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A Note about Analytic Solvability of Complex Planar Vector Fields with Degeneracies

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Abstract. This paper deals with the analytic solvability of a special class of complex vector fields defined on the real plane, where they are tangent to a closed real curve, while off the real curve, they are elliptic.

1 Introduction and Some Known Results

Let

$$\mathcal{L} = a(x, y)\partial/\partial x + b(x, y)\partial/\partial y \tag{1.1}$$

be a complex vector field, defined on $\mathbb{R}^2$, where a and b are real-analytic complex-valued functions that do not vanish simultaneously. Consider $\overline{\mathcal{L}} = \overline{a(x, y)}\partial/\partial x + \overline{b(x, y)}\partial/\partial y$.

We say that $\mathcal{L}$ is elliptic at $(x, y) \in \mathbb{R}^2$ if $\mathcal{L}_{(x,y)}$ and $\overline{\mathcal{L}}_{(x,y)}$ are linearly independent. If $\mathcal{L}$ is elliptic at (x, y), then $\mathcal{L}$ is equivalent to the Cauchy–Riemann operator $\partial/\partial\bar{z}$ in some neighborhood of (x, y).

In this paper we shall consider complex vector fields $\mathcal{L}$, given by (1.1), which are tangent to a closed real curve Σ, while off the real curve, they are elliptic and such that $\mathcal{L} \wedge \overline{\mathcal{L}}$ vanishes of constant finite order $m \geq 1$ along Σ.

At points $(x, y) \in \Sigma$, the vector field (1.1) is not elliptic. However, at points $(x, y) \in \Sigma$ and in suitable local coordinates, (1.1) can be rewritten in the form

$$\partial/\partial t - ix^m c(x, t)\partial/\partial x, \quad \Re c(0, 0) \neq 0, \tag{1.2}$$

where c is a real-analytic complex-valued function. As a consequence of the Cauchy–Kowalevsky theorem, we have that (1.2) is locally equivalent to a multiple of

$$\partial/\partial t - ix^m\partial/\partial x.$$

Note that $\partial/\partial\bar{z}$ and $\partial/\partial t - ix^m\partial/\partial x$ satisfy the well-known *Nirenberg–Treves* condition $(\mathcal{P})$ and, consequently, the local solvability is well understood (see [16]).

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However, the problem of solvability of (1.1) is still interesting if we deal with the solvability in a full neighborhood of Σ.

After a change of coordinates in a tubular neighborhood of Σ, we can assume that (1.1) is given by

$$\mathcal{L} = \partial/\partial t + (a(x,t) + ib(x,t))\partial/\partial x \tag{1.3}$$

defined on $\Omega_\epsilon = (-\epsilon,\epsilon) \times S^1$, $\epsilon > 0$, where a and b are real-analytic real-valued functions and $t \longrightarrow (a+ib)(0,t) \equiv 0$; moreover, $\Sigma = \{0\} \times S^1$. Note that $\mathcal{L}$ satisfies condition $(\mathcal{P})$ on Ω_ϵ, since $\mathcal{L}$ is tangent to Σ and elliptic away from Σ.

For operators given by (1.3), condition $(\mathcal{P})$ has a simple statement: $\mathcal{L}$ satisfies condition $(\mathcal{P})$ if and only if the function b does not change sign on any integral curve of $\Re(\mathcal{L}) = \partial/\partial t + a(x,t)\partial/\partial x$ (see [13]).

A simple calculus shows that $\mathcal{L} \wedge \overline{\mathcal{L}} = 2ib(x,t)\partial/\partial x \wedge \partial/\partial t$ and, consequently, we can write $b(x,t) = x^m b_0(x,t)$, where $b_0(0,t) \neq 0$, for all $t \in S^1$. Hence (1.3) can be rewritten in the form

$$\mathcal{L} = \partial/\partial t + (x^n a_0(x,t) + ix^m b_0(x,t))\partial/\partial x, \tag{1.4}$$

where either $n \geq m$ or $n < m$ and $a_0(0,t) \neq 0$, for all $t \in S^1$.

We say that an operator $\mathcal{L}$ of the form (1.4) is C^w-solvable at Σ if given a real-analytic function f defined in a neighborhood of Σ and satisfying

$$\int_0^{2\pi} \frac{\partial^{(j)} f}{\partial x^j}(0,t)dt = 0, \quad j = 0,\dots,k-1, \quad k = \min\{m,n\},$$

there exists a real-analytic function u solution of $\mathcal{L}u = f$ in a neighborhood of Σ.

When $m = 1$, it was shown in [14] that

$$\lambda = \frac{1}{2\pi}\int_0^{2\pi} (b_0 - ix^{n-1}a_0)(0,t)dt$$

is an invariant of $\mathcal{L}$ and, if $\lambda \in \mathbb{C} \setminus \mathbb{R}$, then $\mathcal{L}$ is equivalent to $T_\lambda = \partial/\partial t - i\lambda\partial/\partial x$. The C^w-solvability of the model operator T_λ was studied in [8] when $\lambda \in \mathbb{C} \setminus \mathbb{R}$, and in [5] when $\lambda \in \mathbb{R}$ (see also [6]). Moreover, if $\lambda \in \mathbb{R} \setminus \mathbb{Q}$, the following was proved in [10].

Proposition 1.1 *Let $\mathcal{L}$ be given by* (1.4) *with $m = 1$ and irrational λ. Then, the operator $\mathcal{L}$ is C^w-solvable at Σ if and only if $\mathcal{L}$ is real-analytically equivalent to T_λ and* $(\log|e^{2\pi ni\lambda} - 1|/n)$ *is bounded.*

When $m = 1$ and $\lambda \in \mathbb{Q}$, if we consider that the functions a_0 and b_0, given in (1.4), depend only on the x-variable, then there are infinitely many compatibility conditions for the equation $\mathcal{L}u = f$ and, consequently, $\mathcal{L}$ is not C^w-solvable at Σ.

From now on, we shall assume $m \geq 2$. It was proved in [15] that if $n \geq m$, then (1.4) is real-analytically equivalent to

$$R_m = \partial/\partial\theta - i\frac{r^m}{rP'(r) - (m-1)P(r) + \mu r^{m-1}}\partial/\partial r,$$

where $\mu \in \mathbb{C}$ and $P(r)$ is a polynomial with degree at most $m-2$ and $\Re P(0) < 0$, provided a certain function

$$f(r,\theta) = \frac{\alpha_{-(m-1)}}{r^{m-1}} + \cdots + \frac{\alpha_{-1}}{r} + \mu \log|r| + i\theta + \sum_{j\geq 1} f_j(\theta) r^j$$

converges. It was proved in [6] that R_m is not C^w-solvable at Σ.

When $n < m$, the study of C^w-solvability at Σ of (1.4) is untouched. In this paper we shall consider $m, n \geq 2$ and we shall allow $n < m$; however, we shall consider that the functions a and b depend only on the x-variable.

For related papers see, for instance, [1–4, 7, 9, 11, 12].

2 Analytic Solvability at Σ

Let

$$\mathcal{L}_n = \partial/\partial t - x^{n+1} c_0(x) \partial/\partial x, \quad c_0 \in C^w, \quad c_0(0) \neq 0, \quad n \geq 1, \tag{2.1}$$

be a complex vector field defined on $\Omega_\epsilon = (-\epsilon, \epsilon) \times S^1$, $\epsilon > 0$, where $c_0(x) = a_0(x) + ib_0(x)$, and a_0, b_0 are real-analytic real-valued functions.

In this section, we will deal with the study of the C^w-solvability of

$$\mathcal{L}_n u = f, \tag{2.2}$$

in a full neighborhood of Σ, that is, the C^w-solvability at Σ.

The lemma below is due to Bergamasco and Meziani [6].

Lemma 2.1 ([6, Lemma 2.1]) *Given $n \in \mathbb{Z}_+$, $a_0 \in \mathbb{C}^*$, and $a_n \in \mathbb{C}$, there is $p \in \mathbb{Z}_+$ such that the differential equation*

$$z^{n+1} dv/dz - (a_0 + a_n z^n) v = z^p$$

has no holomorphic solution in any neighborhood of $0 \in \mathbb{C}$.

Now, we are ready to prove that $\mathcal{L}_n$, given by (2.1), is not C^w-solvable at Σ for every $n \geq 1$.

Theorem 2.2 *Let $\mathcal{L}_n$ be given by* (2.1). *For every $n \geq 1$, there exists $f \in C^w(\Sigma)$ satisfying*

$$\int_0^{2\pi} \frac{\partial^{(j)} f}{\partial x^j}(0,t)\, dt = 0, \quad j = 0, \ldots, n$$

such that the equation (2.2) *has no solution $u \in C^w$ in any neighborhood of Σ.*

Proof Define $f(x,t) = \varphi(x) e^{it}$, where φ is a real-analytic function defined on an interval of center $x = 0$. For a fixed $n \geq 1$, assume that the equation $\mathcal{L}_n u = f$ has a

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solution $u \in C^w$ in a neighborhood of Σ. The solution u can be written, using partial Fourier series, in the form

$$u(x,t) = \sum_{j\in\mathbb{Z}} \hat{u}_j(x) e^{ijt}, \quad \text{where} \quad \hat{u}_j(x) = \frac{1}{2\pi}\int_0^{2\pi} u(x,t) e^{-ijt} dt;$$

moreover, since u is a solution of (2.2), we have that $\hat{u}_1$ must satisfy the ordinary differential equation

$$i\hat{u}_1(x) - x^{n+1}(a_0(x) + ib_0(x))\frac{d\hat{u}_1}{dx}(x) = \varphi(x). \tag{2.3}$$

Consider the complexification of (2.3):

$$i\hat{U}_1(\zeta) - \zeta^{n+1}(A_0(\zeta) + iB_0(\zeta))\frac{d\hat{U}_1}{d\zeta}(\zeta) = \Phi(\zeta), \tag{2.4}$$

where $\zeta = x + i\tilde{x} \in \mathbb{C}$ and $\hat{U}_1$, A_0, B_0, and Φ are complexifications of $\hat{u}_1$, a_0, b_0, and φ, respectively.

We claim that there is a holomorphic change of coordinates in a neighborhood of $0 \in \mathbb{C}^2$ that transforms the equation (2.4) in an equation as in Lemma 2.1. To prove this claim, we define

$$F(\zeta,\eta) = -\frac{\alpha_0}{n(1+\eta)^n} + \frac{\alpha_0}{n} + \frac{\alpha_1\zeta}{n-1} + \frac{\alpha_2\zeta^2}{n-2} + \cdots + \alpha_{n-1}\zeta^{n-1} + h_0\zeta^n \ln(1+\eta) - \sum_{j\geq 1}\frac{h_j}{j}\zeta^{j+n},$$

where α_k and h_j, $k = 0,\dots,n$, and $j \in \mathbb{Z}_+$ are such that

$$\frac{1}{A_0(\zeta) + iB_0(\zeta)} = \alpha_0 + \alpha_1\zeta + \alpha_2\zeta^2 + \cdots + \alpha_{n-1}\zeta^{n-1} + \zeta^n h(\zeta)$$

and $h(\zeta) = \sum_{j\geq 0} h_j\zeta^j$. We have that $F(0,0) = 0$ and $\frac{\partial F}{\partial\eta}(0,0) = \alpha_0 \neq 0$; note that $\alpha_0 = \frac{1}{a_0(0)+ib_0(0)}$. Hence, by the implicit function theorem, there is a holomorphic function $\eta(\zeta)$ defined on a disk $|\zeta| < \delta$, with $\eta(0) = 0$, such that

$$F(\zeta,\eta(\zeta)) = 0, \quad \text{if} \quad |\zeta| < \delta.$$

Now, define $z(\zeta) = \zeta(1+\eta(\zeta))$. It follows that

$$\frac{dz}{d\zeta} = 1 + \eta - \frac{\frac{\alpha_1\zeta}{n-1} + \frac{2\alpha_2\zeta^2}{n-2} + \cdots + (n-1)\alpha_{n-1}\zeta^{n-1} + nh_0\zeta^n\ln(1+\eta)}{\frac{\alpha_0 + h_0\zeta^n(1+\eta)^n}{(1+\eta)^{n+1}}} - \frac{\sum_{j\geq 1}\binom{j+n}{j} h_j\zeta^{j+n}}{\frac{\alpha_0+h_0\zeta^n(1+\eta)^n}{(1+\eta)^{n+1}}}.$$

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Since $F(\zeta,\eta(\zeta))=0$, we have

$$-h_0\zeta^n\ln(1+\eta)=-\frac{\alpha_0}{n(1+\eta)^n}+\frac{\alpha_0}{n}+\frac{\alpha_1\zeta}{n-1}+\frac{\alpha_2\zeta^2}{n-2}+\cdots+\alpha_{n-1}\zeta^{n-1}-\sum_{j\geq 1}\frac{h_j}{j}\zeta^{j+n},$$

and thus

$$\frac{dz}{d\zeta}=1+\eta+\frac{\alpha_0+\alpha_1\zeta+\alpha_2\zeta^2+\cdots+\alpha_{n-1}\zeta^{n-1}+\sum_{j\geq 1}h_j\zeta^{j+n}-\frac{\alpha_0}{(1+\eta)^n}}{\frac{\alpha_0+h_0\zeta^n(1+\eta)^n}{(1+\eta)^{n+1}}}.$$

Hence,

$$\frac{\alpha_0+h_0\zeta^n(1+\eta)^n}{\zeta^{n+1}(1+\eta)^{n+1}}\frac{dz}{d\zeta}=\frac{\alpha_0+\alpha_1\zeta+\alpha_2\zeta^2+\cdots+\alpha_{n-1}\zeta^{n-1}+\sum_{j\geq 0}h_j\zeta^{j+n}}{\zeta^{n+1}},$$

which implies

$$\frac{\alpha_0+h_0\zeta^n(1+\eta)^n}{\zeta^{n+1}(1+\eta)^{n+1}}\frac{dz}{d\zeta}=\frac{1}{\zeta^{n+1}(A_0+iB_0)(\zeta)}.$$

Therefore, equation (2.4) is transformed to

$$z^{n+1}\frac{dw}{dz}-i(\alpha_0+h_0z^n)w=-\Phi(\zeta(z))(\alpha_0+h_0z^n);$$

equivalently,

$$z^{n+1}\frac{dw}{dz}-(\tilde{\alpha}_0+\tilde{h}_0z^n)w=f(z), \tag{2.5}$$

where $\tilde{\alpha}_0=i\alpha_0$, $\tilde{h}_0=ih_0$ and $f(z)=-\Phi(\zeta(z))(\tilde{\alpha}_0+\tilde{h}_0z^n)$.

Choosing $\varphi(x)=-\frac{x^p(1+\eta(x))^p}{\tilde{\alpha}_0+\tilde{h}_0x^n(1+\eta(x))^n}$, with $p\in\mathbb{Z}_+$, we obtain $f(z)=z^p$.

It follows from Lemma 2.1, with a suitable choice of $p\in\mathbb{Z}_+$, that equation (2.5) has no holomorphic solution, w, in any neighborhood of $z=0$. Therefore, there is no $\hat{u}_1(x)$ real-analytic solution of (2.3), and, consequently, there is no $u(x,t)$ real-analytic solution of (2.2) in any neighborhood of Σ. ■

Remark 2.3 Note that in Theorem 2.2 it is permitted $b_0(0)=0$ and, in particular, $b_0\equiv 0$.

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References

[1] A. P. Bergamasco, P. D. Cordaro, and G. Petronilho, *Global solvability for a class of complex vector fields on the two-torus.* Comm. Partial Differential Equations **29**(2004), no. 5–6, 785–819. doi:10.1081/PDE-120037332

[2] A. P. Bergamasco and P. L. Dattori da Silva, *Global solvability for a special class of vector fields on the torus.* In: Recent progress on some problems in several complex variables and partial differential equations, Contemp. Math., 400, American Mathematical Society, Providence, RI, 2006, pp. 11–20.

[3] ______, *Solvability in the large for a class of vector fields on the torus.* J. Math. Pures Appl. **86**(2006), no. 5, 427–447.

[4] A. P. Bergamasco, P. L. Dattori da Silva, and M. R. Ebert, *Gevrey solvability near the characteristic set for a class of planar complex vector fields of infinite type.* J. Differential Equations **246**(2009), no. 4, 1673–1702. doi:10.1016/j.jde.2008.10.028

[5] A. P. Bergamasco and A. Meziani, *Semiglobal solvability of a class of planar vector fields of infinite type.* Mat. Contemp.**18**(2000), 31–42.

[6] ______, *Solvability near the characteristic set for a class of planar vector fields of infinite type.* Ann. Inst. Fourier (Grenoble) **55**(2005), no. 1, 77–112.

[7] A. P. Bergamasco and G. Petronilho, *Closedness of the range for vector fields on the torus.* J. Differential Equations **154**(1999), no. 1, 132–139. doi:10.1006/jdeq.1998.3563

[8] S. Berhanu and A. Meziani, *Global properties of a class of planar vector fields of infinite type.* Comm. Partial Differential Equations **22**(1997), no. 1–2, 99–142. doi:10.1080/03605309708821288

[9] W. A. Cerniauskas and A. Kirilov, *C^k solvability near the characteristic set for a class of vector fields of infinite type.* Mat. Contemp. **36**(2009), 91–106.

[10] P. D. Cordaro and X. Gong, *Normalization of complex-valued planar vector fields which degenerate along a real curve.* Adv. Math. **184**(2004), no. 1, 89–118. doi:10.1016/S0001-8708(03)00139-7

[11] P. L. Dattori da Silva, *Nonexistence of global solutions for a class of complex vector fields on two-torus.* J. Math. Anal. Appl. **351**(2009), no. 2, 543–555. doi:10.1016/j.jmaa.2008.10.039

[12] ______, *C^k-Solvability near the characteristic set for a class of planar complex vector fields of infinite type.* Ann. Mat. Pura Appl. **189**(2010), no. 3, 403–413. doi:10.1007/s10231-009-0115-8

[13] L. Hörmander, *Pseudodifferential operators of principal type.* In: Singularities in boundary value problems (Proc. NATO Adv. Study Inst., Maratea, 1980), NATO Adv. Study Inst. Ser. C: Math. Phys. Sci., 65, Reidel, Dordrecht-Boston, MA, 1981, pp. 69–96.

[14] A. Meziani, *On planar elliptic structures with infinite type degeneracy.* J. Funct. Anal. **179**(2001), no. 2, 333–373. doi:10.1006/jfan.2000.3695

[15] ______, *Elliptic planar vector fields with degeneracies.* Trans. Amer. Math. Soc. **357**(2005), no. 10, 4225–4248. doi:10.1090/S0002-9947-04-03658-X

[16] L. Nirenberg and F. Treves, *Solvability of a first order linear partial differential equation.* Comm. Pure Appl. Math. **16**(1963), 331–351. doi:10.1002/cpa.3160160308

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