left[l\left(\frac{1}{\tau}\|C(s)\| + ML(s)\right) - \gamma(s)\right] ds\right), \tag{4.3}$$

where $L(t) = \|A(t)\|\|C(t)\| + \|C'(t)\| + \|B(t)\|$.

PROOF. By the method of variation of parameters, the solution of (4.2) is given by

$$x(t) = R(t, t_0)x(t_0) + \int_{t_0}^t R(t, s)B(s)x(s - \tau)\,ds + \int_{t_0}^t R(t, s)C(s)x'(s - \tau)\,ds. \tag{4.4}$$

Integrating by parts, the third term on the right-hand side is

$$\begin{aligned}
&\int_{t_0}^t R(t, s)C(s)x'(s - \tau)\,ds \\
&\quad = \int_{t_0}^t R(t, s)C(s)\,dx(s - \tau) \\
&\quad = R(t, t)C(t)x(t - \tau) - R(t, t_0)C(t_0)x(t_0 - \tau) \\
&\qquad - \int_{t_0}^t [R'_s(t, s)C(s) + R(t, s)C'(s)]x(s - \tau)\,ds \\
&\quad = C(t)x(t - \tau) - R(t, t_0)C(t_0)x(t_0 - \tau) \\
&\qquad - \int_{t_0}^t [-R(t, s)A(s)C(s) + R(t, s)C'(s)]x(s - \tau)\,ds
\end{aligned}$$

since $R(t, t) = E$. So

$$\begin{aligned}
x(t) = R(t, t_0)x(t_0) &+ \int_{t_0}^t R(t, s)B(s)x(s - \tau)\,ds + C(t)x(t - \tau) \\
&- R(t, t_0)C(t_0)x(t_0 - \tau) + \int_{t_0}^t R(t, s)[A(s)C(s) - C'(s)]x(s - \tau)\,ds.
\end{aligned}$$

Thus

$$\|x(t)\| \le \|R(t, t_0)\|\|x(t_0)\| + \int_{t_0}^t \|R(t, s)\|\|B(s)\|\|x(s - \tau)\|\,ds$$

$$+\|C(t)\|\|x(t-\tau)\|+\|R(t,t_0)\|\|C(t_0)\|\|x(t_0-\tau)\|$$
$$+\int_{t_0}^{t}\|R(t,s)\|[\|A(s)\|\|C(s)\|+\|C'(s)\|]\|x(s-\tau)\|\,ds$$
$$\le Me^{-\int_{t_0}^{t}\gamma(s)\,ds}\|x_{t_0}\|+\int_{t_0}^{t}Me^{-\int_s^t\gamma(\eta)\,d\eta}\|B(s)\|\|x(s-\tau)\|\,ds$$
$$+\|C(t)\|\|x(t-\tau)\|+Me^{-\int_{t_0}^{t}\gamma(s)\,ds}\|C(t_0)\|\|x_{t_0}\|$$
$$+\int_{t_0}^{t}Me^{-\int_s^t\gamma(\eta)\,d\eta}[\|A(s)\|\|C(s)\|+\|C'(s)\|]\|x(s-\tau)\|\,ds$$
$$=Me^{-\int_{t_0}^{t}\gamma(s)\,ds}\|x_{t_0}\|(1+\|C(t_0)\|)+\|C(t)\|\|x(t-\tau)\|$$
$$+\int_{t_0}^{t}Me^{-\int_s^t\gamma(\eta)\,d\eta}[\|A(s)\|\|C(s)\|+\|C'(s)\|+\|B(s)\|]\|x(s-\tau)\|\,ds$$
$$=Me^{-\int_{t_0}^{t}\gamma(s)\,ds}\|x_{t_0}\|(1+\|C(t_0)\|)+\|C(t)\|\|x(t-\tau)\|$$
$$+\int_{t_0}^{t}Me^{-\int_s^t\gamma(\eta)d\eta}L(s)\|x(s-\tau)\|\,ds.$$

Multiplying both sides by $e^{\int_{t_0}^{t}\gamma(s)\,ds}$, we have

$$e^{\int_{t_0}^{t}\gamma(s)\,ds}\|x(t)\|\le M\|x_{t_0}\|(1+\|C(t_0)\|)+\|C(t)\|e^{\int_{t_0}^{t}\gamma(s)\,ds}\|x(t-\tau)\|$$
$$+\int_{t_0}^{t}Me^{\int_{t_0}^{s}\gamma(\eta)d\eta}L(s)\|x(s-\tau)\|\,ds.$$

Let

$$y(t)=\sup_{t_0-\tau\le s\le t}e^{\int_{t_0}^{s}\gamma(\eta)d\eta}\|x(s)\|.$$

Then $y(t)\ge e^{\int_{t_0}^{t}\gamma(s)\,ds}\|x(t)\|$, $t\ge t_0-\tau$, $y(t)$ is a nondecreasing function and

$$e^{\int_{t_0}^{t}\gamma(s)\,ds}\|x(t)\|$$
$$\le M\|x_{t_0}\|(1+\|C(t_0)\|)+\|C(t)\|e^{\int_{t-\tau}^{t}\gamma(s)\,ds}e^{\int_{t_0}^{t-\tau}\gamma(s)\,ds}\|x(t-\tau)\|$$
$$+\int_{t_0}^{t}Me^{\int_{s-\tau}^{s}\gamma(\eta)\,d\eta}e^{\int_{t_0}^{s-\tau}\gamma(\eta)\,d\eta}L(s)\|x(s-\tau)\|\,ds$$
$$\le M\|x_{t_0}\|(1+\|C(t_0)\|)+\|C(t)\|ly(t-\tau)+\int_{t_0}^{t}MlL(s)y(s-\tau)\,ds$$
$$\le M\|x_{t_0}\|(1+\|C(t_0)\|)+\frac{l}{\tau}\int_{t-\tau}^{t}\|C(s)\|y(s)\,ds$$
$$+\int_{t_0}^{t}MlL(s)y(s-\tau)\,ds,$$

https://doi.org/10.1017/S1446181100009998 Published online by Cambridge University Press

in view of the fact that $\|C(t)\|$ is nonincreasing and $y(t)$ is nondecreasing. Thus

$$\begin{aligned}
e^{\int_{t_0}^t \gamma(s)\,ds}\|x(t)\| &\le M\|x_{t_0}\|(1+\|C(t_0)\|) + \int_{t_0-\tau}^{t_0} \frac{l}{\tau}\|C(s)\|y(s)\,ds \\
&\quad + \int_{t_0}^{t} \frac{l}{\tau}\|C(s)\|y(s)\,ds + \int_{t_0}^{t} MlL(s)y(s)\,ds \\
&\le M\|x_{t_0}\|(1+\|C(t_0)\|) + l\|C(t_0-\tau)\|\,\|x_{t_0}\| \\
&\quad + \int_{t_0}^{t} \frac{l}{\tau}\|C(s)\|y(s)\,ds + \int_{t_0}^{t} MlL(s)y(s)\,ds \\
&\le \|x_{t_0}\|[(1+\|C(t_0)\|)M + \|C(t_0-\tau)\|l] \\
&\quad + \int_{t_0}^{t} l\left[\frac{1}{\tau}\|C(s)\| + ML(s)\right] y(s)\,ds.
\end{aligned}$$

The right-hand side of the last inequality is nondecreasing and it yields that

$$\begin{aligned}
y(t) &\le \|x_{t_0}\|\left[(1+\|C(t_0)\|)M + \|C(t_0-\tau)\|l\right] \\
&\quad + \int_{t_0}^{t} l\left[\frac{1}{\tau}\|C(s)\| + ML(s)\right] y(s)\,ds.
\end{aligned}$$

The Gronwall-Bellman inequality implies

$$y(t) \le \|x_{t_0}\|[(1+\|C(t_0)\|)M + \|C(t_0-\tau)\|l]e^{\int_{t_0}^t l[\frac{1}{\tau}\|C(s)\|+ML(s)]ds}$$

and so (4.3) holds. The proof is complete. □

LEMMA 4.2. *Assume that conditions* (2)–(3) *of Lemma* 4.1 *hold and there exists* $\tilde{C} > 0$ *such that* $\|C(t)\| \le \tilde{C} < 1/l$. *Then the solution* $x(t)$ *of* (4.2) *satisfies*

$$\|x(t)\| \le \frac{M(1+\|C(t_0)\|)}{1-\tilde{C}l}\|x_{t_0}\| \exp\left(\int_{t_0}^{t}\left[\frac{MlL(s)}{1-\tilde{C}l} - \gamma(s)\right]ds\right), \tag{4.5}$$

where $L(t) = \|A(t)\|\|C(t)\| + \|C'(t)\| + \|B(t)\|$.

PROOF. From the first part of the proof of Lemma 4.1, it follows that

$$\begin{aligned}
e^{\int_{t_0}^t \gamma(s)\,ds}\|x(t)\| &\le M\|x_{t_0}\|(1+\|C(t_0)\|) \\
&\quad + \|C(t)\|e^{\int_{t-\tau}^t \gamma(s)\,ds}e^{\int_{t_0}^{t-\tau} \gamma(s)\,ds}\|x(t-\tau)\| \\
&\quad + \int_{t_0}^{t} Me^{\int_{s-\tau}^t \gamma(\eta)d\eta}e^{\int_{t_0}^{s-\tau} \gamma(\eta)\,d\eta}L(s)\|x(s-\tau)\|\,ds \\
&\le M\|x_{t_0}\|(1+\|C(t_0)\|) + \|C(t)\|ly(t-\tau)
\end{aligned}$$

https://doi.org/10.1017/S1446181100009998 Published online by Cambridge University Press

$$+\int_{t_0}^{t} MlL(s)y(s-\tau)\,ds$$
$$\leq M\|x_{t_0}\|(1+\|C(t_0)\|)+\tilde{C}ly(t)+\int_{t_0}^{t} MlL(s)y(s)\,ds,$$

where $y(t) \geq e^{\int_{t_0}^{t}\gamma(s)\,ds}\|x(t)\|$, $t \geq t_0-\tau$, is a nondecreasing function. Furthermore, since the right-hand side of the last inequality is nondecreasing, it follows that

$$y(t) \leq M\|x_{t_0}\|(1+\|C(t_0)\|)+\tilde{C}ly(t)+\int_{t_0}^{t} MlL(s)y(s)\,ds,$$

thus

$$y(t) \leq \frac{M\|x_{t_0}\|(1+\|C(t_0)\|)}{1-\tilde{C}l}+\frac{1}{1-\tilde{C}l}\int_{t_0}^{t} MlL(s)y(s)\,ds.$$

The Gronwall-Bellman inequality implies

$$y(t) \leq \frac{M(1+\|C(t_0)\|)}{1-\tilde{C}l}\|x_{t_0}\|e^{\int_{t_0}^{t}\frac{MlL(s)}{1-\tilde{C}l}\,ds}$$

and so (4.5) holds. The proof is complete. □

THEOREM 4.3. *Assume that the conditions of Lemma* 4.1 *hold and* $x(t)=x(t,t_0,\phi)$ *is the solution of the system*

$$\begin{cases} dx/dt = A(t)x(t)+B(t)x(t-\tau)+C(t)x'(t-\tau), & t\neq t_k,\\ x(t_k)=D_kx(t_k-0), & \\ x(\theta)=\phi(\theta), & \theta\in[t_0-\tau,t_0]. \end{cases} \tag{4.6}$$

Then for $t_k \leq t < t_{k+1}$,

$$\|x(t)\| \leq \|x_{t_0}\|M_0\prod_{1\leq j\leq k} M_j\max\left\{\sup_{t_j-\tau\leq t<t_j} e^{\int_{t_j}^{t}\alpha(s)\,ds},\ \|D_j\|\right\}e^{\int_{t_0}^{t}\alpha(s)\,ds},$$

where $\alpha(t)=l\left(\frac{1}{\tau}\|C(s)\|+ML(s)\right)-\gamma(s)$, $M_k=M(1+\|C(t_k)\|)+\|C(t_k-\tau)\|l$, $L(t)=\|A(t)\|\|C(t)\|+\|C'(t)\|+\|B(t)\|$ *and* $l \geq e^{\int_{t-\tau}^{t}\gamma(s)\,ds}$, $t\geq t_0$.

PROOF. From Lemma 4.1, for any $k=1,\dots,$ we have

$$\|x(t)\| \leq M_k\|x_{t_k}\|e^{\int_{t_k}^{t}\alpha(s)\,ds},\quad t\in[t_k,t_{k+1}).$$

Since $x(t_k)=D_kx(t_k-0)$, it follows that

$$\|x_{t_k}\| = \sup_{t_k-\tau\leq t\leq t_k}\|x(t)\| = \max\left\{\sup_{t_k-\tau\leq t<t_k}\|x(t)\|,\ \|x(t_k)\|\right\}$$

$$
\begin{aligned}
&\le \max\left\{\sup_{t_k-\tau\le t<t_k}\|x(t)\|,\ \|D_k\|\|x(t_k-0)\|\right\}\\
&\le \max\left\{\sup_{t_k-\tau\le t<t_k} M_{k-1}\|x_{t_{k-1}}\|e^{\int_{t_{k-1}}^{t}\alpha(s)\,ds},\ M_{k-1}\|D_k\|\|x_{t_{k-1}}\|e^{\int_{t_{k-1}}^{t_k}\alpha(s)\,ds}\right\}\\
&\le M_{k-1}\|x_{t_{k-1}}\|\max\left\{\sup_{t_k-\tau\le t<t_k} e^{\int_{t_k}^{t}\alpha(s)\,ds},\ \|D_k\|\right\}e^{\int_{t_{k-1}}^{t_k}\alpha(s)\,ds}.
\end{aligned}
$$

Using a similar argument, we have that

$$
\|x_{t_{k-1}}\|\le M_{k-2}\|x_{t_{k-2}}\|\max\left\{\sup_{t_{k-1}-\tau\le t<t_{k-1}} e^{\int_{t_{k-1}}^{t}\alpha(s)\,ds},\ \|D_{k-1}\|\right\}e^{\int_{t_{k-2}}^{t_{k-1}}\alpha(s)\,ds}
$$

and so on. Thus we get, for $t_k\le t<t_{k+1}$,

$$
\begin{aligned}
\|x(t)\| &\le M_k\|x_{t_k}\|e^{\int_{t_k}^{t}\alpha(s)\,ds}\\
&\le M_kM_{k-1}\|x_{t_{k-1}}\|\max\left\{\sup_{t_k-\tau\le t<t_k} e^{\int_{t_k}^{t}\alpha(s)\,ds},\ \|D_k\|\right\}e^{\int_{t_{k-1}}^{t_k}\alpha(s)\,ds}e^{\int_{t_k}^{t}\alpha(s)\,ds}\\
&= M_kM_{k-1}\|x_{t_{k-1}}\|\max\left\{\sup_{t_k-\tau\le t<t_k} e^{\int_{t_k}^{t}\alpha(s)\,ds},\ \|D_k\|\right\}e^{\int_{t_{k-1}}^{t}\alpha(s)\,ds}\\
&\le\cdots\le \|x_{t_0}\|M_0\prod_{1\le j\le k}M_j\max\left\{\sup_{t_j-\tau\le t<t_j} e^{\int_{t_j}^{t}\alpha(s)\,ds},\ \|D_j\|\right\}e^{\int_{t_0}^{t}\alpha(s)\,ds}.
\end{aligned}
$$

The proof is complete. □

THEOREM 4.4. *Assume that the conditions of Lemma* 4.2 *hold and that* $x(t)$ *is the solution of* (4.6). *Then for* $t_k\le t<t_{k+1}$,

$$
\|x(t)\|\le \|x_{t_0}\|M_0\prod_{1\le j\le k}M_j\max\left\{\sup_{t_j-\tau\le t<t_j} e^{\int_{t_j}^{t}\alpha(s)\,ds},\ \|D_j\|\right\}e^{\int_{t_0}^{t}\alpha(s)\,ds},
$$

where $\alpha(t)=MlL(s)/(1-\tilde{C}l)-\gamma(s)$, $M_k=M(1+\|C(t_k)\|)/(1-\tilde{C}l)$, $L(t)=\|A(t)\|\|C(t)\|+\|C'(t)\|+\|B(t)\|$ *and* $l\ge\int_{t-\tau}^{t}\gamma(s)\,ds$.

PROOF. The proof is similar to that of Theorem 4.3, and we therefore omit it. □

It is easy to obtain the following result from Theorems 4.3 and 4.4.

COROLLARY 4.5. *Assume that the conditions of Theorems* 4.3 *or* 4.4 *hold. Then*

(1) $\limsup_{t\to\infty}\left(e^{\int_{t_0}^{t}\alpha(s)\,ds}\prod_{j\in\Delta(t)}M_j\max\left\{\|D_j\|,\sup_{t_j-\tau\le t\le t_j}e^{\int_{t_j}^{t}\alpha(s)\,ds}\right\}\right)<\infty$ *implies system* (4.6) *is stable*;

https://doi.org/10.1017/S1446181100009998 Published online by Cambridge University Press

(2) $\limsup_{t\to\infty}\left(e^{\int_{t_0}^t \alpha(s)\,ds}\prod_{j\in\Delta(t)} M_j \max\left\{\|D_j\|, \sup_{t_j-\tau\le t\le t_j} e^{\int_{t_j}^t \alpha(s)\,ds}\right\}\right) = 0$ *implies system* (4.6) *is asymptotically stable,*

where M_j and $\alpha(t)$ are as in Theorem 4.3 *or Theorem* 4.4 *respectively according to which conditions hold.*

COROLLARY 4.6. *Assume that the conditions of Theorems* 4.3 *or* 4.4 *hold and* $\alpha(t) = -\alpha =$ constant. *Furthermore, there exist a constant $\eta > \tau$ and $M > 0$ such that $t_k - t_{k-1} = \eta$ and $M_j \max\{e^{\alpha\tau}, \|D_j\|\} \le M$. Then*

(1) *$M \le e^{\alpha\eta}$ implies system* (4.6) *is stable;*
(2) *$M < e^{\alpha\eta}$ implies system* (4.6) *is exponentially stable.*

Next, we will consider a large-scale impulsive neutral system (4.1). Assume that $R_{ii}(t,s)$ satisfies $R_{ii}(t,t) = E$ and

$$\frac{\partial R_{ii}(t,s)}{\partial t} = A_{ii}(t)R_{ii}(t,s), \quad t \in J\backslash I,\ i = 1,\dots,r.$$

THEOREM 4.7. *Assume that*

(1) *there exist a scalar function $\alpha \in C[J,R]$ and constant $M_i \ge 1$ such that*

$$\|R_{ii}(t,t_0)\| \le M_i e^{-\int_{t_0}^t \alpha(\xi)\,d\xi};$$

(2) *$\|C_{ij}(t)\|$ are nonincreasing functions;*

(3) *there exist $N > 0$, $l > 0$ and a scalar function $\beta \in C[I,R]$ such that $l \ge \sup_{t\ge t_0} e^{\int_{t-\tau}^t \alpha(s)\,ds}$ and the fundamental matrix solution $Q(t,t_0)$ of the ordinary differential equations*

$$\frac{d\eta_i}{dt} = \sum_{j=1}^r \nu_{ij}(t)\eta_j, \quad i = 1,\dots,r,$$

satisfies $\|Q(t,t_0)\| \le Ne^{\int_{t_0}^t \beta(s)\,ds}$, where

$$\nu_{ij} = (1-\delta_{ij})M_i\|A_{ij}(s)\| + M_i l L(s) + \frac{l}{\tau}\|C_{ij}(s)\| \quad \textit{and}$$
$$L(t) = \|B_{ij}(t)\| + \|C'_{ij}(t)\| + \|A_{ii}(t)\|\|C_{ij}(t)\|.$$

If $M = \max_{1\le i\le r} M_i$, $\tilde{M}_k = N[M + (M+l)\|C_{t_k}\|]$, then

(1) *system* (4.1) *is stable if*

$$\limsup_{t\to\infty} \frac{\prod_{j\in\Delta(t)} \tilde{M}_j \max\left\{\|D_j\|, \sup_{t_j-\tau\le t<t_j} e^{\int_{t_j}^t (\beta(s)-\alpha(s))\,ds}\right\}}{e^{\int_{t_0}^t (\alpha(s)-\beta(s))\,ds}} < \infty;$$

(2) *system* (4.1) *is asymptotically stable if*

$$\limsup_{t\to\infty} \frac{\prod_{j\in\Delta(t)} \tilde{M}_j \max\left\{\|D_j\|, \sup_{t_j-\tau\le t<t_j} e^{\int_{t_j}^{t}(\beta(s)-\alpha(s))\,ds}\right\}}{e^{\int_{t_0}^{t}(\alpha(s)-\beta(s))\,ds}} = 0;$$

(3) *if there exists a positive number* μ *such that*

$$\limsup_{t\to\infty} \frac{\prod_{j\in\Delta(t)} \tilde{M}_j \max\left\{\|D_j\|, \sup_{t_j-\tau\le t<t_j} e^{\int_{t_j}^{t}(\beta(s)-\alpha(s))\,ds}\right\}}{e^{\int_{t_0}^{t}(\alpha(s)-\beta(s)+\mu)\,ds}} = 0$$

then system (4.1) *is exponentially stable.*

PROOF. Using the method of variation of parameters, the solution of system (4.1) can be written as

$$\begin{aligned} x_i(t) &= R_{ii}(t,t_0)x_i(t_0) + \int_{t_0}^{t} R_{ii}(t,s)\sum_{j\ne i} A_{ij}(s)x_j(s)\,ds \\ &\quad + \int_{t_0}^{t} R_{ii}(t,s)\sum_{j=1}^{r} B_{ij}(s)x_j(s-\tau)\,ds + \int_{t_0}^{t} R_{ii}(t,s)\sum_{j=1}^{r} C_{ij}(s)x_j'(s-\tau)\,ds. \end{aligned}$$

Since

$$\begin{aligned} &\int_{t_0}^{t} R_{ii}(t,s)\sum_{j=1}^{r} C_{ij}(s)x_j'(s-\tau)\,ds \\ &\quad = \sum_{j=1}^{r} C_{ij}(t)x_j(t-\tau) - R_{ii}(t,t_0)\sum_{j=1}^{r} C_{ij}(t_0)x_j(t_0-\tau) \\ &\qquad - \int_{t_0}^{t} R_{ii}(t,s)\sum_{j=1}^{r} C_{ij}'(s)x_j(s-\tau)\,ds \\ &\qquad + \int_{t_0}^{t} R_{ii}(t,s)A_{ii}(s)\sum_{j=1}^{r} C_{ij}(s)x_j(s-\tau)\,ds, \end{aligned}$$

then

$$\begin{aligned} x_i(t) &= R_{ii}(t,t_0)x_i(t_0) + \int_{t_0}^{t} R_{ii}(t,s)\sum_{j\ne i} A_{ij}(s)x_j(s)\,ds \\ &\quad + \int_{t_0}^{t} R_{ii}(t,s)\sum_{j=1}^{r} B_{ij}(s)x_j(s-\tau)\,ds \end{aligned}$$

https://doi.org/10.1017/S1446181100009998 Published online by Cambridge University Press

$$
\begin{aligned}
&+\sum_{j=1}^{r} C_{ij}(t)x_j(t-\tau) - R_{ii}(t,t_0)\sum_{j=1}^{r} C_{ij}(t_0)x_j(t_0-\tau)\\
&-\int_{t_0}^{t} R_{ii}(t,s)\sum_{j=1}^{r} C'_{ij}(s)x_j(s-\tau)\,ds\\
&+\int_{t_0}^{t} R_{ii}(t,s)A_{ii}(s)\sum_{j=1}^{r} C_{ij}(s)x_j(s-\tau)\,ds.
\end{aligned}
$$

Thus we have

$$
\begin{aligned}
\|x_i(t)\| \le{}& M_i e^{-\int_{t_0}^{t}\alpha(s)\,ds}\|x_{it_0}\|\\
&+\sum_{j=1}^{r}\int_{t_0}^{t}(1-\delta_{ij})M_i\|A_{ij}(s)\|e^{-\int_s^t\alpha(\xi)\,d\xi}\|\dot{x}_j(s)\|\,ds\\
&+\sum_{j=1}^{r}\int_{t_0}^{t} M_i\|B_{ij}(s)\|e^{-\int_s^t\alpha(\xi)\,d\xi}\|x_j(s-\tau)\|\,ds\\
&+\sum_{j=1}^{r}\|C_{ij}(t)\|\|x_j(t-\tau)\| + M_i e^{-\int_{t_0}^{t}\alpha(s)\,ds}\sum_{j=1}^{r}\|C_{ij}(t_0)\|\|x_j(t_0-\tau)\|\\
&+\int_{t_0}^{t} M_i e^{-\int_s^t\alpha(\xi)\,d\xi}\sum_{j=1}^{r}\|C'_{ij}(s)\|x_j(s-\tau)\|\,ds\\
&+\int_{t_0}^{t} M_i e^{-\int_s^t\alpha(\xi)\,d\xi}\|A_{ii}(s)\|\sum_{j=1}^{r}\|C_{ij}(s)\|\|x_j(s-\tau)\|\,ds
\end{aligned}
$$

and

$$
\begin{aligned}
\|x_i(t)\|e^{\int_{t_0}^{t}\alpha(s)\,ds} \le{}& M_i\|x_{it_0}\| + \sum_{j=1}^{r}\int_{t_0}^{t}(1-\delta_{ij})M_i\|A_{ij}(s)\|e^{\int_{t_0}^{s}\alpha(\xi)\,d\xi}\|x_j(s)\|\,ds\\
&+\sum_{j=1}^{r}\int_{t_0}^{t} M_i\|B_{ij}(s)\|e^{\int_{t_0}^{s}\alpha(\xi)\,d\xi}\|x_j(s-\tau)\|\,ds\\
&+e^{\int_{t_0}^{t}\alpha(s)\,ds}\sum_{j=1}^{r}\|C_{ij}(t)\|\|x_j(t-\tau)\| + M_i\sum_{j=1}^{r}\|C_{ij}(t_0)\|\|x_{jt_0}\|\\
&+\int_{t_0}^{t} M_i e^{\int_{t_0}^{s}\alpha(\xi)\,d\xi}\sum_{j=1}^{r}\|C'_{ij}(s)\|x_j(s-\tau)\|\,ds\\
&+\int_{t_0}^{t} M_i e^{\int_{t_0}^{s}\alpha(\xi)\,d\xi}\|A_{ii}(s)\|\sum_{j=1}^{r}\|C_{ij}(s)\|\|x_j(s-\tau)\|\,ds
\end{aligned}
$$

https://doi.org/10.1017/S1446181100009998 Published online by Cambridge University Press

$$
\begin{aligned}
&= M_i \|x_{it_0}\| + M_i \sum_{j=1}^{r} \|C_{ij}(t_0)\| \|x_{jt_0}\| \\
&\quad + \sum_{j=1}^{r} \int_{t_0}^{t} (1-\delta_{ij}) M_i \|A_{ij}(s)\| e^{\int_{t_0}^{s} \alpha(\xi)\,d\xi} \|x_j(s)\|\,ds \\
&\quad + \sum_{j=1}^{r} \int_{t_0}^{t} M_i e^{\int_{t_0}^{s} \alpha(\xi)\,d\xi} \big[\|B_{ij}(s)\| + \|C'_{ij}(s)\| \\
&\quad + \|A_{ii}(s)\| \|C_{ij}(s)\|\big] \|x_j(s-\tau)\|\,ds \\
&\quad + e^{\int_{t_0}^{t} \alpha(s)\,ds} \sum_{j=1}^{r} \|C_{ij}(t)\| \|x_j(t-\tau)\|. \qquad (4.7)
\end{aligned}
$$

Let $y_i(t) = \max_{t_0-\tau \le \xi \le t} \|x_i(\xi)\| e^{\int_{t_0}^{\xi} \alpha(s)\,ds}$, then $y_i(t)$ is a nondecreasing function, $i = 1, \dots, r$. As $y_i(t)$ is nondecreasing, $\|C_{ij}(t)\|$ is nonincreasing, and because $l \ge \sup_{t \ge t_0} e^{\int_{t-\tau}^{t} \alpha(s)\,ds}$, we have

$$
\begin{aligned}
&\sum_{j=1}^{r} \|C_{ij}(t)\| \|x_j(t-\tau)\| e^{\int_{t_0}^{t} \alpha(s)\,ds} \\
&\quad \le \sum_{j=1}^{r} \|C_{ij}(t)\| y_j(t-\tau) e^{\int_{t-\tau}^{t} \alpha(s)\,ds} \\
&\quad \le \sum_{j=1}^{r} \frac{l}{\tau} \int_{t-\tau}^{t} \|C_{ij}(t)\| y_j(t-\tau)\,ds \le \sum_{j=1}^{r} \frac{l}{\tau} \int_{t-\tau}^{t} \|C_{ij}(s)\| y_j(s)\,ds \\
&\quad \le \sum_{j=1}^{r} \frac{l}{\tau} \int_{t_0}^{t} \|C_{ij}(s)\| y_j(s)\,ds + \sum_{j=1}^{r} \frac{l}{\tau} \int_{t_0-\tau}^{t_0} \|C_{ij}(s)\| y_j(s)\,ds \\
&\quad \le \sum_{j=1}^{r} \frac{l}{\tau} \int_{t_0}^{t} \|C_{ij}(s)\| y_j(s)\,ds + \sum_{j=1}^{r} l \|C_{ij}(t_0-\tau)\| \|x_{jt_0}\|.
\end{aligned}
$$

Thus

$$
\begin{aligned}
&\|x_i(t)\| e^{\int_{t_0}^{t} \alpha(s)\,ds} \\
&\quad \le M_i \|x_{it_0}\| + M_i \sum_{j=1}^{r} \|C_{ij}(t_0)\| \|x_{jt_0}\| \\
&\qquad + \sum_{j=1}^{r} \int_{t_0}^{t} (1-\delta_{ij}) M_i \|A_{ij}(s)\| y_j(s)\,ds + \sum_{j=1}^{r} \int_{t_0}^{t} M_i l L(s) \|y_j(s)\,ds \\
&\qquad + \sum_{j=1}^{r} \frac{l}{\tau} \int_{t_0}^{t} \|C_{ij}(s)\| y_j(s)\,ds + \sum_{j=1}^{r} l \|C_{ij}(t_0-\tau)\| \|x_{jt_0}\|
\end{aligned}
$$

https://doi.org/10.1017/S1446181100009998 Published online by Cambridge University Press

$$= M_i\|x_{it_0}\| + \sum_{j=1}^{r}\left[M_i\|C_{ij}(t_0)\| + l\|C_{ij}(t_0-\tau)\|)\right]\|x_{jt_0}\|$$
$$+ \sum_{j=1}^{r}\int_{t_0}^{t}\left[(1-\delta_{ij})M_i\|A_{ij}(s)\| + M_i l L(s) + \frac{l}{\tau}\|C_{ij}(s)\|\right]y_j(s)\,ds.$$

The right-hand side of the last inequality is increasing and hence

$$y_i(t) \le M_i\|x_{it_0}\| + \sum_{j=1}^{r}\left[M_i\|C_{ij}(t_0)\| + l\|C_{ij}(t_0-\tau)\|\right]\|x_{jt_0}\|$$
$$+ \sum_{j=1}^{r}\int_{t_0}^{t}\left[(1-\delta_{ij})M_i\|A_{ij}(s)\| + M_i l L(s) + \frac{l}{\tau}\|C_{ij}(s)\|\right]y_j(s)\,ds$$
$$\le M_i\|x_{it_0}\| + (M_i + l)\sum_{j=1}^{r}\|C_{ijt_0}\|\|x_{jt_0}\|$$
$$+ \sum_{j=1}^{r}\int_{t_0}^{t}\left[(1-\delta_{ij})M_i\|A_{ij}(s)\| + M_i l L(s) + \frac{l}{\tau}\|C_{ij}(s)\|\right]y_j(s)\,ds$$
$$\le M\|x_{it_0}\| + (M + l)\sum_{j=1}^{r}\|C_{ijt_0}\|\|x_{jt_0}\|$$
$$+ \sum_{j=1}^{r}\int_{t_0}^{t}\left[(1-\delta_{ij})M_i\|A_{ij}(s)\| + M_i l L(s) + \frac{l}{\tau}\|C_{ij}(s)\|\right]y_j(s)\,ds$$
$$= \overline{M}_i + \sum_{j=1}^{r}\int_{t_0}^{t}\nu_{ij}(s)y_j(s)\,ds.$$

Let $P_i(t) = \overline{M}_i + \sum_{j=1}^{r}\int_{t_0}^{t}\nu_{ij}(s)y_j(s)\,ds$. Then $P_i(t) \ge y_i(t)$, $P_i(t_0) = \overline{M}_i$ and

$$P_i'(t) = \sum_{j=1}^{r}\nu_{ij}(t)y_j(t) \le \sum_{j=1}^{r}\nu_{ij}(t)P_j(t).$$

Consider the system

$$\begin{cases} d\xi_i/dt = \sum_{j=1}^{r}\nu_{ij}(s)\xi_j(s), & t\in[t_0,t_1),\\ \xi(t_0) = \overline{M}, \end{cases} \tag{4.8}$$

where $\xi(s) = \mathrm{col}(\xi_1(s),\dots,\xi_r(s))$ and $\overline{M} = \mathrm{col}(\overline{M}_1,\dots,\overline{M}_r)$. It is obvious that

$$\xi_i(t) \ge P_i(t) \ge y_i(t) \ge \|x_i(t)\|e^{\int_{t_0}^{t}\alpha(s)\,ds}, \quad t\in[t_0-\tau,t_1)$$

and thus $\|\xi(t)\| \geq \|x(t)\|e^{\int_{t_0}^{t}\alpha(s)\,ds}$. Furthermore, from condition (3), it follows that the solution $\xi(t)$ of (4.8) satisfies $\|\xi(t)\| \leq N\|\xi(t_0)\|e^{\int_{t_0}^{t}\beta(s)\,ds}$ on $[t_0, t_1)$ and

$$\begin{aligned}\|\xi(t_0)\| = \|\overline{M}\| = \sum_{i=1}^{r}\overline{M}_i &= \sum_{i=1}^{r}\left[M\|x_{it_0}\| + (M+l)\sum_{j=1}^{r}\|C_{ijt_0}\|\|x_{jt_0}\|\right] \\ &= M\|x_{t_0}\| + (M+l)\sum_{i=1}^{r}\left[\sum_{j=1}^{r}\|C_{ijt_0}\|\right]\|x_{jt_0}\| \\ &\leq M\|x_{t_0}\| + (M+l)\|C_{t_0}\|\|x_{t_0}\|.\end{aligned}$$

Thus

$$\begin{aligned}&\|\xi(t)\| \leq N\big[M + (M+l)\|C_{t_0}\|\big]\|x_{t_0}\|e^{\int_{t_0}^{t}\beta(s)\,ds} \quad \text{and} \\ &\|x(t)\| \leq \|\xi(t)\|e^{-\int_{t_0}^{t}\alpha(s)\,ds} \leq N\big[M + (M+l)\|C_{t_0}\|\big]\|x_{t_0}\|e^{\int_{t_0}^{t}[\beta(s)-\alpha(s)]ds}.\end{aligned}$$

Using the same argument, it is easy to get that

$$x(t) \leq N\big[M + (M+l)\|C_{t_k}\|\big]\|x_{t_k}\|e^{\int_{t_k}^{t}[\beta(s)-\alpha(s)]\,ds}, \quad t \in [t_{k-1}, t_k).$$

Furthermore,

$$\begin{aligned}\|x_{t_k}\| &= \sup_{t_k-\tau\leq t\leq t_k}\|x(t)\| = \max\left\{\|x(t_k)\|, \sup_{t_k-\tau\leq t<t_k}\|x(t)\|\right\} \\ &\leq \max\left\{\|D_k\|\|x(t_k-0)\|, \sup_{t_k-\tau\leq t<t_k}\tilde{M}_{k-1}\|x(t)\|\right\} \\ &\leq \max\left\{\tilde{M}_{k-1}\|D_k\|\|x_{t_{k-1}}\|e^{\int_{t_{k-1}}^{t_k}(\beta(s)-\alpha(s))\,ds}, \sup_{t_k-\tau\leq t<t_k}\tilde{M}_{k-1}\|x_{t_{k-1}}\|e^{\int_{t_{k-1}}^{t}(\beta(s)-\alpha(s))\,ds}\right\} \\ &\leq \max\left\{\|D_k\|, \sup_{t_k-\tau\leq t<t_k}e^{\int_{t_k}^{t}(\beta(s)-\alpha(s))\,ds}\right\}\tilde{M}_{k-1}\|x_{t_{k-1}}\|e^{\int_{t_{k-1}}^{t_k}(\beta(s)-\alpha(s))\,ds}.\end{aligned}$$

Thus when $t \in [t_k, t_{k+1})$,

$$\begin{aligned}\|x(t)\| &\leq \tilde{M}_k\|x_{t_k}\|\,e^{\int_{t_k}^{t}[\beta(s)-\alpha(s)]ds} \\ &\leq \tilde{M}_k\tilde{M}_{k-1}\max\left\{\|D_k\|, \sup_{t_k-\tau\leq t<t_k}e^{\int_{t_k}^{t}(\beta(s)-\alpha(s))\,ds}\right\}\|x_{t_{k-1}}\|e^{\int_{t_{k-1}}^{t}(\beta(s)-\alpha(s))\,ds} \\ &\leq \cdots \leq \tilde{M}_0\prod_{j\in\Delta(t)}\tilde{M}_j\max\left\{\|D_j\|, \sup_{t_j-\tau\leq t<t_j}e^{\int_{t_j}^{t}(\beta(s)-\alpha(s))\,ds}\right\}\|x_{t_0}\|e^{\int_{t_0}^{t}(\beta(s)-\alpha(s))\,ds}.\end{aligned}$$

The inequality implies all of the results of Theorem 4.7. The proof is complete. □

https://doi.org/10.1017/S1446181100009998 Published online by Cambridge University Press

THEOREM 4.8. *Assume that conditions* (1)–(2) *of Theorem* 4.7 *hold. Let*

$$\tilde{C}(t) = (\|C_{ij}(t)\|), \quad \tilde{A}(t) = ((1-\delta_{ij})\|A_{ij(t)}\|), \quad L(t) = (L_{ij}(t)),$$

$$M = \max_{1\le i\le r} M_i, \quad \tilde{M}_k = \frac{M(1+\|\tilde{C}(t_k)\|)}{1-\|\tilde{C}(t_k)\|}, \quad \beta(t) = \frac{M(\|\tilde{A}(t)\| + l\|L(t)\|)}{1-\|\tilde{C}(t_0)\|},$$

$$l = \sup_{t\ge t_0} e^{\int_{t-\tau}^{t}\alpha(s)\,ds} < +\infty \quad \text{and} \quad L_{ij}(t) = \|B_{ij}(t)\| + \|C'_{ij}(t)\| + \|A_{ii}(t)\|\|C_{ij}(t)\|.$$

Then

(1) *system* (4.1) *is stable if*

$$\limsup_{t\to\infty} \frac{\prod_{j\in\Delta(t)} \tilde{M}_j \max\left\{\|D_j\|, \sup_{t_j-\tau\le t<t_j} e^{\int_{t_j}^{t}(\beta(s)-\alpha(s))\,ds}\right\}}{e^{\int_{t_0}^{t}(\alpha(s)-\beta(s))\,ds}} < \infty;$$

(2) *system* (4.1) *is asymptotically stable if*

$$\limsup_{t\to\infty} \frac{\prod_{j\in\Delta(t)} \tilde{M}_j \max\left\{\|D_j\|, \sup_{t_j-\tau\le t<t_j} e^{\int_{t_j}^{t}(\beta(s)-\alpha(s))\,ds}\right\}}{e^{\int_{t_0}^{t}(\alpha(s)-\beta(s))\,ds}} = 0;$$

(3) *if there exists a positive number* μ *such that*

$$\limsup_{t\to\infty} \frac{\prod_{j\in\Delta(t)} \tilde{M}_j \max\left\{\|D_j\|, \sup_{t_j-\tau\le t<t_j} e^{\int_{t_j}^{t}(\beta(s)-\alpha(s))\,ds}\right\}}{e^{\int_{t_0}^{t}(\alpha(s)-\beta(s)+\mu)\,ds}} = 0$$

then system (4.1) *is exponentially stable.*

PROOF. By the first part of the proof of Theorem 4.7, we have (4.7). Let

$$y_i(t) = \max_{t_0-\tau\le\xi\le t} \|x_i(\xi)\| e^{\int_{t_0}^{\xi}\alpha(s)\,ds},$$

then $y_i(t)$ is a nondecreasing function, $i = 1, \dots, r$. As $y_i(t)$ is nondecreasing, $\|C_{ij}(t)\|$ is nonincreasing, and because $l = \sup_{t\ge t_0} e^{\int_{t-\tau}^{t}\alpha(s)\,ds} < +\infty$, we have

$$\begin{aligned}\sum_{j=1}^{r} \|C_{ij}(t)\|\|x_j(t-\tau)\| e^{\int_{t_0}^{t}\alpha(s)\,ds} &\le \sum_{j=1}^{r} \|C_{ij}(t)\| y_j(t-\tau) e^{\int_{t-\tau}^{t}\alpha(s)\,ds} \\ &\le \sum_{j=1}^{r} \|C_{ij}(t)\| l y_j(t).\end{aligned}$$

Thus from (4.7), we can get the following estimate:

$$\|x_i(t)\| e^{\int_{t_0}^{t}\alpha(s)\,ds} \le M_i\|x_{it_0}\| + M_i \sum_{j=1}^{r} \|C_{ij}(t_0)\|\|x_{jt_0}\| + \sum_{j=1}^{r} \|C_{ij}(t)\| l y_j(t)$$

https://doi.org/10.1017/S1446181100009998 Published online by Cambridge University Press

$$+\sum_{j=1}^{r}\int_{t_0}^{t}(1-\delta_{ij})M_i\|A_{ij}(s)\|y_j(s)\,ds$$
$$+\sum_{j=1}^{r}\int_{t_0}^{t}M_i l L_{ij}(s)y_j(s)\,ds$$

since $y(t)$ is increasing and $\|C_{ij}(t)\|$ is decreasing. It is easy to see that the right-hand side of the last inequality is increasing and hence

$$y_i(t)\le M_i\|x_{it_0}\|+M_i\sum_{j=1}^{r}\|C_{ij}(t_0)\|\|x_{jt_0}\|+\sum_{j=1}^{r}\|C_{ij}(t)\|l y_j(t)$$
$$+\sum_{j=1}^{r}\int_{t_0}^{t}\big[(1-\delta_{ij})M_i\|A_{ij}(s)\|+M_i l L_{ij}(s)\big]y_j(s)\,ds.$$

Thus

$$\sum_{i=1}^{r}y_i(t)\le\sum_{i=1}^{r}M_i\|x_{it_0}\|+\sum_{i=1}^{r}M_i\sum_{j=1}^{r}\|C_{ij}(t_0)\|\|x_{jt_0}\|+\sum_{i=1}^{r}\sum_{j=1}^{r}\|C_{ij}(t)\|l y_j(t)$$
$$+\sum_{i=1}^{r}\sum_{j=1}^{r}\int_{t_0}^{t}\big[(1-\delta_{ij})M_i\|A_{ij}(s)\|+M_i l L_{ij}(s)\big]y_j(s)\,ds$$
$$\le M\sum_{i=1}^{r}\|x_{it_0}\|+M\sum_{j=1}^{r}\left[\sum_{i=1}^{r}\|C_{ij}(t_0)\|\right]\|x_{jt_0}\|$$
$$+\sum_{j=1}^{r}\left[\sum_{i=1}^{r}\|C_{ij}(t)\|\right]l y_j(t)$$
$$+\int_{t_0}^{t}M\sum_{j=1}^{r}\left[\sum_{i=1}^{r}(1-\delta_{ij})\|A_{ij}(s)\|+\sum_{i=1}^{r}lL_{ij}(s)\right]y_j(s)\,ds$$
$$\le M\|x_{t_0}\|+M\|\tilde{C}(t_0)\|\|x_{t_0}\|+\int_{t_0}^{t}M\big[\|\tilde{A}(s)\|+l\|L(s)\|\big]\|y(s)\|\,ds$$
$$+l\|\tilde{C}(t)\|\|y(t)\|.$$

Since $\|\tilde{C}(t)\|$ is nondecreasing,

$$\|y(t)\|\le M\|x_{t_0}\|(1+\|\tilde{C}(t_0)\|)+\int_{t_0}^{t}M\big[\|\tilde{A}(s)\|+l\|L(s)\|\big]\|y(s)\|\,ds$$
$$+l\|\tilde{C}(t_0)\|\|y(t)\|$$

https://doi.org/10.1017/S1446181100009998 Published online by Cambridge University Press

and thus

$$\begin{aligned}\|y(t)\| &\le \frac{M(1+\|\tilde{C}(t_0)\|)}{1-\|\tilde{C}(t_0)\|}\|x_{t_0}\| + \frac{M}{1-\|\tilde{C}(t_0)\|}\int_{t_0}^{t}\left[\|\tilde{A}(s)\| + l\|L(s)\|\right]\|y(s)\|\,ds \\ &= \tilde{M}_0\|x_{t_0}\| + \int_{t_0}^{t}\beta(s)\|y(s)\|\,ds.\end{aligned}$$

The Gronwall-Bellman inequality implies $\|y(t)\| \le \tilde{M}_0\|x_{t_0}\|e^{\int_{t_0}^{t}\beta(s)\,ds}$. In view of the relations

$$\|y(t)\| = \sum_{i=1}^{r}\sup_{s\le t}\|x_i(s)\|e^{\int_{t_0}^{s}\alpha(\eta)d\eta} \ge \sum_{i=1}^{r}\|x_i(t)\|e^{\int_{t_0}^{t}\alpha(s)\,ds},$$

we obtain

$$\|x(t)\| \le \tilde{M}_0\|x_{t_0}\|e^{\int_{t_0}^{t}[\beta(s)-\alpha(s)]\,ds}, \quad t\in[t_0,t_1).$$

Using the same argument, it is easy to get that

$$\|x(t)\| \le \tilde{M}_k\|x_{t_k}\|e^{\int_{t_k}^{t}[\beta(s)-\alpha(s)]\,ds}, \quad t\in[t_k,t_{k+1}).$$

Furthermore,

$$\|x_{t_k}\| \le \tilde{M}_{k-1}\max\left\{\|D_k\|,\ \sup_{t_k-\tau\le t<t_k} e^{\int_{t_k}^{t}(\beta(s)-\alpha(s))\,ds}\right\}\|x_{t_{k-1}}\|e^{\int_{t_{k-1}}^{t_k}(\beta(s)-\alpha(s))\,ds}.$$

Thus when $t\in[t_k,t_{k+1})$,

$$\begin{aligned}\|x(t)\| &\le \tilde{M}_k\|x_{t_k}\|\,e^{\int_{t_k}^{t}[\beta(s)-\alpha(s)]\,ds} \\ &\le \tilde{M}_k\tilde{M}_{k-1}\max\left\{\|D_k\|,\ \sup_{t_k-\tau\le t<t_k} e^{\int_{t_k}^{t}(\beta(s)-\alpha(s))\,ds}\right\}\|x_{t_{k-1}}\|e^{\int_{t_{k-1}}^{t}(\beta(s)-\alpha(s))\,ds} \\ &\le \cdots \le \tilde{M}_0\prod_{j\in\Delta(t)}\tilde{M}_j\max\left\{\|D_j\|,\ \sup_{t_j-\tau\le t<t_j} e^{\int_{t_j}^{t}(\beta(s)-\alpha(s))\,ds}\right\}\|x_{t_0}\|e^{\int_{t_0}^{t}(\beta(s)-\alpha(s))\,ds}.\end{aligned}$$

The inequality implies all of the results of Theorem 4.8. The proof is complete. □

THEOREM 4.9. *Assume that*

(1) *there exist* $l>0$, $M>1$ *and* $\alpha\in C[R,R]$ *such that* $e^{\int_{t-\tau}^{t}\alpha(s)\,ds}\le l$ *and, for* $i=1,\dots,r$, $\|R_{ii}(t,s)\|\le Me^{\int_{s}^{t}\alpha(\eta)d\eta}$;

(2) *there exist* $c_{ij}\ge 0$ *such that* $\|C_{ij}(t)\|\le c_{ij}$, $i,j=1,\dots,r$;

(3) *we denote by* $\tilde{C}=(c_{ij})$, $\tilde{A}(t)=(\|A_{ij}(t)\|)$, $L(t)=(L_{ij}(t))$, $L_{ij}(t)=\|B_{ij}(t)\|+\|C'_{ij}(t)\|+\|A_{ii}(t)\|\|C_{ij}(t)\|$, $\tilde{M}=M(1+\|\tilde{C}\|)/(1-\|\tilde{C}\|)$ *and* $\beta(t)=M(\|\tilde{A}(t)\|+l\|L(t)\|)/(1-\|\tilde{C}\|)$.

https://doi.org/10.1017/S1446181100009998 Published online by Cambridge University Press

Then

(1) *system* (4.1) *is stable if*

$$\limsup_{t\to\infty} \frac{\prod_{j\in\Delta(t)} \tilde{M}^k \max\left\{\|D_j\|, \sup_{t_j-\tau\le t<t_j} e^{\int_{t_j}^{t}(\beta(s)-\alpha(s))\,ds}\right\}}{e^{\int_{t_0}^{t}(\alpha(s)-\beta(s))\,ds}} < \infty;$$

(2) *system* (4.1) *is asymptotically stable if*

$$\limsup_{t\to\infty} \frac{\prod_{j\in\Delta(t)} \tilde{M}^k \max\left\{\|D_j\|, \sup_{t_j-\tau\le t<t_j} e^{\int_{t_j}^{t}(\beta(s)-\alpha(s))\,ds}\right\}}{e^{\int_{t_0}^{t}(\alpha(s)-\beta(s))\,ds}} = 0;$$

(3) *if there exists a positive number* μ *such that*

$$\limsup_{t\to\infty} \frac{\prod_{j\in\Delta(t)} \tilde{M}^k \max\left\{\|D_j\|, \sup_{t_j-\tau\le t<t_j} e^{\int_{t_j}^{t}(\beta(s)-\alpha(s))\,ds}\right\}}{e^{\int_{t_0}^{t}(\alpha(s)-\beta(s)+\mu)\,ds}} = 0,$$

then system (4.1) *is exponentially stable.*

PROOF. The proof is similar to that of Theorem 4.8 and thus we omit it here. □

THEOREM 4.10. *Assume that*

(S1) *there exist scalar functions* $\alpha_i \in C[I, R^+]$ *and constants* c_i, $M_i \ge 1$, *such that*

$$\|R_{ii}(t,s)\| \le M_i e^{-\int_s^t \alpha_i(\xi)\,d\xi} \quad \textit{and} \quad \sum_{j=1}^{r} \|C_{ij}(t)\| \le c_i e^{-\int_{t-\tau}^{t} \alpha_i(\xi)\,d\xi};$$

(S2)
$$\begin{aligned} \bar{a}_{ij}(t) &= -\delta_{ij}\alpha_i(t) + (1-\delta_{ij})M_i\|A_{ij}(t)\|, \\ \bar{b}_{ij}(t) &= M_i\|B_{ij}(t)\| + M_i\|A_{ii}(t)\|\|C_{ij}(t)\| + M_i\|C'_{ij}(t)\|, \end{aligned}$$

then the stability properties of the system

$$\begin{cases} \xi_i'(t) = \sum_{j=1}^{r} \bar{a}_{ij}(t)\xi_j(t) + \sum_{j=1}^{r} \bar{b}_{ij}(t)\xi_j(t-\tau), & t\in J\backslash I,\ j=1,\dots,r, \\ \xi_i(t_k) = \sum_{j=1}^{r} \|D_{ij}(t_k)\|\xi_j(t_k-0), & t_k \in I, \end{cases}$$

imply the corresponding stability properties of (4.1).

PROOF. Using the same argument of Theorem 4.7, for $t\in J_{k+1}$, we have

$$\|x_i(t)\| \le M_i e^{-\int_{t_k}^{t}\alpha_i(s)\,ds}\|x_{t_k}\| + \sum_{j=1}^{r}\int_{t_k}^{t}(1-\delta_{ij})M_i e^{-\int_{\xi}^{t}\alpha_i(s)\,ds}\|A_{ij}(\xi)\|\|x_j(\xi)\|\,d\xi$$

https://doi.org/10.1017/S1446181100009998 Published online by Cambridge University Press

$$
\begin{aligned}
&+\sum_{j=1}^{r}\int_{t_k}^{t} M_i e^{-\int_\xi^t \alpha_i(s)\,ds}\|B_{ij}(\xi)\|\|x_j(\xi-\tau)\|\,d\xi \\
&+ M_i e^{-\int_{t_k}^t \alpha_i(s)\,ds} c_i\|x_{t_k}\| + c_i e^{-\int_{t_k}^t \alpha_i(s)\,ds}\|x_{t_k}\| \\
&+\sum_{j=1}^{r}\int_{t_k}^{t} M_i e^{-\int_\xi^t \alpha_i(s)\,ds}\|A_{ii}(\xi)\|\|C_{ij}(\xi)\|\|x_j(\xi-\tau)\|\,d\xi \\
&+\sum_{j=1}^{r}\int_{t_k}^{t} M_i e^{-\int_\xi^t \alpha_i(s)\,ds}\|C'_{ij}(\xi)\|\|x_j(\xi-\tau)\|\,d\xi \\
=:\ & P_i(t).
\end{aligned}
$$

Then $\|x_i(t)\| \le P_i(t), t \in J_{k+1}$. Let $P_i(t_k) = \sum_{j=1}^r \|D_{ij}(t_k)\| P_j(t_k - 0)$, $t_k \in I$, then $\|x_i(t)\| \le P_i(t), t \ge t_0$. Furthermore,

$$
\begin{aligned}
P'_i(t) &\le -\alpha_i(t)P_i(t) + \sum_{j=1}^{r}(1-\delta_{ij})M_i\|A_{ij}(t)\|\|x_j(t)\| \\
&\quad + \sum_{j=1}^{r} M_i\|B_{ij}(t)\|\|x_j(t-\tau)\| + \sum_{j=1}^{r} M_i\|A_{ii}(t)\|\|C_{ij}(t)\|\|x_j(t-\tau)\| \\
&\quad + \sum_{j=1}^{r} M_i\|C'_{ij}(t)\|\|x_j(t-\tau)\| \\
&\le -\alpha_i(t)P_i(t) + \sum_{j=1}^{r}(1-\delta_{ij})M_i\|A_{ij}(t)\|P_j(t) + \sum_{j=1}^{r} M_i\|B_{ij}(t)\|P_j(t-\tau) \\
&\quad + \sum_{j=1}^{r} M_i\|A_{ii}(t)\|\|C_{ij}(t)\|P_j(t-\tau) + \sum_{j=1}^{r} M_i\|C'_{ij}(t)\|P_j(t-\tau) \\
&= \sum_{j=1}^{r}\bar{a}_{ij}(t)P_j(t) + \sum_{j=1}^{r}\bar{b}_{ij}(t)P_j(t-\tau), \quad t \in J_{k+1}.
\end{aligned}
$$

Let $P(t) = \mathrm{col}(P_1(t), \dots, P_r(t))$, $P_i(t_k) = \sum_{j=1}^r \|D_{ij}(t_k)\| P_j(t_k - 0)$, then $\|x_i(t)\| \le P_i(t), t \in J$. Consider the comparison system

$$
\begin{cases}
P'_i(t) \le \sum_{j=1}^r \bar{a}_{ij}(t)P_j(t) + \sum_{j=1}^r \bar{b}_{ij}(t)P_j(t-\tau), & t_k \in J\backslash I, \\
P_i(t_k) = \sum_{j=1}^r \|D_{ij}(t_k)\|P_j(t_k - 0), & t_k \in I;
\end{cases}
$$

$$
\begin{cases}
\xi'_i(t) = \sum_{j=1}^r \bar{a}_{ij}(t)\xi_j(t) + \sum_{j=1}^r \bar{b}_{ij}(t)\xi_j(t-\tau), & t \in J\backslash I, \\
\xi_i(t_k) = \sum_{j=1}^r \|D_{ijk}\|\xi_j(t_k - 0), & t_k \in I.
\end{cases}
$$

Lemma 2.2 and $\|x(t)\| \le \|P(t)\|$ imply that $\|x(t)\| \le \|\xi(t)\|$, which implies that the conclusions of the theorem are true. The proof is complete. □

EXAMPLE 3. Consider the neutral impulsive system

$$\begin{cases} \dfrac{dx_1(t)}{dt} = A_{11}(t)x_1(t) + A_{12}(t)x_2(t) + B_{11}(t)x_1(t-1) \\ \qquad + C_{12}(t)x_2'(t-1), \\ \dfrac{dx_2(t)}{dt} = A_{21}(t)x_1(t) + A_{22}(t)x_2(t) + B_{22}(t)x_2(t-1) \\ \qquad + C_{21}(t)x_1'(t-1), \quad t \neq t_k, \\ x(t_k) = D_k x(t_k - 0), \quad k = 1, 2, \ldots, \end{cases} \tag{4.9}$$

where $t_0 \geq 0$, $x = (x_1, x_2)^T \in R^4$, $x_1, x_2 \in R^2$, $D_k = (D_{ijk}) \in R^{4\times4}$, $D_{ijk} \in R^{2\times2}$, $i, j = 1, 2$ and

$$A_{11}(t) = \begin{bmatrix} -4 & (1+t^2)^{-1} \\ -(1+t^2)^{-1} & -4 \end{bmatrix}, \qquad A_{12}(t) = \begin{bmatrix} (\cos^2 t)/2 & 1/6 \\ (\sin^2 t)/2 & (\cos^2 t)/4 \end{bmatrix},$$

$$A_{21}(t) = \begin{bmatrix} 1/4 & 0 \\ 1/14 & -1/3 \end{bmatrix}, \qquad A_{22}(t) = \begin{bmatrix} -4 - \cos^2 t & \sin^2 t \\ -\sin^2 t & -4 \end{bmatrix},$$

$$B_{11}(t) = \begin{bmatrix} (4e)^{-1} & [4e(1+t^2)]^{-1} \\ 0 & 0 \end{bmatrix}, \qquad B_{22}(t) = \begin{bmatrix} 0 & (4e^2)^{-1} \\ (4e)^{-1} & 0 \end{bmatrix},$$

$$C_{12}(t) = \begin{bmatrix} e^{-4(t+1)}/2 & 0 \\ 0 & e^{-4(t+1)}/2 \end{bmatrix}, \qquad C_{21}(t) = \begin{bmatrix} (4e^{4(t+1)})^{-1} & 0 \\ (9e^{3(t+1)})^{-1} & 0 \end{bmatrix},$$

$$C_{12}'(t) = \begin{bmatrix} -4e^{-4(t+1)} & 0 \\ 0 & -2e^{-4(t+1)} \end{bmatrix}, \qquad C_{21}'(t) = \begin{bmatrix} -(e^{4(t+1)})^{-1} & 0 \\ -(3e^{3(t+1)})^{-1} & 0 \end{bmatrix}.$$

Then the fundamental matrix solutions $R_{ii}(t, t_0)$ and $R_{22}(t, t_0)$ of systems $x_1'(t) = A_{11}(t)x_1(t)$ and $x_2'(t) = A_{22}(t)x_2(t)$ satisfy

$$\|R_{11}(t, t_0)\| \leq \sqrt{2}e^{-4(t-t_0)}, \quad \|R_{22}(t, t_0)\| \leq \sqrt{2}e^{-3(t-t_0)}$$

and

$$\begin{gathered} \|A_{11}(t)\| \leq 5, \quad \|A_{22}(t)\| \leq 5, \quad \|A_{12}(t)\| = 1/2, \quad \|A_{21}(t)\| \leq 1/3, \\ \|B_{11}(t)\| \leq (4e)^{-1}, \quad \|B_{22}(t)\| \leq (4e)^{-1}, \quad \|C_{12}(t)\| \leq e^{-4(t+1)}/2, \\ \|C_{21}(t)\| \leq e^{-4(t+1)} \quad \text{and} \quad \|B_{12}\| = \|B_{21}\| = \|C_{11}\| = \|C_{22}\| = 0. \end{gathered}$$

Using the notation of Theorem 4.10 and a simple argument, we have

$$\begin{gathered} t_k = k, \quad \tau = 1, \quad \alpha_1 = 4, \quad \alpha_2 = 3, \quad M_1 = M_2 = \sqrt{2}, \\ \tilde{a}_{11}(t) = -\alpha_1 = -4, \quad \tilde{a}_{12}(t) = M_1\|A_{12}(t)\| \leq \sqrt{2}/2, \\ \tilde{a}_{21}(t) = M_2\|A_{21}(t)\| \leq \sqrt{2}/3, \quad \tilde{a}_{22}(t) = -\alpha_2 = -3, \\ \tilde{b}_{11}(t) \leq (2e)^{-1}, \quad \tilde{b}_{12}(t) \leq 1/e, \quad \tilde{b}_{21}(t) \leq (2e)^{-1} \quad \text{and} \quad \tilde{b}_{22}(t) \leq 1/e. \end{gathered}$$

Let the comparison system be

$$\begin{cases} \xi'(t) = \begin{bmatrix} -4 & \sqrt{2}/2 \\ \sqrt{2}/3 & -3 \end{bmatrix} \xi(t) + \begin{bmatrix} 1/2e & 1/2e \\ 1/2e & 1/2e \end{bmatrix} \xi(t-1), \quad t \neq t_k, \\ \xi(t_k) = (\|D_{ijk}\|)_{2\times 2}\xi(t_k - 0). \end{cases} \tag{4.10}$$

From (4.10), using the notation of Theorem 3.1, we have $\alpha_1 = 4, \alpha_2 = 3, c_1 = c_2 = 1$, $\gamma = 2.1$, $\|\tilde{B}\| = e^{-1}$, $\tau = 1$ and $\alpha = 1$. Let

$$D_k = \begin{bmatrix} D_{11k} & D_{12k} \\ D_{21k} & D_{22k} \end{bmatrix} = \begin{bmatrix} e-3/2 & 0 & 0 & 0 \\ 1/2 & 0 & 0 & 0 \\ 0 & 1 & 1/2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

then $\|D_k\| = \|\tilde{D}_k\| = e - 1$ and

(1) if $t_k - t_{k-1} \geq \tau = 1$, then $e^{-t}\prod_{j\in\Delta(t)} \max\{\|\tilde{D}_j\|, e^{\alpha\tau}\} \leq 1$,

(2) if $t_k - t_{k-1} \geq \tau + 1/k$, then $\lim_{t\to\infty}\left(e^{-t}\prod_{j\in\Delta(t)} \max\{\|\tilde{D}_j\|, e^{\alpha\tau}\}\right) = 0$,

(3) if $t_k - t_{k-1} \geq \tau + \eta$, $\eta > 0$, then $e^{-t/(1+\eta)}\prod_{j\in\Delta(t)} \max\{\|\tilde{D}_j\|, e^{\alpha\tau}\} \leq e$.

By Theorem 4.4, system (4.9) is stable in case (1); asymptotically stable in case (2); and exponentially stable in case (3).

5. Conclusion

In this paper, we have studied the stability issue for both linear and nonlinear impulsive functional systems with delay. Our approach has utilised the comparison principle and an inequality for the establishment of stability criteria. Although only a single delay has been considered in this paper, the study can be extended to the case with multiple delays.

Acknowledgements

The authors wish to thank the referees whose comments improved the quality of the paper. This work was supported by the Natural Sciences and Engineering Research Council of Canada.

References

[1] A. Anokhin, L. Berezansky and E. Braverman, "Exponential stability of linear delay impulsive differential equations", *J. Math. Anal. Appl.* **193** (1995) 923–941.

[2] G. Ballinger and X. Liu, "On boundedness of solutions of impulsive systems", *Nonlinear Stud.* **4** (1997) 121–131.

[3] G. Ballinger and X. Liu, "Existence and uniqueness results for impulsive delay differential equations", *Dynam. Contin. Discrete Impuls. Systems* **5** (1999) 579–591.

[4] L. Berezansky and L. Idels, "Exponential stability of some scalar impulsive delay differential equation", *Commun. Appl. Anal.* **2** (1998) 301–309.

[5] S. G. Deo and S. G. Pandit, *Differential systems involving impulses*, Lecture Notes 954 (Springer, Berlin, 1982).

[6] Z. Guan, "Decentralized stabilization for impulsive large-scale systems with delays", *Dynam. Contin. Discrete Impuls. Systems* **6** (1999) 367–379.

[7] D. Guo and X. Liu, "First order impulsive integro-differential equations on unbounded domain in a Banach space", *Dynam. Contin. Discrete Impuls. Systems* **2** (1996) 381–394.

[8] S. V. Krishna and A. V. Anokhin, "Delay differential systems with discontinuous initial data and existence and uniqueness theorems for systems with impulse and delay", *J. Appl. Math. Stochastic Anal.* **7** (1994) 49–67.

[9] V. Lakshmikantham, D. D. Bainov and P. S. Simeonov, *Theory of impulsive differential equations* (World Scientific, Teaneck, NJ, 1989).

[10] V. Lakshmikantham and S. Leela, *Differential and integral inequalities* (Academic Press, New York, 1969).

[11] V. Lakshmikantham and X. Liu, "Stability criteria for impulsive differential equations in term of two measures", *J. Math. Anal. Appl.* **137** (1989) 591–604.

[12] X. Liu, "Stability results for impulsive differential systems with applications to population growth models", *Dynam. Stability Systems* **9** (1994) 163–174.

[13] X. Liu and X. Liao, "Comparison method and robust stability of large-scale dynamic systems", *Dynam. Contin. Discrete Impuls. Systems Ser. A Math. Anal.* **11** (2004) 413–430.

[14] X. Liu and J. Shen, "Asymptotic behavior of solutions of impulsive neutral differential equations", *Appl. Math. Lett.* **12** (1999) 51–58.

[15] V. D. Mil'man and A. D. Myshkis, "On the stability of motion in the presence of impulses", *Siberian Math. J.* **1** (1960) 233–237.

[16] T. Pavlids, "Stability of systems described by differential equations containing impulses", *IEEE Trans. Automatic Control* **12** (1967) 43–45.

[17] J. Shen, Z. Luo and X. Liu, "Impulsive stabilization of functional differential equations via Liapunov functionals", *J. Math. Anal. Appl.* **240** (1999) 1–15.

[18] J. Shen and J. Yan, "Razumikhim type stability theorems for impulsive functional differential equations", *Nonlinear Anal.* **33** (1998) 519–531.

[19] L. Wen and P. Weng, "Weakly exponentially asymptotic stability of functional differential equation with impulses", *Dynam. Contin. Discrete Impuls. Systems* **6** (1999) 251–269.

[20] T. Yang, *Impulsive control theory*, Lecture Notes in Control and Information Sciences 272 (Springer, Berlin, 2000).