Can. J. Math. Vol. **50** (6), 1998 pp. 1138–1162.

COMPOUND INVARIANTS AND MIXED F-, DF-POWER SPACES

P. A. CHALOV, T. TERZIOĞLU AND V. P. ZAHARIUTA

ABSTRACT. The problems on isomorphic classification and quasiequivalence of bases are studied for the class of mixed F-, DF-power series spaces, *i.e.* the spaces of the following kind

$$G(\lambda, a) = \lim_{p\to\infty} \operatorname{proj} \left(\lim_{q\to\infty} \operatorname{ind} \left(\ell_1 \left(a_i(p,q) \right) \right) \right),$$

where $a_i(p,q) = \exp\Big((p - \lambda_i q)a_i\Big)$, $p, q \in \mathbb{N}$, and $\lambda = (\lambda_i)_{i\in\mathbb{N}}$, $a = (a_i)_{i\in\mathbb{N}}$ are some sequences of positive numbers. These spaces, up to isomorphisms, are basis subspaces of tensor products of power series spaces of F- and DF-types, respectively. The m-rectangle characteristic $\mu_m^{\lambda,a}(\delta, \varepsilon; \tau, t)$, $m \in \mathbb{N}$ of the space $G(\lambda, a)$ is defined as the number of members of the sequence $(\lambda_i, a_i)_{i\in\mathbb{N}}$ which are contained in the union of m rectangles $P_k = (\delta_k, \varepsilon_k] \times (\tau_k, t_k]$, $k = 1, 2, \ldots, m$. It is shown that each m-rectangle characteristic is an invariant on the considered class under some proper definition of an equivalency relation. The main tool are new compound invariants, which combine some version of the classical approximative dimensions (Kolmogorov, Pełczynski) with appropriate geometrical and interpolational operations under neighborhoods of the origin (taken from a given basis).

1. **Introduction.** Pełczynski ([42]) and Kolmogorov ([31]) introduced first important linear topological invariants (approximative dimension), dealing with non-normable locally convex spaces. These fundamental invariants as well as their more or less direct developments ([7, 8, 9, 36, 37, 43, 44, 23, 24, 5, 6, 17, 18, 2, 3, 32, 33, 47] *et al.*) proved to be powerful instruments for studying locally convex spaces, especially those with some homogeneous linear topological structure: for example, these invariants gave a complete isomorphic classification of the class of all Fréchet spaces with a regular absolute basis ([23, 2, 17, 32, 33, 18]). Nevertheless the classical invariants and their traditional modifications could give only quite coarse differentiation of spaces even for such simple (at least from the first view) classes as Cartesian or tensor products of power series spaces of finite and infinite type (see, *e.g.*, [36, 43, 23, 55, 57, 58]). The reason is that the combination of spaces, so different in topological sense, might bring some subtle differences between resulting spaces, non-distinguishable for those invariants. In an effort to get more distinguishing tools for isomorphic classification of above-mentioned and other more general classes of locally convex spaces, some new linear topological invariants were suggested in [56, 59, 60, 61] and later on (in some new geometrical

Received by the editors June 16, 1997.

The first author was supported by TÜBİTAK-NATO Fellowship Program.

AMS subject classification: 46A04, 46A45, 46M05.

respect) in [62, 64, 66]; in this connection the initiative influence of Mityagin's results [38, 39, 40] must be emphasized.

The aim of this paper is to study the topological structure (in particular, the problems of isomorphic classification and quasiequivalence of absolute bases) of another intriguing class of spaces for which an interference of two different topological structures (of F- and DF-types, in this case) results again some slight effects, requiring a very scrupulous analysis of invariant properties of spaces to distinguish them. Namely, we study the class of *mixed* F-, DF-*power series spaces*, *i.e.* spaces of the following kind:

$$G(\lambda, a) = \lim_{p\to\infty} \operatorname{proj}\Big(\lim_{q\to\infty} \operatorname{ind}\Big(\ell_1\big(a_i(p,q)\big)\Big)\Big), \tag{1}$$

where $a_i(p,q) = \exp\big((p - \lambda_i q)a_i\big)$, $p, q \in \mathbb{N}$, and $\lambda = (\lambda_i)_{i\in\mathbb{N}}$, $a = (a_i)_{i\in\mathbb{N}}$ are some sequences of positive numbers.

This class, up to isomorphisms, consists of basis subspaces (step subspaces [25]) of projective tensor products (with respect to the canonical bases of them)

$$E_\infty(c)\hat{\otimes}E'_\infty(d) \tag{2}$$

of a *power series space of infinite type*:

$$E_\infty(c) = \lim_{p} \operatorname{proj} \ell_1\big(\exp(pc_i)\big), \tag{3}$$

with a *dual power series space of infinite type*:

$$E'_\infty(d) = \lim_{p} \operatorname{ind} \ell_1\big(\exp(-pd_i)\big), \tag{4}$$

where $c = (c_i)$, $d = (d_i)$ may be arbitrary sequences of positive numbers.

Tensor products (2), belonging to the class (1), were investigated in [28, 26, 27], where some necessary and sufficient conditions of isomorphism for such spaces were obtained.

The main difficulty in studying of the spaces (1) is how to separate two collided features of their nature: F- and DF-topological structures. To this end we consider some more complicated linear topological invariants (following [12, 11] we use the term *compound invariants* for them). With these invariants, described in the end of Section 4 and in Section 5 (within the proofs of Lemma 8 and Theorem 9), we show that any m-rectangle characteristic is an invariant on the class (1). Following [1], the term *m-rectangle characteristic* is used for the function (*cf.* [10, 13, 14, 15, 16, 12, 11]) $\mu_m^{\lambda,a}(\delta,\varepsilon;\tau,t)$, which calculates how many points of the sequence $(\lambda, a) = \{(\lambda_i, a_i)\}_{i\in\mathbb{N}}$ are situated in the union of m rectangles $P_k = (\delta_k, \varepsilon_k] \times (\tau_k, t_k]$, $k = 1, 2, \ldots, m$. The notions of equivalence (for individual m-rectangle characteristics: $\mu_m^X \approx \mu_m^{\tilde{X}}$ and for the systems of these characteristics: $(\mu_m^X) \approx (\mu_m^{\tilde{X}})$), considered in Section 4, play an important part in the concept of invariance. It should be pointed out that the applying of interpolational constructions in compound invariants considerations (for example using of "interpolational blocks" in Section 5) linked to a row of results connected with generalizations of Dragilev classes d_1, d_2 [23, 25, 45, 46, 49, 50, 51, 52, 53, 55, 57, 58, 63, 65] *et al*.

In Section 5 it is shown that the system $(\mu_m)_{m\in\mathbb{N}}$ forms a complete invariant in respect to quasidiagonal isomorphisms (Theorem 7). Although it remains a gap between results on the isomorphic and quasidiagonally isomorphic invariants (1) (see Problem 1 below), we prove the invariance of the system of m-rectangle characteristics under some weakened relation of equivalence $\overset{w}{\approx}$ (Section 6). It should be noticed that all the results mentioned in the beginning paragraph (see also [21, 30]) were within the framework of one-rectangular characteristic considerations.

Some applications to concrete subclasses of spaces (1) are considered in Section 7. Applying of many-rectangular characteristics gives some new results for tensor products (2). The subclass of the class (1), consisting (up to isomorphisms) of all Cartesian products of spaces (3), (4), admits (Theorem 11) the complete isomorphic characterization by the use of two-rectangle characteristics only (*cf.* [12, 11]).

2. Preliminaries.

2.1. Let X, Y be locally convex spaces and $\{x_i\}_{i\in\mathbb{N}}$ and $\{y_i\}_{i\in\mathbb{N}}$ unconditional bases for the spaces X and Y, respectively. We say that these bases are *quasiequivalent* if there exists an isomorphism $T\colon X \longrightarrow Y$ such that $Tx_i = t_i y_{\sigma(i)}$, where (t_i) is a sequence of scalars and $\sigma\colon \mathbb{N} \longrightarrow \mathbb{N}$ is a bijection. In this case the isomorphism T is called *quasidiagonal*, the spaces X and Y are called *quasidiagonally isomorphic* (with respect to those fixed bases) and we write shortly $X \overset{qd}{\simeq} Y$; in the particular case $t_i \equiv 1$ for all $i \in \mathbb{N}$, the operator T is said to be *permutational*, the spaces are called *permutationally isomorphic* and we write $X \overset{p}{\simeq} Y$.

2.2. Let E be a class of locally convex spaces and Γ a set with an equivalence relation $\sim$. We say that $\gamma\colon E \longrightarrow \Gamma$ is a (linear topological) invariant if $X \simeq \tilde{X} \Rightarrow \gamma(X) \sim \gamma(\tilde{X})$, X, $\tilde{X} \in E$. The invariants to be studied are based on the following well-known characteristic of a couple of absolutely convex sets.

Let X be a linear space, U, V absolutely convex subsets in X. Consider

$$\beta(V,U) := \sup\{\dim L : U \cap L \subset V\}, \tag{5}$$

where L runs along the set of all finite-dimensional subspaces of $X_V = \overline{\operatorname{span} V}$. This characteristic relates with so-called Bernstein diameters $b_n(V,U)$ [48] in the following way:

$$\beta(V,U) = \big|\{n : b_n(V,U) \geq 1\}\big|.$$

The following properties follow immediately from the definition (5):

$$(a)\ \text{ if } V_1 \subset V \text{ and } U \subset U_1, \quad \text{then } \beta(V_1,U_1) \leq \beta(V,U); \tag{6}$$

$$(b)\ \ \beta(\alpha V,U) = \beta\Big(V, \frac{1}{\alpha}U\Big), \quad \alpha > 0. \tag{7}$$

https://doi.org/10.4153/CJM-1998-055-0 Published online by Cambridge University Press

Let X be a locally convex space, $e = \{e_i\}_{i\in\mathbb{N}}$ an unconditional basis in X. A set

$$B^e(a) := \Big\{x = \sum_{i=1}^{\infty} \xi_i e_i \in X \colon \sum_{i=1}^{\infty} |\xi_i| a_i \le 1\Big\}$$

is the weighted ℓ_1-ball in X, defined with a given weight sequence of positive numbers $a = (a_i)_{i\in\mathbb{N}}$. For weighted balls the characteristic (5) admits a simple computation.

PROPOSITION 1 (SEE, *e.g.*, [36, 20]). *For a couple of weights a, b we have*

$$\beta\big(B^e(b), B^e(a)\big) = |\{i : b_i \le a_i\}|.$$

2.3 In the construction of compound invariants (see Section 5) we shall use the following geometrical facts.

For a couple $A_\nu = B^e(a^{(\nu)})$, $\nu = 0, 1$, we consider the following one-parameter family of weighted balls

$$(A_0)^{1-\alpha}(A_1)^{\alpha} := B^e(a^{(\alpha)}),$$

where $a^{(\alpha)} := \big((a_i^{(0)})^{1-\alpha}(a_i^{(1)})^{\alpha}\big)_{i\in\mathbb{N}}$, $\alpha \in \mathbb{R}$. The following elementary fact is well-known (see, for example, [4, 34, 41]

PROPOSITION 2. *Let e and f be unconditional bases of locally convex space X and $A_\nu = B^e(a^{(\nu)})$, $\tilde{A}_\nu = B^f(\tilde{a}^{(\nu)})$, $\nu = 0, 1$. Then*

$$A_\nu \subset \tilde{A}_\nu, \quad \nu = 0, 1$$

implies

$$(A_0)^{1-\alpha}(A_1)^{\alpha} \subset (\tilde{A}_0)^{1-\alpha}(\tilde{A}_1)^{\alpha}, \quad \alpha \in (0, 1).$$

PROPOSITION 3. *Let e be an unconditional basis of a locally convex space X, $a^{(j)} = (a_i^{(j)})$, $j = 1, \dots, r$, sequences of positive numbers and $c = (c_i)$, $d = (d_i)$ sequences, defined by the following formulae: $c_i = \max\{a_i^{(j)} : j = 1, \dots, r\}$, $d_i = \min\{a_i^{(j)} : j = 1, \dots, r\}$, $i \in \mathbb{N}$. Then the following relations hold:*

$$B^e(c) \subset \bigcap_{j=1}^{r} B^e(a^{(j)}) \subset rB^e(c), \quad B^e(d) = \operatorname{conv}\Big(\bigcup_{j=1}^{r} B^e(a^{(j)})\Big),$$

where $\operatorname{conv}(M)$ *means the convex hull of a set M.*

2.4 For two sequences of positive numbers $a = (a_i)$ and $\tilde{a} = (\tilde{a}_i)$ we shall write $a \asymp \tilde{a}$ or $a_i \asymp \tilde{a}_i$ if there exists a constant $c > 1$ such that $\frac{1}{c} a_i \le \tilde{a}_i \le c a_i$. Using the notion of counting function $m_a(t) := |\{i : a_i \le t\}|$, we can write the relation $a \asymp \tilde{a}$ in the equivalent form $\exists c \colon m_{\tilde{a}}(\frac{t}{c}) \le m_a(t) \le m_{\tilde{a}}(ct)$ if both a and $\tilde{a}$ tend to ∞ monotonically. The following statement is well known ([36, 37]).

PROPOSITION 4. *Let $a = (a_i)$ and $\tilde{a} = (\tilde{a}_i)$ be sequences of positive numbers and both of them are tending to ∞ monotonically. Suppose $X = E_\infty(a)$ (or $X = E'_\infty(a)$) and $\tilde{X} = E_\infty(\tilde{a})$ ($\tilde{X} = E'_\infty(\tilde{a})$, respectively). Then $X \simeq \tilde{X}$ if and only if $a_i \asymp \tilde{a}_i$. Moreover, the spaces $E_\infty(a)$ and $E'_\infty(\tilde{a})$ cannot be isomorphic if at least one of the sequences a or $\tilde{a}$ is not bounded.*

2.5 Here we give some facts about spaces $G(\lambda, a)$. Without loss of generality we will assume that parameters of spaces (1) satisfy the following requirements:

$$a_i \geq 1, \quad \frac{1}{a_i} \leq \lambda_i \leq 1. \tag{8}$$

Indeed, one can replace any space (1) with an isomorphic space $G(\tilde{\lambda}, \tilde{a})$, satisfying those conditions: it is sufficient for this to put $\tilde{a}_i = 1 + a_i$, $\tilde{\lambda}_i = \max\{\frac{1}{\tilde{a}_i}, \lambda_i\}$, if $\lambda_i \leq 1$ and $\tilde{a}_i = 1 + \lambda_i a_i$, $\tilde{\lambda}_i = 1$ if $\lambda_i > 1$.

There are the following possibilities for a space $X = G(\lambda, a)$:

(i) $X \simeq E'_\infty(a) \Leftrightarrow \inf\{\lambda_i : i \in \mathbb{N}\} > 0$;
(ii) $X \simeq E_\infty(a) \Leftrightarrow \lim \lambda_i = 0$;
(iii) X is *mixed*, *i.e.* (i), (ii) do not hold.

In the case (iii) the space X is isomorphic to a Cartesian product of spaces (3) and (4), if and only if the set $\mathbb{N}$ can be divided into the sum of two non-intersecting subsequences $\{i_k\}$ and $\{j_k\}$ such that $\lim \lambda_{i_k} = 0$ and $\inf\{\lambda_{j_k}\} > 0$; otherwise we say that the space $X = G(\lambda, a)$ is *properly mixed*.

Each tensor product (2) is, up to quasidiagonal isomorphism, a space of the kind (1). Indeed, it can be represented in the form (1) with the matrix

$$a_i(p, q) = \exp\left(pc_{k(i)} - qd_{l(i)}\right),$$

where $i \longrightarrow \big(k(i), l(i)\big)$ is any bijection from $\mathbb{N}$ onto $\mathbb{N} \times \mathbb{N}$. It is obvious that this space is isomorphic to the space $G(\lambda, a)$ with $a_i := \max\{c_{k(i)}, d_{l(i)}\}$ and $\lambda_i := \frac{d_{k(i)}}{a_i}$.

3. **Criterion of quasidiagonal isomorphism.** Here we study necessary and sufficient conditions for quasidiagonal isomorphism of two given spaces of the kind (1). For this purpose, first we describe some properties of basis subspaces of the space (1) as follows (compare with [60, 66]).

PROPOSITION 5. *Let $X = G(\lambda, a)$, $\nu = \{i_k\}$ be any subsequence of $\mathbb{N}$,*

$$\lambda^{(\nu)} := \{\lambda_{i_k}\}, \quad a^{(\nu)} := \{a_{i_k}\},$$

and $X^{(\nu)}$ be the subspace of X, generated by the corresponding subbasis $\{e_{i_k}\}$. Then
(i) $X^{(\nu)} \simeq E_\infty(a^{(\nu)})$ if $\lambda_i \longrightarrow 0$, $i \in \nu$;
(ii) $X^{(\nu)} \simeq E'_\infty(a^{(\nu)})$ if $\inf\{\lambda_i : i \in \nu\} > 0$.
If $X^{(\nu)}$ is Montel space then "if" in the both items can be changed to "iff".

PROOF. If $\lambda_i \longrightarrow 0$, $i \in \nu$ then $p - 1 \leq p - \lambda_i q \leq p$ for all $i \in \nu$, $i \geq i_0 = i_0(q)$. Therefore we get that

$$X^{(\nu)} \simeq \lim_p \operatorname{proj} \ell_1\big(\exp(pa_{i_k})\big) = E_\infty(a^{(\nu)}).$$

If $\lambda_i \geq \delta > 0$ for all $i \in \nu$, then under the condition $q > \frac{2p}{\delta}$ we have

$$-q < p - q \leq p - \lambda_i q \leq p - \delta q < \frac{\delta}{2}q - \delta q = -\frac{\delta}{2}q.$$

https://doi.org/10.4153/CJM-1998-055-0 Published online by Cambridge University Press

Thus,

$$X^{(\nu)} \simeq \lim_{q} \operatorname{ind} \ell_1\big(\exp(-qa_{i_k})\big) = E'_\infty(a^{(\nu)}).$$

To finish the proof we have to use what have been proved above together with the fact that $E_\infty(a^{(\nu)})$ cannot be isomorphic to $E'_\infty(a^{(\nu)})$ as $X^{(\nu)}$ is a Montel space, *i.e.* as the sequence $a^{(\nu)}$ tends to ∞.

REMARK. If X is Montel and both (i), (ii) do not hold then the subspace $X^{(\nu)}$ is as complicated as the whole space.

THEOREM 6. *Let $X = G(\lambda, a)$, $\tilde{X} = G(\tilde{\lambda}, \tilde{a})$ be Montel spaces. Then the following conditions are equivalent:*

(a) $X \overset{p}{\simeq} \tilde{X}$;

(b) $X \overset{qd}{\simeq} \tilde{X}$

(c) there exists a bijection $\sigma\colon \mathbb{N} \longrightarrow \mathbb{N}$ such that

$$a_i \asymp \tilde{a}_{\sigma(i)} \tag{9}$$

and for any subsequence (i_k)

$$(\lambda_{i_k}) \longrightarrow 0 \Leftrightarrow (\tilde{\lambda}_{\sigma(i_k)}) \longrightarrow 0. \tag{10}$$

PROOF. The relation (a) $\Rightarrow$ (b) is evident. Let us show that (b) $\Rightarrow$ (c). Let $T\colon X \longrightarrow \tilde{X}$ be an isomorphism, defined by

$$Te_i = t_i e_{\sigma(i)}. \tag{11}$$

It is sufficient to prove that the condition (c) holds with the same bijection σ.

First we show (10). Assuming that (10) is not true, we see that there exists a subsequence $\nu = (i_k)$ such that one of the sequences $\lambda^{(\nu)}$ or $\tilde{\lambda}^{(\tilde{\nu})} := \{\tilde{\lambda}_{\sigma(i_k)}\}$ tends to 0, while another is restricted from 0. Then by Propositions 5, 4 the corresponding subspaces $X^{(\nu)}$ and $\tilde{X}^{(\tilde{\nu})}$ cannot be isomorphic, which contradicts the assumption that (11) defines an isomorphism.

Let us show that for the same bijection σ the relation (9) holds. Supposing it fails, we find a subsequence $\nu = \{i_k\}$ such that $a_{i_k} \not\asymp \tilde{a}_{\sigma(i_k)}$, and both the sequences $\lambda^{(\nu)} = \{\lambda_{i_k}\}$ and $\tilde{\lambda}^{(\sigma(\nu))} = \{\tilde{\lambda}_{\sigma(i_k)}\}$ simultaneously tend or not to 0. Then both of the corresponding basis subspaces $X^{(\nu)}$ and $\tilde{X}^{(\tilde{\nu})}$ are isomorphic spaces of the same kind (3) or (4), which by Proposition 4 contradicts our supposition.

Now let us show that (c) $\Rightarrow$ (a). Obviously it is enough to prove that under conditions $\sigma(i) \equiv i$ and $\tilde{a}_i = a_i$ the operator $I\colon G(\lambda, a) \longrightarrow G(\tilde{\lambda}, \tilde{a})$ is an isomorphism. First we prove that I is continuous, *i.e.*

$$\forall r \exists p\, \forall q \exists s \exists C\colon \quad \exp\big((r - \tilde{\lambda}_i s)\tilde{a}_i\big) \leq C \exp\big((p - \lambda_i q)a_i\big). \tag{12}$$

It follows from (10) that there exists a function $\varphi\colon (0,1] \longrightarrow (0,1]$ such that $\varphi(t) \downarrow 0$ as $t \downarrow 0$ and for every $\delta \in (0,1]$ the inequality $\lambda_i \geq \delta$ implies the inequality $\tilde{\lambda}_i \geq \varphi(\delta)$.

Let us take an arbitrary r, p, q, $(r < p < q)$, any $\delta \in (0, \frac{p-r}{q})$, and an arbitrary $s > \frac{q-p+r}{\varphi(\delta)}$.

Suppose $N_1 = \{i : \lambda_i \geq \delta\}$, $N_2 = \mathbb{N} \backslash N_1$; then for $i \in N_1$ we have

$$r - \tilde{\lambda}_i s \leq r - \varphi(\delta)s < r - \varphi(\delta)\frac{q-p+r}{\varphi(\delta)} = p - q \leq p - \lambda_i q,$$

and

$$r - \tilde{\lambda}_i s \leq r = p - \frac{p-r}{q}q < p - \delta q < p - \lambda_i q \quad \text{for } i \in N_2.$$

From here we obtain (12) with $C = 1$. In view of symmetry, we get that the operator I^{-1} is also continuous. Thus the operator I is an isomorphism.

This completes the proof.

4. *m*-rectangle characteristics and compound invariants. For given $\lambda = (\lambda_i)_{i\in\mathbb{N}}$, $a = (a_i)_{i\in\mathbb{N}}$ and $m \in \mathbb{N}$ we introduce the following function:

$$\mu_m^{(\lambda,a)}(\delta, \varepsilon; \tau, t) = \Big|\bigcup_{k=1}^{m}\{i : \delta_k < \lambda_i \leq \varepsilon_k, \tau_k < a_i \leq t_k\}\Big|, \tag{13}$$

defined for $\delta = (\delta_k)$, $\varepsilon = (\varepsilon_k)$, $\tau = (\tau_k)$, $t = (t_k)$, such that

$$0 \leq \delta_k < \varepsilon_k \leq 2, \quad 0 < \tau_k < t_k < \infty, \quad k = 1, 2, \ldots, m.$$

Hereafter $|M|$ denotes the cardinality for a finite set M and $+\infty$ for an infinite set M. We may also write μ_m^X instead of $\mu_m^{(\lambda,a)}$, if $X = G(\lambda, a)$.

The function (13) will be called *m-rectangle characteristic* of the pair (λ, a) or of the corresponding space $G(\lambda, a)$. This name can be justified by the following relation

$$\mu_m^{(\lambda,a)}(\delta, \varepsilon; \tau, t) = \Big|\bigcup_{k=1}^{m}\{i : (\lambda_i, a_i) \in P_k\}\Big| = \Big|\Big\{i : (\lambda_i, a_i) \in \bigcup_{k=1}^{m} P_k\Big\}\Big|, \tag{14}$$

where $P_k = (\delta_k, \varepsilon_k] \times (\tau_k, t_k]$, $k = 1, 2, \ldots, m$. Hence the function (13) calculates how many points (λ_i, a_i) are contained in the union of m rectangles.

Let $\lambda = (\lambda_i)$, $a = (a_i)$, $\tilde{\lambda} = (\tilde{\lambda}_i)$, and $\tilde{a} = (\tilde{a}_i)$ be arbitrary sequences, m a fixed natural number. We shall say that *the functions* $\mu_m^{(\lambda,a)}$ *and* $\mu_m^{(\tilde{\lambda},\tilde{a})}$ *are equivalent* and write $\mu_m^{(\lambda,a)} \approx \mu_m^{(\tilde{\lambda},\tilde{a})}$ if there exists a strictly increasing function $\varphi \colon [0, 2] \longrightarrow [0, 1]$, $\varphi(0) = 0$, $\varphi(2) = 1$, and a positive constant α (in general, φ and α depend on m) such that the following inequalities

$$\mu_m^{(\lambda,a)}(\delta, \varepsilon; \tau, t) \leq \mu_m^{(\tilde{\lambda},\tilde{a})}\Big(\varphi(\delta), \varphi^{-1}(\varepsilon); \frac{\tau}{\alpha}, \alpha t\Big), \tag{15}$$

$$\mu_m^{(\tilde{\lambda},\tilde{a})}(\delta, \varepsilon; \tau, t) \leq \mu_m^{(\lambda,a)}\Big(\varphi(\delta), \varphi^{-1}(\varepsilon); \frac{\tau}{\alpha}, \alpha t\Big), \tag{16}$$

with $\varphi(\delta) = \big(\varphi(\delta_k)\big)$, $\varphi^{-1}(\varepsilon) = \big(\varphi^{-1}(\varepsilon_k)\big)$, $\frac{\tau}{\alpha} = (\frac{\tau_k}{\alpha})$, $\alpha t = (\alpha t_k)$ hold for all collections of parameters δ, ε, τ, t.

https://doi.org/10.4153/CJM-1998-055-0 Published online by Cambridge University Press

If, moreover, the function φ and the constant α can be chosen so that the inequalities (15), (16) hold for all $m \in \mathbb{N}$ (*i.e.* φ and α are independent of m), then we say that *the systems of characteristics* $(\mu_m^{(\lambda,a)})_{m\in\mathbb{N}}$ *and* $(\mu_m^{(\tilde\lambda,\tilde a)})_{m\in\mathbb{N}}$ *are equivalent* and write $(\mu_m^{(\lambda,a)}) \approx (\mu_m^{(\tilde\lambda,\tilde a)})$.

The following statement describes the quasidiagonal isomorphisms of spaces (1) in the terms of m-rectangle characteristics.

THEOREM 7. *For spaces* $X = G(\lambda, a)$ *and* $\tilde X = G(\tilde\lambda, \tilde a)$, *the following statements are equivalent:*

(a) $X \overset{qd}{\simeq} \tilde X$;

(b) $(\mu_m^X) \approx (\mu_m^{\tilde X})$.

The proof is omitted, since it is basically the same that in the case of power Köthe spaces of the first type [16]; but, instead of Proposition 2 from [16], we have to use Proposition 6.

Theorem 7 means that the system of all m-rectangle characteristics is a complete invariant with respect to quasidiagonal isomorphisms.

The following problem arises:

PROBLEM 1 (*cf.* PROBLEM 13, [1]). *Is the statement of Theorem 7 true if the quasidiagonal isomorphism* $\overset{qd}{\simeq}$ *is replaced by the usual isomorphism* $\simeq$?

This problem still remains open. Even the question about invariance of any individual m-rectangle characteristic (13) turns out to be quite complicated. We are studying this question in the next section with the use of *compound invariants*.

Let us explain the main idea of compound invariants, as applied to the studied class of spaces (1). Let $T: \tilde X \longrightarrow X$ be an isomorphism. We take the following two absolute bases of the space X: the canonical basis $e = \{e_i\}_{i\in\mathbb{N}}$ and T-image of the canonical basis of $\tilde X$: $f = \{f_i\}$, $f_i = Te_i$, $i \in \mathbb{N}$. Then each $x \in X$ has two basis expansions:

$$x = \sum_{i=1}^{\infty} \xi_i e_i = \sum_{i=1}^{\infty} \eta_i f_i.$$

We consider two systems of sets $(A_{p,q})$ and $(\tilde A_{p,q})$ in X, defined as follows

$$A_{p,q} = \Big\{ x \in X : \sum_{i=1}^{\infty} |\xi_i| a_i(p,q) \le 1 \Big\}, \quad p, q \in \mathbb{N}, \tag{17}$$

$$\tilde A_{p,q} = \Big\{ x \in X : \sum_{i=1}^{\infty} |\eta_i| \tilde a_i(p,q) \le 1 \Big\}, \quad p, q \in \mathbb{N}. \tag{18}$$

By Grothendieck's factorization theorem ([29], I, p. 16) the systems (17) and (18) are equivalent in the following sense:

$$\forall r \exists p\, \forall q \exists s \exists C\colon \quad \tilde A_{p,q} \subset C A_{r,s}, \quad A_{p,q} \subset C \tilde A_{r,s}. \tag{19}$$

https://doi.org/10.4153/CJM-1998-055-0 Published online by Cambridge University Press

To prove the estimate (15) we build two pairs of special absolutely convex sets U, V and $\tilde{U}$, $\tilde{V}$ in the form of certain geometrical constructions, using the sets (17) and (18) as raw materials. On the basis of the relations (19) we provide the inclusions

$$U \supset \tilde{U}, \quad V \subset \tilde{V}, \tag{20}$$

which, due to the properties of the characteristic β, get the estimate

$$\beta(V, U) \leq \beta(\tilde{V}, \tilde{U}). \tag{21}$$

The sets U, V, $\tilde{U}$, $\tilde{V}$ will be fitted (applying interpolational and geometrical constructions from subsection 2.3) so that, after some handling, the estimate (21) gives the required inequality (15). This draft program gets its concrete realization within the proofs of Lemma 8 and Theorem 9 in the next section.

5. **Invariance of *m*-rectangle characteristic.** The main difficulties are surmounted in the proof of Lemma 8, where, in fact, the invariance of μ_m is obtained with another definition of the equivalence: namely, instead of (15), (16) the given below relation (22), considered together with the symmetrical relation, obtained by interchanging X with $\tilde{X}$). Then, applying this lemma, we obtain the invariance of μ_m in the terms of Section 4 (Theorem 9).

LEMMA 8. *Let $X = G(\lambda, a)$, $\tilde{X} = G(\tilde{\lambda}, \tilde{a})$, $m \in \mathbb{N}$. If $X \simeq \tilde{X}$, then there exists an increasing function $\gamma\colon [0, 2] \longrightarrow [0, 1]$, $\gamma(0) = 0$, $\gamma(2) = 1$, a decreasing function $M\colon (0, 1] \longrightarrow (0, \infty)$, and a constant $\alpha > 1$ such that for each $\delta = (\delta_k)$, $\varepsilon = (\varepsilon_k) \in (0, 1]^m$ and $\tau = (\tau_k)$, $t = (t_k) \in \mathbb{R}_+^m$ the following estimate holds:*

$$\mu_m^X(\delta, \varepsilon; \tau, t) \leq \mu_m^{\tilde{X}}\left(\gamma(\delta) - \frac{M(\delta)}{\tau}, \gamma^{-1}(\varepsilon) + \frac{M(\varepsilon)}{\tau}; \frac{\tau}{\alpha}, \alpha t\right) \tag{22}$$

PROOF. We begin to manage the program, drafted in the end of previous section, with a choice of an infinite chain of positive integers

$$\begin{aligned} r_{m+1} < p_{m+1} < r'_{m+1} < r_m < p_m < r'_m < \cdots < r_0 < p_0 < r'_0 \\ < s'_0 < q_0 < s_0 < \cdots < s'_{m+1} < q_{m+1} < s_{m+1} \\ < n_1 < n_2 < \cdots < n_j < \cdots \end{aligned} \tag{23}$$

such that the following inclusions

$$\begin{aligned} &A_{p_k,q_k} \subset C\tilde{A}_{r_k,s_k},\ \tilde{A}_{r'_k,s'_k} \subset CA_{p_k,q_k}, \quad k = 0, 1, \ldots, m+1; \\ &A_{p_{m+1},n_j} \subset C_j\tilde{A}_{r_{m+1},n_{j+1}},\ \tilde{A}_{r'_{m+1},n_j} \subset C_jA_{p_{m+1},n_{j+1}}, \quad j \in \mathbb{N}, \end{aligned} \tag{24}$$

are valid with some constants $C = C(m)$, C_j, $j \in \mathbb{N}$. Without loss of generality, we can assume that each consequent number of the chain (23) is four times more than the preceding one and that the sequence n_j satisfies the condition $2r'_0 n_j < n_{j+1}$.

https://doi.org/10.4153/CJM-1998-055-0 Published online by Cambridge University Press

Let us fix the arguments δ, ε, τ, t in (22). With the aim of dealing with values of the parameters δ_k, ε_k from a given countable set, we use the following sequence

$$\zeta_0 = 1,\ \zeta_j = \frac{1}{n_j}, \quad j \in \mathbb{N}. \tag{25}$$

Thus we can take indices ν_k and j_k such that

$$\zeta_{\nu_k} \le \delta_k < \zeta_{\nu_k - 1},\ \zeta_{j_k+1} < \varepsilon_k \le \zeta_{j_k}, \quad k = 1, 2, \dots, m. \tag{26}$$

We can suppose that

$$\delta_1 \le \delta_2 \le \cdots \le \delta_m. \tag{27}$$

Now we define the sets serving as elementary blocks in the construction of the sets U, V, $\tilde{U}$, $\tilde{V}$. We begin with the first couple of the sets U, V. The estimates for λ_i from above and from below in (13) are linked, respectively, with the following two series of "interpolational" blocks for U ($k = 1, 2, \dots, m$):

$$W_1^{(k)} = \begin{cases} A^{\frac{1}{2}}_{p_{m+1}, n_{j_k}} A^{\frac{1}{2}}_{p_0, q_0} & \text{if } j_k > 2, \\ A_{p_0, q_0} & \text{if } j_k = 1, 2, \end{cases}$$

$$\bar{W}_1^{(k)} = A^{\frac{1}{2}}_{p_{m+1}, n_{\nu_k+1}} A^{\frac{1}{2}}_{p_0, q_0}.$$

The estimates of a_i by the parameters τ_k and t_k in (13) are connected with other series of blocks for U ($k = 1, 2, \dots, m$):

$$W_2^{(k)} = \exp\Big(\frac{p_0 \tau_k}{2}\Big) A_{p_0, q_0}, \quad \bar{W}_2^{(k)} = \exp\Big(-\frac{p_k \tau_k}{2}\Big) A_{p_{m+1}, q_{m+1}},$$

$$W_3^{(k)} = \exp(-q_{m+1} t_k) A_{p_{m+1}, q_{m+1}}, \quad \bar{W}_3^{(k)} = \exp(q_k t_k) A_{p_0, q_0}.$$

The following blocks serve both V and U:

$$W_4^{(k)} = A_{p_k, q_k},\ \bar{W}_4^{(k)} = A_{p_k, q_k}, \quad k = 1, 2, \dots, m.$$

Each above-mentioned block W is a weighted ℓ_1-ball $B^e(w)$ (we denote its weight by the same but small letter). For example, $\bar{W}_3^{(k)} = B^e(\bar{w}_3^{(k)})$.

To construct the set $\tilde{U}$ we also use two series of interpolational blocks, responsible for the estimates of $\tilde{\lambda}_i$ ($k = 1, 2, \dots, m$):

$$\tilde{W}_1^{(k)} = \begin{cases} \frac{1}{\sqrt{CC_{j_k-1}}} \tilde{A}^{\frac{1}{2}}_{r'_{m+1}, n_{j_k-1}} \tilde{A}^{\frac{1}{2}}_{r'_0, s'_0} & \text{if } j_k > 2, \\ \frac{1}{C} \tilde{A}_{r'_0, s'_0} & \text{if } j_k = 1, 2, \end{cases}$$

$$\tilde{\bar{W}}_1^{(k)} = \sqrt{CC_{\nu_k+1}} \tilde{A}^{\frac{1}{2}}_{r_{m+1}, n_{\nu_k+2}} \tilde{A}^{\frac{1}{2}}_{r_0, s_0}$$

and four series, associated with the estimates for $\tilde{a}_i$ ($k = 1, 2, \dots, m$):

$$\tilde{W}_2^{(k)} = \frac{1}{C} \exp\Big(\frac{p_0 \tau_k}{2}\Big) \tilde{A}_{r'_0, s'_0}, \quad \tilde{\bar{W}}_2^{(k)} = C \exp\Big(-\frac{p_k \tau_k}{2}\Big) \tilde{A}_{r_{m+1}, s_{m+1}},$$

$$\tilde{W}_3^{(k)} = \frac{1}{C} \exp(-q_{m+1} t_k) \tilde{A}_{r'_{m+1}, s'_{m+1}}, \quad \tilde{\bar{W}}_3^{(k)} = C \exp(q_k t_k) \tilde{A}_{r_0, s_0}.$$

Finally, we consider the blocks:

$$\tilde{W}_4^{(k)} = \frac{1}{C}\tilde{A}_{r'_k,s'_k},\ \tilde{\bar{W}}_4^{(k)} = C\tilde{A}_{r_k,s_k}, \quad k = 1,2,\dots,m.$$

Each above-defined set is again a weighted ℓ_1-ball in X, associated with the second basis f. To denote the corresponding weight we again repeat the name of the set with small letter w instead of W, for example, $\tilde{\bar{W}}_4^{(k)} = B^f(\tilde{\bar{w}}_4^{(k)})$.

Putting

$$U^{(k)} = \operatorname{conv}\Big(\bigcup_{l=1}^{4} W_l^{(k)}\Big), \quad \tilde{U}^{(k)} = \operatorname{conv}\Big(\bigcup_{l=1}^{4} \tilde{W}_l^{(k)}\Big), \quad V^{(k)} = \bigcap_{l=1}^{4} \bar{W}_l^{(k)}, \quad \tilde{V}^{(k)} = \bigcap_{l=1}^{4} \tilde{\bar{W}}_l^{(k)},$$

where $k = 1,2,\dots,m$, we are ready to define the sets

$$U = \bigcap_{k=1}^{m} U^{(k)}, \quad \tilde{U} = \bigcap_{k=1}^{m} \tilde{U}^{(k)}, \quad V = \operatorname{conv}\Big(\bigcup_{k=1}^{m} V^{(k)}\Big), \quad \tilde{V} = \operatorname{conv}\Big(\bigcup_{k=1}^{m} \tilde{V}^{(k)}\Big).$$

By the construction, from (24) and Proposition 2 we have the inclusions

$$W_l^{(k)} \supset \tilde{W}_l^{(k)},\ \bar{W}_l^{(k)} \subset \tilde{\bar{W}}_l^{(k)},\ l = 1,2,3,4; \quad k = 1,2,\dots,m.$$

Consequently, we get the inclusions (20) and then the estimate (21).

Unlike elementary blocks, the sets U, V, $\tilde{U}$ and $\tilde{V}$ are not weighted ℓ_1-balls; it is why Proposition 1 cannot be used directly for the calculation of $\beta(V,U)$ and $\beta(\tilde{V},\tilde{U})$. Still, using Proposition 3, we approximate these sets with some appropriate weighted ℓ_1-balls. To this end we consider the sequences: $c^{(k)} = (c_i^{(k)})$, $\tilde{c}^{(k)} = (\tilde{c}_i^{(k)})$, $d^{(k)} = (d_i^{(k)})$, $\tilde{d}^{(k)} = (\tilde{d}_i^{(k)})$, $k = 1,2,\dots,m$, and the sequences $c = (c_i)$, $\tilde{c} = (\tilde{c}_i)$, $d = (d_i)$, $\tilde{d} = (\tilde{d}_i)$, defined as follows:

$$\begin{aligned} c_i^{(k)} &= \min\{w_{i,l}^{(k)} : l = 1,2,3,4\}, & \tilde{c}_i^{(k)} &= \min\{\tilde{w}_{i,l}^{(k)} : l = 1,2,3,4\},\\ d_i^{(k)} &= \max\{\bar{w}_{i,l}^{(k)} : l = 1,2,3,4\}, & \tilde{d}_i^{(k)} &= \max\{\tilde{\bar{w}}_{i,l}^{(k)} : l = 1,2,3,4\},\\ c_i &= \min\{d_i^{(k)} : k = 1,2,\dots,m\}, & \tilde{c}_i &= \min\{\tilde{d}_i^{(k)} : k = 1,2,\dots,m\},\\ d_i &= \max\{c_i^{(k)} : k = 1,2,\dots,m\}, & \tilde{d}_i &= \max\{\tilde{c}_i^{(k)} : k = 1,2,\dots,m\}. \end{aligned}$$

Applying Proposition 3, we get

$$B^e(c^{(k)}) = U^{(k)}, \quad B^f(\tilde{c}^{(k)}) = \tilde{U}^{(k)}, \quad B^e(d^{(k)}) \subset V^{(k)}, \quad \tilde{V}^{(k)} \subset 4B^f(\tilde{d}^{(k)})$$

and then

$$B^e(c) \subset V, \quad U \subset mB^e(d), \quad \tilde{V} \subset 4B^f(\tilde{c}), \quad B^f(\tilde{d}) \subset \tilde{U}.$$

Therefore, using the relations (6), (7), we get

$$\beta\big(B^e(c), B^e(d)\big) \le \beta\Big(V, \frac{1}{m}U\Big), \quad \beta(\tilde{V},\tilde{U}) \le \beta\big(4B^f(\tilde{c}), B^f(\tilde{d})\big). \tag{28}$$

Combining (28), (21) and using (7), we obtain

$$\beta\big(B^e(c), B^e(d)\big) \leq \beta(4mB^f\big(\tilde{c}), B^f(\tilde{d})\big). \tag{29}$$

Now we are going to estimate the left side of this inequality from below. Namely, we prove the following inequality

$$\beta\big(B^e(c), B^e(d)\big) \geq \mu_m^X(\delta, \varepsilon; \tau, t). \tag{30}$$

By Proposition 1 we have

$$\beta\big(B^e(c), B^e(d)\big) = |\{i : c_i \leq d_i\}|.$$

Taking into account the definitions of the sequences c and d, we obtain

$$\beta\big(B^e(c), B^e(d)\big) = \Big|\bigcup_{k=1}^{m}\bigcup_{l=1}^{m}\{i : d_i^{(k)} \leq c_i^{(l)}\}\Big|.$$

Therefore the following estimate

$$\beta\big(B^e(c), B^e(d)\big) \geq \Big|\bigcup_{k=1}^{m}\{i : d_i^{(k)} \leq c_i^{(k)}\}\Big| \tag{31}$$

is true.

By the definition of the sequences $d^{(k)}$ and $c^{(k)}$, $k = 1, 2, \ldots, m$, we get

$$\{i : d_i^{(k)} \leq c_i^{(k)}\} = \big\{i : \max_{1\leq l\leq 4} \overline{w}_{i,l}^{(k)} \leq \min_{1\leq l\leq 4} w_{i,l}^{(k)}\big\}. \tag{32}$$

In view of the fact that $\overline{w}_4^{(k)} = w_4^{(k)}$ the set in the right-hand side of (32) can be written in the following form:

$$\{i : d_i^{(k)} \leq c_i^{(k)}\} = \bigcap_{l=1}^{3}\{i : \overline{w}_{i,l}^{(k)} \leq w_{i,4}^{(k)}, \overline{w}_{i,4}^{(k)} \leq w_{i,l}^{(k)}\}. \tag{33}$$

To prove the estimate (30) we need to bring out the following inclusions ($k = 1, 2, \ldots, m$):

$$\{i : \overline{w}_{i,1}^{(k)} \leq w_{i,4}^{(k)}\} \supset \{i : \lambda_i > \delta_k\}, \tag{34}$$

$$\{i : \overline{w}_{i,4}^{(k)} \leq w_{i,1}^{(k)}\} \supset \{i : \lambda_i \leq \varepsilon_k\}, \tag{35}$$

$$\{i : \overline{w}_{i,2}^{(k)} \leq w_{i,4}^{(k)}\} \supset \{i : a_i > \tau_k\}, \tag{36}$$

$$\{i : \overline{w}_{i,4}^{(k)} \leq w_{i,2}^{(k)}\} \supset \{i : a_i > \tau_k\}, \tag{37}$$

$$\{i : \overline{w}_{i,3}^{(k)} \leq w_{i,4}^{(k)}\} \supset \{i : a_i \leq t_k\}, \tag{38}$$

$$\{i : \overline{w}_{i,4}^{(k)} \leq w_{i,3}^{(k)}\} \supset \{i : a_i \leq t_k\}. \tag{39}$$

First we consider (34). By the definitions of the weights and the matrix $\big(a_i(p, q)\big)$ the inequality in the left member of (34) is equivalent to the following inequality

$$\frac{p_{m+1} + p_0}{2} - p_k \leq \lambda_i\Big(\frac{n_{\nu_k+1} + q_0}{2} - q_k\Big). \tag{40}$$

By the assumption about the chain (23) and by (25), (26) we have the following relations

$$\frac{p_{m+1}+p_0}{2}-p_k<\frac{p_0}{2},\qquad \frac{n_{\nu_k+1}+q_0}{2}-q_k>\frac{n_{\nu_k+1}}{4}>\frac{p_0 n_{\nu_k}}{2}=\frac{p_0}{2\zeta_{\nu_k}}>\frac{p_0}{2\delta_k}.$$

From here and (40) we gain the inclusion (34).

The inclusion (37) can be obtained by similar arguments in the case $j_k>2$, and it is trivial in the case $j_k\le 2$.

It remains only to check the inclusion (36), since the rest can be obtained similarly. The left-side inequality in (36) is equivalent to the inequality

$$\frac{p_k\tau_k}{2}\le[(p_k-p_{m+1})+\lambda_i(q_{m+1}-q_k)]a_i.$$

Since

$$(p_k-p_{m+1})+\lambda_i(q_{m+1}-q_k)>\frac{p_k}{2},$$

we get (36).

It follows from (33), (34)–(39) that

$$\{i: d_i^{(k)}\le c_i^{(k)}\}\supset\{i:\delta_k<\lambda_i\le\varepsilon_k,\tau_k<a_i\le t_k\}.$$

Combining this with (31), we obtain (30).

Now we begin to estimate the right side of the inequality (29) from above. The application of Proposition 1 yields

$$\beta\big(4mB^f(\tilde c),B^f(\tilde d)\big)=\big|\{i:\tilde c_i\le 4m\tilde d_i\}\big|.$$

Then, due to the definitions of the sequences $\tilde c$ and $\tilde d$, we have

$$\beta\big(4mB^f(\tilde c),B^f(\tilde d)\big)=\Big|\bigcup_{k=1}^{m}\bigcup_{l=1}^{m}\{i:\tilde d_i^{(k)}\le 4m\tilde c_i^{(l)}\}\Big|. \tag{41}$$

Let us take any $k,l=1,2,\ldots,m$. Using the definitions of the sequences $\tilde c^{(k)}$ and $\tilde d^{(l)}$, we deduce that

$$\{i:\tilde d_i^{(k)}\le 4m\tilde c_i^{(l)}\}\subset\bigcap_{\rho=1}^{3}\{i:\tilde{\tilde w}_{i,\rho}^{(k)}\le 4m\tilde w_{i,4}^{(l)},\tilde{\tilde w}_{i,4}^{(k)}\le 4m\tilde w_{i,\rho}^{(l)},\}\cap\{i:\tilde{\tilde w}_{i,4}^{(k)}\le 4m\tilde w_{i,4}^{(l)}\}. \tag{42}$$

Having regard to the expressions for $\tilde{\tilde w}_{i,1}^{(k)}$, $\tilde w_{i,4}^{(l)}$ and the form of matrix $\tilde a_{p,q}$, it is easy to see that the inequality

$$\tilde{\tilde w}_{i,1}^{(k)}\le 4m\tilde w_{i,4}^{(l)}, \tag{43}$$

is equivalent to the inequality

$$\frac{1}{\sqrt{CC_{\nu_k+1}}}\tilde a_{i,r_{m+1},n_{\nu_k+2}}^{\frac12}\tilde a_{i,r_0,s_0}^{\frac12}\le 4mC\tilde a_{i,r_l',s_l'}$$

https://doi.org/10.4153/CJM-1998-055-0 Published online by Cambridge University Press

which, in its turn, is equivalent to the inequality

$$\Big[\Big(\frac{r_{m+1}+r_0}{2}-r_l'\Big)-\tilde{\lambda}_i\Big(\frac{n_{\nu_k+2}+s_0}{2}-s_l'\Big)\Big]\tilde{a}_i \le \ln(4mC\sqrt{CC_{\nu_k+1}}).$$

By the choice of the chain (23) we have the following relations

$$\frac{r_{m+1}+r_0}{2}-r_l' > \frac{r_0}{4},\ \frac{n_{\nu_k+2}+s_0}{2}-s_l' < \frac{n_{\nu_k+2}}{2},\ \frac{r_0}{2n_{\nu_k+2}} > \frac{1}{n_{\nu_k+2}} = \zeta_{\nu_k+2}.$$

Therefore the inequality (43) is stronger than the inequality

$$\tilde{\lambda}_i \ge \zeta_{\nu_k+2} - \frac{2\zeta_{\nu_k+2}\ln(4mC^2_{\nu_k+1})}{\tilde{a}_i}.$$

Hence,

$$\{i : \tilde{w}^{(k)}_{i,1} \le 4m\tilde{w}^{(l)}_{i,4}\} \subset \Big\{i : \tilde{\lambda}_i > \zeta_{\nu_k+2} - \frac{2\zeta_{\nu_k+2}\ln(4mC^2_{\nu_k+1})}{\tilde{a}_i}\Big\}. \tag{44}$$

Using similar arguments we get the inclusion

$$\{i : \tilde{w}^{(k)}_{i,4} \le 4m\tilde{w}^{(l)}_{i,1}\} \subset \Big\{i : \tilde{\lambda}_i \le \zeta_{j_l-2} + \frac{4\zeta_{j_l-1}\ln(4mC^2_{j_l-1})}{\tilde{a}_i}\Big\}, \quad \text{if } j_l > 2. \tag{45}$$

In the case $j_l \le 2$, the inequality $\tilde{w}^{(k)}_{i,4} \le 4m\tilde{w}^{(l)}_{i,1}$ is equivalent to the inequality $\frac{1}{C}\tilde{a}_{i,r_k,s_k} \le 4mC\tilde{a}_{i,r_0',s_0'}$. Hence we get

$$[(r_k - r_0') + \tilde{\lambda}_i(s_0' - s_k)]\tilde{a}_i \le \ln(4mC^2). \tag{46}$$

Since $r_k < r_0'$ and $s_0' < s_k$, the inequality (46) holds for all $i \in \mathbb{N}$. Thus the inclusion

$$\{i : \tilde{w}^{(k)}_{i,4} \le 4m\tilde{w}^{(l)}_{i,1}\} \subset \{i : \tilde{\lambda}_i \le \zeta_0\} \tag{47}$$

holds if $j_l = 1, 2$.

Let us define an increasing function $\gamma\colon [0,2] \longrightarrow [0,1]$ so that

$$\gamma(0) = 0,\ \gamma(2) = 1,\ \gamma(\zeta_j) = \zeta_{j+3}, \quad j = 0, 1, \ldots$$

and a decreasing function $M\colon (0,1] \longrightarrow (0,\infty)$ so that

$$M(\zeta_j) \ge 2\alpha\zeta_{j+2}\ln(4mC^2_{j+1}), \quad j = 1, 2;$$
$$M(\zeta_j) \ge 2\alpha\max\{\zeta_{j+2}\ln(4mC^2_{j+1}),\ 2\zeta_{j-1}\ln(4mC^2_{j-1})\}, \quad j = 3, 4, \ldots.$$

Then from (44), (45) and (47) it follows that

$$\{i : \tilde{w}^{(k)}_{i,1} \le 4m\tilde{w}^{(l)}_{i,4}\} \subset \Big\{i : \tilde{\lambda}_i > \gamma(\zeta_{\nu_k-1}) - \frac{M(\zeta_{\nu_k})}{\alpha\tilde{a}_i}\Big\},$$
$$\{i : \tilde{w}^{(k)}_{i,4} \le 4m\tilde{w}^{(l)}_{i,1}\} \subset \Big\{i : \tilde{\lambda}_i \le \gamma^{-1}(\zeta_{j_l+1}) + \frac{M(\zeta_{j_l})}{\alpha\tilde{a}_i}\Big\}.$$

https://doi.org/10.4153/CJM-1998-055-0 Published online by Cambridge University Press

Hence, bringing to mind (26), we obtain

$$\{i : \tilde{\tilde{w}}_{i,1}^{(k)} \le 4m\tilde{w}_{i,4}^{(l)}\} \subset \left\{i : \tilde{\lambda}_i > \gamma(\delta_k) - \frac{M(\delta_k)}{\alpha\tilde{a}_i}\right\}, \tag{48}$$

$$\{i : \tilde{\tilde{w}}_{i,4}^{(k)} \le 4m\tilde{w}_{i,1}^{(l)}\} \subset \left\{i : \tilde{\lambda}_i \le \gamma^{-1}(\varepsilon_l) + \frac{M(\varepsilon_l)}{\alpha\tilde{a}_i}\right\}. \tag{49}$$

It will be shown below that any constant satisfying the following condition:

$$\alpha > \max\{\ln(4mC^2), s_{m+1}\} \tag{50}$$

can be taken as a constant α in (22). First we note that the following inclusions hold:

$$\{i : \tilde{\tilde{w}}_{i,2}^{(k)} \le 4m\tilde{w}_{i,4}^{(l)}\} \subset \left\{i : \tilde{a}_i > \frac{\tau_k}{\alpha}\right\}, \tag{51}$$

$$\{i : \tilde{\tilde{w}}_{i,4}^{(k)} \le 4m\tilde{w}_{i,2}^{(l)}\} \subset \left\{i : \tilde{a}_i > \frac{\tau_l}{\alpha}\right\}, \tag{52}$$

$$\{i : \tilde{\tilde{w}}_{i,3}^{(k)} \le 4m\tilde{w}_{i,4}^{(l)}\} \subset \{i : \tilde{a}_i \le \alpha t_k\}, \tag{53}$$

$$\{i : \tilde{\tilde{w}}_{i,4}^{(k)} \le 4m\tilde{w}_{i,3}^{(l)}\} \subset \{i : \tilde{a}_i \le \alpha t_l\}. \tag{54}$$

We prove only the inclusion (51), since the rest can be obtained analogously. Having regard to the concrete form of the weights, we see that the inequality in the left-hand side of (51) is equivalent to the inequality

$$\frac{p_k\tau_k}{2} \le \ln(4mC^2) + [(r_l' - r_{m+1}) + \tilde{\lambda}_i(s_{m+1} - s_l')]\tilde{a}_i. \tag{55}$$

Taking into account (23), (50) and the assumption $\tilde{a}_i \ge 1$ (see subsection 2.5), we get that the inequality (55) remains true after replacing its right-hand side by $2\alpha\tilde{a}_i$. Since, by (23), $p_k > 4$, we get (51).

Using the notation: $T_1 = \max\{\tau_k, \tau_l\}$ and $T_2 = \min\{t_k, t_l\}$, after combining (51)–(54), (48) and (49), we obtain

$$\bigcap_{\rho=1}^{3}\{i : \tilde{\tilde{w}}_{i,\rho}^{(k)} \le 4m\tilde{w}_{i,4}^{(l)}, \tilde{\tilde{w}}_{i,4}^{(k)} \le 4m\tilde{w}_{i,\rho}^{(l)}, \} \subset D_{k,l}, \tag{56}$$

where

$$D_{k,l} = \left\{i : \gamma(\delta_k) - \frac{M(\delta_k)}{T_1} < \lambda_i \le \gamma^{-1}(\varepsilon_l) + \frac{M(\varepsilon_l)}{T_1}; \frac{T_1}{\alpha} < \tilde{a}_i \le \alpha T_2\right\}.$$

Taking into account the definitions of the sequences $\tilde{\tilde{w}}_4^{(k)}, \tilde{w}_4^{(l)}$, and of the matrix $\left(\tilde{a}_i(p,q)\right)$ we have

$$\{i : \tilde{\tilde{w}}_{i,4}^{(k)} \le 4m\tilde{w}_{i,4}^{(l)}\} \subset R_{k,l}, \tag{57}$$

where

$$R_{k,l} = \{i : [(r_k - r_l') + \tilde{\lambda}_i(s_l' - s_k)]\tilde{a}_i \le \alpha\}. \tag{58}$$

https://doi.org/10.4153/CJM-1998-055-0 Published online by Cambridge University Press

Combining (41), (42), (56) and (57) we obtain

$$\beta\big(4mB^f(\tilde{c}), B^f(\tilde{d})\big) \le \Big|\bigcup_{k=1}^{m}\bigcup_{l=1}^{m}(D_{k,l}\cap R_{k,l})\Big|. \tag{59}$$

Since, by (24), the left-hand side of the inequality in (58) is negative as $k \ge l$, we have

$$R_{k,l} = \mathbb{N} \quad \text{if } k \ge l.$$

On the other hand, for $k < l$, by (24),

$$(r_k - r_l') + \tilde{\lambda}_i(s_l' - s_k) > 1.$$

Hence,

$$R_{k,l} \subset \{i : \tilde{a}_i \le \alpha\} \quad \text{if } k < l. \tag{60}$$

Let us prove now the inclusion

$$D_{k,l}\cap R_{k,l} \subset D_{l,l}. \tag{61}$$

Since this relation is trivial if either the left-hand side of it is empty or $k = l$, we can assume that $k \ne l$ and

$$D_{k,l}\cap R_{k,l} \ne \emptyset. \tag{62}$$

For convenience we put

$$\Delta_k = \gamma(\delta_k) - \frac{M(\delta_k)}{\tau_k}, \quad E_l = \gamma^{-1}(\varepsilon_l) + \frac{M(\varepsilon_l)}{\tau_l},$$
$$\Delta_{k,l} = \gamma(\delta_k) - \frac{M(\delta_k)}{T_1}, \quad E_{l,k} = \gamma^{-1}(\varepsilon_l) + \frac{M(\varepsilon_l)}{T_1},$$

where $k, l = 1, 2, \ldots, m$. It is clear that

$$\Delta_k \le \Delta_{k,l},\ E_{l,k} \le E_l, \quad k, l = 1, 2, \ldots, m. \tag{63}$$

By (62) we have

$$\Delta_{k,l} < E_{l,k}, \quad \frac{T_1}{\alpha} < \alpha T_2. \tag{64}$$

Suppose, first, $k > l$; then, by assumption (27), $\delta_k \ge \delta_l$. From the definitions of the functions γ and M it follows that $\gamma(\delta_l) \le \gamma(\delta_k)$, $M(\delta_l) \ge M(\delta_k)$. Hence,

$$\Delta_{l,k} = \gamma(\delta_l) - \frac{M(\delta_l)}{T_1} \le \gamma(\delta_k) - \frac{M(\delta_k)}{T_1} = \Delta_{k,l}.$$

From here and (63), (64) it follows that

$$\Delta_l \le \Delta_{l,k} \le \Delta_{k,l} < E_{l,k} \le E_l, \quad \frac{\tau_l}{\alpha} \le \frac{T_1}{\alpha} < \alpha T_2 \le \alpha t_l.$$

Therefore we get (61) if $k > l$.

https://doi.org/10.4153/CJM-1998-055-0 Published online by Cambridge University Press

Now suppose $k < l$; then (62) means that (60), (64) and

$$\frac{T_1}{\alpha} < \alpha \tag{65}$$

hold. Since, as suggested, $\tilde{\lambda}_i \geq \frac{1}{\tilde{a}_i}$ for all $i \in \mathbb{N}$, it follows from (65) that $\tilde{\lambda}_i \geq \frac{1}{\alpha}$. Hence, taking into account (63) and the definitions of the numbers T_1 and T_2, we have

$$D_{k,l} \cap R_{k,l} \subset \left\{ i : \frac{1}{\alpha} \leq \tilde{\lambda}_i \leq E_l, \frac{\tau_l}{\alpha} < \tilde{a}_i \leq \alpha t_l \right\}.$$

On the other hand, by the definitions of γ and Δ_l, we have

$$\Delta_l < \gamma(\delta_l) < \gamma(\zeta_0) = \zeta_3 = \frac{1}{n_3}. \tag{66}$$

Since the constant α, depends only on m, we can assume the number n_3 chosen so that

$$\frac{1}{n_3} \leq \frac{1}{\alpha}. \tag{67}$$

Taking into account (66) and (67), we get (61) in the case $k < l$ as well. Thus the relation (61) is proved. Together with (59) it gives the relation

$$\beta\left(4mB^f(\tilde{c}), B^f(\tilde{d})\right) \leq \left| \bigcup_{k=1}^{m} D_{k,k} \right|. \tag{68}$$

This completes the proof of Lemma.

THEOREM 9. *Let $X = G(\lambda, a)$, $\tilde{X} = G(\tilde{\lambda}, \tilde{a})$, $m \in \mathbb{N}$. If $X \simeq \tilde{X}$, then $\mu_m^X \approx \mu_m^{\tilde{X}}$.*

PROOF. Because of symmetry we need to prove only the first inequality (15). Let us rewrite it, using (14), in the form:

$$\left| \left\{ i : (\lambda_i, a_i) \in \bigcup_{k=1}^{m} P_k \right\} \right| \leq \left| \left\{ i : (\tilde{\lambda}_i, \tilde{a}_i) \in \bigcup_{k=1}^{m} Q_k \right\} \right|, \tag{69}$$

where

$$Q_k = \left(\varphi(\delta_k), \varphi^{-1}(\varepsilon_k)\right] \times \left(\frac{\tau_k}{\alpha}, \alpha t_k\right], \quad k = 1, 2, \ldots, m. \tag{70}$$

We cover each rectangle P_k by an appropriate couple of nonintersecting rectangles P_k' and P_k'' (some of them may be empty) so as to apply Lemma 8 to this new system of $2m$ rectangles. For construction of above-mentioned rectangles we need to define the increasing function $\Psi\colon (0, 1] \longrightarrow (1, +\infty)$ so that

$$\Psi(\xi) > \max\left\{ \frac{2M(\xi)}{\gamma(\xi)}, \frac{1}{2\xi} \right\}, \tag{71}$$

where M and γ are as in Lemma 8, but considered with $2m$ instead of m. By ξ_0 we define the number $\frac{1}{2\Psi(1)}$.

https://doi.org/10.4153/CJM-1998-055-0 Published online by Cambridge University Press

We are acting by different ways in the following three cases:
(a) $\tau_k \geq \Psi(\delta_k)$; (b) $\tau_k < \Psi(\delta_k) < t_k$; (c) $t_k \leq \Psi(\delta_k)$.
Setting the notation

$$\tau_k' := \max\{\Psi(\delta_k), \tau_k\},\ t_k' := \min\{\Psi(\delta_k), t_k\},\ \varepsilon_k' = \begin{cases} \max\{\varepsilon_k, \Psi^{-1}(\tau_k)\}, & \text{if } \varepsilon_k < \xi_0, \\ 1 & \text{otherwise,} \end{cases}$$

we put

$$P_k' = \begin{cases} \emptyset & \text{in the case (c),} \\ (\delta_k, \varepsilon_k] \times (\tau_k', t_k] & \text{otherwise} \end{cases}$$

and

$$P_k'' = \begin{cases} \emptyset & \text{in the case (a),} \\ (\delta_k, \varepsilon_k'] \times (\tau_k, t_k'] & \text{otherwise.} \end{cases}$$

Applying Lemma 8, we get

$$\left|\bigcup_{k=1}^{m}(P_k' \cup P_k'')\right| \leq \left|\bigcup_{k=1}^{m}(\tilde{P}_k' \cup \tilde{P}_k'')\right|,$$

with

$$\tilde{P}_k' = \begin{cases} \emptyset, & \text{in the case (c),} \\ (\Delta_k, E_k] \times (\frac{\tau_k}{\alpha}, \alpha t_k] & \text{otherwise} \end{cases}$$

and

$$\tilde{P}_k'' = \begin{cases} \emptyset, & \text{in the case (a),} \\ (\Delta_k, E_k'] \times (\frac{\tau_k}{\alpha}, \alpha t_k] & \text{otherwise,} \end{cases}$$

where

$$\Delta_k = \gamma(\delta_k) - \frac{M(\delta_k)}{\tau_k}, \quad E_k = \gamma^{-1}(\varepsilon_k) + \frac{M(\varepsilon_k)}{\tau_k}, \quad E_k' = \gamma^{-1}(\varepsilon_k') + \frac{M(\varepsilon_k')}{\tau_k}.$$

From $\tilde{\lambda}_i \geq \frac{1}{\tilde{a}_i}$ and (71) it follows that

$$\{i : (\tilde{\lambda}_i, \tilde{a}_i) \in \tilde{P}_k''\} \subset \left\{i : (\tilde{\lambda}_i, \tilde{a}_i) \in \left(\frac{1}{2\alpha\Psi(\delta_k)}, E_k'\right] \times \left(\frac{\tau_k}{\alpha}, \alpha t_k\right]\right\}. \tag{72}$$

The required function φ can be defined now so that $\varphi(1) \leq \xi_0$ and

$$\varphi(\xi) \leq \min\left\{\frac{1}{2}\gamma(\xi), \frac{1}{2\alpha\Psi(\xi)}\right\}$$

if $0 < \xi < \xi_0$. Then, taking into account (72), we get

$$\{i : (\tilde{\lambda}_i, \tilde{a}_i) \in \tilde{P}_k' \cup \tilde{P}_k''\} \subset \{i : (\tilde{\lambda}_i, \tilde{a}_i) \in Q_k\},$$

where Q_k is defined in (70). Thus (69) is proved.

https://doi.org/10.4153/CJM-1998-055-0 Published online by Cambridge University Press

6**. Weak multirectangular invariant.** We say that the systems of characteristics $(\mu_m^{(\lambda,a)})_{m\in\mathbb{N}}$ and $(\mu_m^{(\tilde\lambda,\tilde a)})_{m\in\mathbb{N}}$ are *weakly equivalent* and write

$$(\mu_m^{(\lambda,a)}) \overset{w}{\approx} (\mu_m^{(\tilde\lambda,\tilde a)}) \tag{73}$$

if in the definition of the equivalency $(\mu_m^{(\lambda,a)}) \approx (\mu_m^{(\tilde\lambda,\tilde a)})$ (see Section 4) it is demanded only that the inequalities (15), (16) are valid under the additional restrictions that

$$\varepsilon_k = 1 \quad \text{for each } k = 1, 2, \ldots, m. \tag{74}$$

With so defined equivalency the system

$$(\mu_m^X)_{m\in\mathbb{N}} := (\mu_m^{(\lambda,a)})_{m\in\mathbb{N}}$$

is a linear topological invariant on the class of all spaces $X = G(\lambda, a)$ as it runs out from the following

THEOREM 10. *Let $X = G(\lambda, a)$ and $\tilde X = G(\tilde\lambda, \tilde a)$. Then $X \simeq \tilde X$ implies (73).*

PROOF. Let us analyze the proof of Lemma 8 to fit it to the considered particular case. First we come to recognize that, under the assumption (74), one can escape the most complicated considerations in that proof, connected with the estimation of the right side of (59) by the right side of (68). Therefore we can use the following, shorter than (23), chain of norms' numbers:

$$r_2 < p_2 < r_2' < r_1 < \cdots < s_1 < s_2' < q_2 < s_2 < n_1 < n_2 < \cdots < n_j < \cdots$$

The consequent simplification is that whenever in the proof of Lemma 8 any number r, p, r', s', q, s is indexed by k we have to change its index to 1 if $k = 1, 2, \ldots, m$ and to 2 if $k = m + 1$. By the assumptions (74), (8) we can omit everywhere the blocks, which are responsible for estimates of λ from above. Finally, we consider the following expression for the constructions U, V and $\tilde U$, $\tilde V$:

$$U = A_{p_1,q_1}, \quad V = \operatorname{conv}\Big(\bigcup_{k=1}^{m}\bigcap_{l=1}^{3} \bar W_l^k\Big), \quad \tilde U = \frac{1}{C}\tilde A_{r_1',s_1'}, \quad \tilde V = \operatorname{conv}\Big(\bigcup_{k=1}^{m}\bigcap_{l=1}^{3} \bar{\tilde W}{}_l^k\Big).$$

Therefore it is easy to check that both the function φ and the constant α can be defined as independent of m, what completes the proof.

7**. Applications.** In this section we sketch some applications of multirectangular invariants to the spaces (1). More detailed consideration of such applications will be the object of another paper.

7.1. *Cartesian products.* By analogy with [12] (see also [54, 19, 20, 11]), it can be got the following complete isomorphic classification on the subclass of all spaces (1), isomorphic to Cartesian products of spaces (3) and (4). Therewith, as in [12], the notation $X^{(s)}$ means any subspace of codimension s if $s \geq 0$ in X or any space isomorphic to $X \times \mathbb{R}^{-s}$ if $s < 0$.

https://doi.org/10.4153/CJM-1998-055-0 Published online by Cambridge University Press

THEOREM 11. *Let* $X = G(\lambda, a) \simeq E_\infty(c) \times E'_\infty(d)$ *and* $\tilde{X} = G(\tilde{\lambda}, \tilde{a}) \simeq E_\infty(\tilde{c}) \times E'_\infty(\tilde{d})$. *Then the following statements are equivalent:*

(i) $X \simeq \tilde{X}$;

(ii) $X \overset{qd}{\simeq} \tilde{X}$;

(iii) there exists an integer s such that

$$E_\infty(c) \simeq \big(E_\infty(\tilde{c})\big)^{(s)}, \quad \big(E_\infty(d)\big)^{(s)} \simeq E_\infty(\tilde{d});$$

(iv) $\mu_2^X \approx \mu_2^{\tilde{X}}$.

7.2. *Tensor products.* Each tensor product

$$X = E_\infty(c) \hat{\otimes} E'_\infty(d),$$

is quasidiagonally isomorphic to a space $Y = G(\lambda, a)$ if we put $a_i := \max\{c_{r(i)}, d_{s(i)}\}$ and $\lambda_i := \frac{d_{s(i)}}{a_i}$, where $i \longrightarrow \big(r(i), s(i)\big)$ is any bijection from $\mathbb{N}$ onto $\mathbb{N} \times \mathbb{N}$. Since μ_m^Y does not depend on above-mentioned bijections, we can consider this invariant as defined for X, so that

$$\begin{aligned}\mu_m^X(\delta, \varepsilon; \tau, t) &:= \mu_m^Y(\delta, \varepsilon; \tau, t)\\ &= \left|\bigcup_{k=1}^m \left\{(r,s) : \delta_k < \frac{d_{s(i)}}{\max\{c_{r(i)}, d_{s(i)}\}} \le \varepsilon_k, \tau_k < \max\{c_{r(i)}, d_{s(i)}\} \le t_k\right\}\right|.\end{aligned}$$

Some necessary and sufficient conditions of the isomorphism of tensor products (2) were considered in [27] (see also [28, 26]) by the use of one-rectangular invariants. Here we describe some classes of spaces (2) such that two-rectangular invariant is complete on each class, but one-rectangular invariant fails to be complete on certain of them.

Let a sequence $c = (c_i)$ satisfy the condition: there exists an increasing sequence i_ν, $\nu \in \mathbb{N}$ such that

$$\frac{c_{i_{\nu+1}}}{c_{i_\nu+1}} \uparrow \infty, \quad \frac{c_{i_\nu+1}}{c_{i_\nu}} \uparrow \infty,$$

as $\nu \longrightarrow \infty$. Let $\mathcal{E}_c$ be the class of all spaces $X_d = E_\infty(c) \hat{\otimes} E'_\infty(d)$, where $d = (d_j)$ is any non-decreasing tending to ∞ sequence which has no point in any interval $(c_{i_\nu+1}, c_{i_{\nu+1}})$, $\nu \in \mathbb{N}$.

PROPOSITION 12. *Let* $X_d, X_{\tilde{d}}$ *belong to some class* $\mathcal{E}_c$. *Then the following statements are equivalent:*

(i) $X_d \simeq X_{\tilde{d}}$,

(ii) $d \asymp \tilde{d}$,

(iii) $\mu_2^{X_d} \approx \mu_2^{X_{\tilde{d}}}$.

PROOF. Since (i) ⇒ (iii) follows from Theorem 9 and (ii) ⇒ (i) is trivial, we need to prove (iii) ⇒ (ii) only.

Let (iii) be true. Then, in the framework of the equivalency definition, the estimate (15), (16) hold for $m = 2$ with some function φ and constant α. For given $\nu \in \mathbb{N}$ we put

https://doi.org/10.4153/CJM-1998-055-0 Published online by Cambridge University Press

$\delta(\nu) := \big(\delta_1(\nu), \delta_2(\nu)\big)$, $\varepsilon(\nu) := \big(\varepsilon_1(\nu), \varepsilon_2(\nu)\big)$, $\tau(\nu) := \big(\tau_1(\nu), \tau_2(\nu)\big)$, $t(\nu) := \big(t_1(\nu), t_2(\nu)\big)$, where

$$\delta_1(\nu) = \delta_2(\nu) := \frac{d_1}{2c_{i_\nu}}; \quad \varepsilon_1(\nu) := \frac{c_{i_{\nu-1}+1} + c_{i_\nu}}{2c_{i_\nu}}, \quad \varepsilon_2(\nu) := 1;$$
$$\tau_1(\nu) = \tau_2(\nu) := 0; \quad t_1(\nu) := c_{i_\nu}, \quad t_2(\nu) := \frac{c_{i_{\nu-1}+1} + c_{i_\nu}}{2}.$$

Taking ν_0 such that $\alpha t_2(\nu_0) < t_1(\nu_0)$ and $\varphi(1) > \varepsilon_1(\nu_0)$, we obtain that two inequalities (15), (16) imply the following equality

$$\mu_2^{X_d}\big(\delta(\nu), \varepsilon(\nu); \tau(\nu), t(\nu)\big) = \mu_2^{X_{\tilde d}}\big(\delta(\nu), \varepsilon(\nu); \tau(\nu), t(\nu)\big)$$

for $\nu \geq \nu_0$. From here, after some elementary computations, we get

$$m_c\big(t_1(\nu)\big) m_d\big(t_2(\nu)\big) = m_c\big(t_1(\nu)\big) m_{\tilde d}\big(t_2(\nu)\big)$$

and consequently

$$m_d\big(t_2(\nu)\big) = m_{\tilde d}\big(t_2(\nu)\big), \quad \nu \geq \nu_0. \tag{75}$$

Applying the inequalities (15), (16) for $m = 1$ and with the following values of parameters: $\varepsilon(\nu) := 1$, $\tau(\nu) := \alpha t_2(\nu - 1)$, $\delta(\nu) := \varepsilon_1(\nu_0)$, and any t such that

$$\tau(\nu) < t < \frac{a_{i_{\nu-1}+1}}{\alpha}, \tag{76}$$

we get for $\nu > \nu_0$

$$m_d(t) - m_d\big(\tau(\nu)\big) \leq m_{\tilde d}(\alpha t) - m_{\tilde d}\big(\tau(\nu)\big) \tag{77}$$

if t satisfies (76). It can easily be shown that (75) and (77) imply (ii).

The following example disclosed that one-rectangular invariant is not complete on certain classes E_c.

EXAMPLE. Let $(n_\nu)_0^\infty$ be a sequence of integer such that $n_0 = 0$ and $\frac{n_{\nu+1}}{n_\nu} \uparrow \infty$. Put

$$C := \bigcup_{\nu=0}^{\infty} \{l \in \mathbb{N} : n_{2\nu+1} < l \leq n_{2(\nu+1)}\}, \quad D := \bigcup_{\nu=0}^{\infty} \{l \in \mathbb{N} : n_{2\nu} < l \leq n_{2\nu+1}\}.$$

We define now sequences c, d, $\tilde d$ as follows: $c = (c_i)$ is a sequence which is obtained by enumeration of the set C; $d = (d_j)$ is a sequence such that the set of its elements coincides with D and any number $n_{2\nu+1}$ occurs in it not less that $n_{2\nu+2}$ times, while each of others has to be solitary; finally, $\tilde d = (\tilde d_j)$ is such that $\tilde d_j = d_{j+1}$. One can check that for $m = 1$ the inequalities (15) and (16) hold with $\alpha = 2$ and any function $\varphi\colon [0, 2] \longrightarrow [0, 1]$, satisfying the following conditions:

$$\varphi(\lambda) \leq \frac{\lambda}{2}, \ \lambda \in [0, 2]; \quad \varphi\Big(\frac{n_{2\nu+1}}{n_{2\nu+2}}\Big) < \frac{1}{n_{2\nu+2}}.$$

By construction the sequences d and $\tilde d$ are not weakly equivalent. Thus from Proposition 12 it follows that the spaces X_d and $X_{\tilde d}$ are not isomorphic.

PROBLEM 2. *Is the two-rectangular invariant complete on the class* $E := \bigcup_c E_c$, *where c runs the set of all c considered in Example?*

PROBLEM 3. *Does there exist m such that* μ_m *is complete invariant on the class of tensor products of kind (2)?*

It is worth noting that for the class of all spaces of kind (1) any m-rectangular invariant is not complete and, moreover, each $m+1$-rectangular invariant is properly stronger than m-rectangular one: to be certain of this it is sufficient to consider two spaces $G(\lambda, a)$, $G(\tilde{\lambda}, \tilde{a})$ with (λ, a) and $(\tilde{\lambda}, \tilde{a})$, constructed in [16] as proving Theorem 4 there.

7.3. *Application of weak multirectangular invariant.* In studies of power Köthe spaces of the first type ([16], Theorem 5) it was constructed two sequences (λ, a) and $(\tilde{\lambda}, \tilde{a})$. Applying the same considerations to spaces (1), we obtain that the corresponding pair of spaces $X = G(\lambda, a)$ and $\tilde{X} = G(\tilde{\lambda}, \tilde{a})$ possesses the following properties:

(a) $\mu_m^X \approx \mu_m^{\tilde{X}}$, $m \in \mathbb{N}$;
(b) $(\mu_m^X) \not\approx (\mu_m^{\tilde{X}})$.

Thus the spaces X and $\tilde{X}$ cannot be distinguished with any m-rectangle characteristic although they are not quasidiagonally isomorphic (by Theorem 7); the question about the isomorphism of these spaces (and the corresponding power Köthe spaces from [16]) could not be solved in terms of the equivalence $(\mu_m^X) \approx (\mu_m^{\tilde{X}})$ until its invariance will be proved. However the weak rectangular invariant, introduced in Section 6, is sufficient to answer this question. Indeed, it is easy to check that for these spaces the relation $(\mu_m^X) \overset{w}{\approx} (\mu_m^{\tilde{X}})$ fails and, by Theorem 10, we have $X \not\simeq \tilde{X}$. The same fact takes place also for the above-mentioned power Köthe spaces. In fact, the weak multirectangular invariant can be applied to some quite wide class of spaces (1) (as well as of power Köthe spaces).

In conclusion let us note that there are some more intricate examples showing that the week multirectangular invariant is not complete on the class (1).

7.4. *Quasiequivalence of bases.* After Dragilev's result [22] on quasiequivalence of bases in the space of all analytic functions in the unit disc it arose the problem on quasiequivalence of bases in locally convex spaces [36]. Referring to [24, 25, 33, 35, 36, 66, 1] for information about preceding results, we display only some results concerning with the class of spaces (1).

PROPOSITION 13. *Let a space* $X = G(\lambda, a)$ *satisfy one of two conditions:*

(i) $X \simeq E_\infty(c) \times E_\infty^{'}(d)$,
(ii) $X \overset{qd}{\simeq} X^2$.

Then all absolute bases in X are pairwise quasiequivalent.

To prove this proposition we have to apply basically the same considerations as in [57] for the case (i) or as in [66] for the second case. Therewith we use the relation

(i) ⇔ (ii) from Theorem 11 in the first case and the following statement in the latter case

PROPOSITION 14. *Let* $X = G(\lambda, a)$ *and* $X \overset{qd}{\simeq} X^2$. *Then* $X \simeq \tilde{X} \Leftrightarrow X \overset{qd}{\simeq} \tilde{X}$.

PROOF. The inclusion ⇐ is trivial. To prove the inverse inclusion one can check that one-rectangular invariant is complete on the class of all spaces (1) such that $X \overset{qd}{\simeq} X^2$ (basically in the same manner as in [27], Theorems 3.8, 3.9 and their proofs).

REFERENCES

1. A. Aytuna, P. B. Djakov, A. P. Goncharov, T. Terzioğlu and V. P. Zahariuta, *Some open problems in the theory of locally convex spaces*. In: Linear Topological Spaces and Complex Analysis, I, METU-TÜBITAK, Ankara, 1994, 147–165.

2. V. I. Baran, *Quasiequivalence of absolute bases in Cartesian products of some Köthe spaces*. In: Qualitative and approximative methods for operator equations, Yaroslavl **2**, 1977, 8–24 (in Russian).

3. ———, *On quasiequivalence of absolute bases in Cartesian products of Köthe spaces*. In: Actual Problems of Mathematical Analysis, Rostov State University, 1978, 13–21 (in Russian).

4. J. Bergh and J. Löfström, *Interpolation spaces*. Springer-Verlag, 1976.

5. Cz. Bessaga, *Some remarks on Dragilev's theorem*. Studia Math. **31**(1968), 307–318.

6. ———, *Geometrical methods of the theory of Fréchet spaces*. World Scientific, Singapore, New Jersey, Hong Kong, 1986.

7. Cz. Bessaga and A. Pełczynski, *Approximative dimension of linear topological spaces and some its applications*. In: Reports of Conference on Functional Analysis, Warsaw, 1960.

8. Cz. Bessaga, A. Pełczynski and S. Rolewicz, *Some properties of the space S*. Colloq. Math. **7**(1959), 45–51.

9. ———, *On diametral approximative dimension and linear homogeneity of F-spaces*. Bull. Polish Acad. Sci. **9**(1961), 677–683.

10. P. A. Chalov, *Triples of Hilbert spaces*. Manuscript 1387–81 Dep., deposited at VINITI, 1981 (in Russian).

11. P. A. Chalov, P. B. Djakov, T. Terzioğlu and V. P. Zahariuta, *On Cartesian products of locally convex spaces*. In: Linear Topological Spaces and Complex Analysis, II, METU-TÜBITAK, Ankara, 1995, 9–33.

12. P. A. Chalov, P. B. Djakov and V. P. Zahariuta, *Compound invariants and embeddings of Cartesian products*. preprint.

13. P. A. Chalov and V. P. Zahariuta, *On linear topological invariants*. Manuscript No. 5941-85, deposited at VINITI, 1985 (in Russian).

14. ———, *On linear topological invariants on some class of families of Hilbert spaces*. Manuscript No.3862-B 86, deposited at VINITI, 1986 (in Russian).

15. ———, *On uniqueness of unconditional basis in families of Banach spaces*. Preprint Series in Pure and Applied Math. **26**, Marmara Research Center, 1995.

16. ———, *On quasi-diagonal isomorphisms of generalized power spaces*. In: Linear Topological Spaces and Complex Analysis, II, METU-TÜBITAK, Ankara, 1995, 35–44.

17. L. Crone and W. Robinson. *Every nuclear Fréchet space with a regular basis has the quasi-equivalence property*. Studia Math. **52**(1974), 203–207.

18. P. B. Djakov, *A short proof of the theorem on quasi-equivalence of regular bases*. Studia Math. **55**(1975), 269–271.

19. P. B. Djakov, M. Yurdakul and V. P. Zahariuta, *On Cartesian products of Köthe spaces*. Bull. Polish Acad. Sci. Math. **43**(1995), 113–117.

20. ———, *Isomorphic classification of Cartesian products of power series spaces*. Michigan Math. J. **43**(1996), 221–229.

21. P. B. Djakov and V. P. Zahariuta, *On Dragilev-type power Köthe spaces*. Studia Math. **120**(1996), 219–234.

22. M. M. Dragilev, *The canonical form of a basis in a space of analytic functions*. Uspekhi Mat. Nauk **15**(1960), 181–188 (in Russian).

23. ———, *On regular bases in nuclear spaces*. Mat. Sb. **68**(1965), 153–173 (in Russian).

24. ______, *Bases in Köthe spaces*. Rostov State University, Rostov-on-Don, 1983 (in Russian).

25. E. Dubinsky, *The structure of nuclear Fréchet spaces*. Lecture Notes in Math. **720**, 1979.

26. A. Goncharov, T. Terzioğlu and V. Zahariuta, *On isomorphic classification of spaces* $s\hat{\otimes}E'_\infty(a)$. In: Linear Topological Spaces and Complex Analysis, I, METU-TÜBITAK, Ankara, 1994, 14–24.

27. ______, *On isomorphic classification of tensor products* $E_\infty(a)\hat{\otimes}E'_\infty(b)$. Dissertationes Math., CCCL, Warszawa, 1996, 1–27.

28. A. P. Goncharov and V. P. Zahariuta, *Linear topological invariants for tensor products of power F- and* DF-*spaces*. Turkish J. Math. **19**(1995), 90–101.

29. A. Grothendieck, *Produits tensoriels topologiques et espaces nucléaires*. Mem. Amer. Math. Soc. **16**(1955).

30. M. Kocatepe and V. Zahariuta, *Köthe spaces modeled on spaces of* C^∞ *functions*. Studia Math. **121**(1996), 1–14.

31. A. N. Kolmogorov, *On the linear dimension of vector topological spaces*. Dokl. Acad. Nauk SSSR **120**(1958), 239–241 (in Russian).

32. V. P. Kondakov, *On quasi-equivalence of regular bases in Köthe spaces*. Mat. Anal. i ego prilozheniya, Rostov State University, Rostov-on-Don **5**, 1974, 210–213 (in Russian).

33. ______, *Problems of geometry of non-normable spaces*. Rostov State University, Rostov-on-Don, 1983 (in Russian).

34. S. G. Krein, Yu. I. Petunin and E. M. Semenov, *Interpolation of linear operators*. Nauka, Moscow, 1978 (in Russian).

35. J. Lindenstrauss and L. Tzafriri, *Classical Banach spaces*. Springer-Verlag, Berlin, Heidelberg, New York, 1977.

36. B. S. Mityagin, *Approximative dimension and bases in nuclear spaces*. Russian Math. Surveys **16**(1961), 59–127.

37. ______, *Nuclear Riesz scales*. Dokl. Akad. Nauk SSSR **137**(1961), 519–522 (in Russian).

38. ______, *Sur l'equivalence des bases unconditional dans les echelles de Hilbert*. C. R. Acad. Sci. Paris **269**(1969), 426–428.

39. ______, *Equivalence of bases in Hilbert scales*. Studia Math. **37**(1970), 111–137 (in Russian).

40. ______, *Non-Schwartzian power series spaces*. Math. Z. **182**(1983), 303–310.

41. J. Peetre, *On interpolation functions*. Acta Sci. Math. **27**(1966), 167–171; II, ibid. **29**(1968), 91–92; III ibid. **30**(1969), 235–239.

42. A. Pełczynski, *On the approximation of S-spaces by finite-dimensional spaces*. Bull. Polish Acad. Sci. **5**(1957), 879–881.

43. S. Rolewicz, *On spaces of holomorphic functions*. Studia Math. **21**(1962), 135–160.

44. ______, *Metric linear Spaces*. Polish Scientific Publishers, Warsaw, 1984.

45. T. Terzioğlu, *Smooth sequence spaces*. In: Proceedings of Symp. on Funct. Anal., Silivri, 1974, 31–41.

46. ______, *Unstable Köthe spaces and the functor Ext*. Tr. J. Math. **10**(1986), 227–231.

47. ______, *Some invariants of Fréchet spaces and imbeddings of smooth sequence spaces*. Advances in the Theory of Frèchet Spaces, Kluwer Academic Publishers, Dordrecht-Boston-London, 1988, 305–324.

48. V. M. Tikhomirov, *Some problems of approximation theory*. Moscow University, Moscow, 1976 (in Russian).

49. D. Vogt, *Charakterisierung der Unterräume von s*. Math. Z. **155**(1977), 109–117.

50. ______, *Eine Charakterisierung der Potenzreihenräume von endlichem Typ und ihre Folgerungen*. Manuscripta Math. **37**(1982), 269–301.

51. ______, *Frécheträume, zwischen denen jede stetige lineare Abdildung beschränkt ist*. J. Reine. Angew. Math. **345**(1983), 182–200.

52. ______, *Power series spaces representations of nuclear Fréchet spaces*. Trans. Amer. Math. Soc. **319**(1990), 191–208.

53. M. J. Wagner, *Quotienträume von stabilen Potenzreihenräumen unendlichen Typs*. Manuscripta Math. **31**(1980), 97–109.

54. M. Yurdakul and V. P. Zahariuta, *Linear topological invariants and isomorphic classification of Cartesian products of locally convex spaces*. Tr. J. Math. **19**(1995), 37–47.

55. V. P. Zahariuta, *On isomorphisms of Cartesian products of linear topological spaces*. Funk. Anal. i ego Pril. **4**(1970), 87–88 (in Russian).

56. ______, *Linear topological invariants and isomorphisms of spaces of analytic functions*. Matem. analiz i ego pril. **2**, Rostov Univ., Rostov-on-Don, 1970, 3–13; ibid. **3**, 1971, 176–180 (in Russian).

57. ———, *On the isomorphism of Cartesian products of locally convex spaces*. Studia Math. **46**(1973), 201–221.
58. ———, *Some linear topological invariants and isomorphisms of tensor products of scale's centers*. Izv. Severo-Kavkaz. Nauchn. Tsentra Vyssh. Shkoly Tekhn. Nauk **4**(1974), 62–64 (in Russian).
59. ———, *On isomorphisms and quasi-equivalence of bases of power Köthe spaces*. Soviet Math. Dokl. **16**(1975), 411–414.
60. ———, *On isomorphisms and quasi-equivalence of bases of power Köthe spaces*. In: Proceedings of 7th winter school in Drogobych, CEMI, Moscow, 1976, 101–126 (in Russian).
61. ———, *Generalized Mityagin's invariants and continuum pairwise nonisomorphic spaces of analytic functions*. Funktsional Anal. i Prilozhen **11**(1977), 24–30 (in Russian).
62. ———, *Synthetic diameters and linear topological invariants*. School on Operator Theory in Function Spaces, abstracts of reports, Minsk, 1978, 51–52 (in Russian).
63. ———, *Compact operators and isomorphisms of Köthe spaces*. In: Actual Problems of Mathematical Analysis, Rostov State University, Rostov-on-Don, 1978, 62–71 (in Russian).
64. ———, *Linear topological invariants and their application to generalized power spaces*. Rostov State University, 1979 (in Russian).
65. ———, *Isomorphisms of spaces of analytic functions*. Soviet Math. Dokl. **22**(1980), 631–634.
66. ———, *Linear topological invariants and their application to isomorphic classification of generalized power spaces*. Turkish J. Math. **20**(1996), 237–289.

Department of Mechanics and Mathematics
Rostov State University
Rostov-on-Don
Russia
e-mail: chalov@ns.math.rsu.ru

Sabanci University
Istanbul
Turkey

Current address:
Feza Gürsey Institute
Çengelköy-Istanbul
Turkey
e-mail: zaha@mam.gov.tr

Permanent address:
Department of Mechanics and Mathematics
Rostov State University
Rostov-on-Don
Russia

https://doi.org/10.4153/CJM-1998-055-0 Published online by Cambridge University Press