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SERIES EXPANSIONS FOR RANDOM DISC-POLYGONS IN SMOOTH PLANE CONVEX BODIES

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Abstract

We establish power-series expansions for the asymptotic expectations of the vertex number and missed area of random disc-polygons in planar convex bodies with C^{k+1}_+-smooth boundaries. These results extend asymptotic formulas proved in Fodor et al. (2014).

Keywords: Expectation of missed area; expected vertex number; random polytopes; set estimation; spindle convexity

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1. Introduction and results

Reconstructing a possibly unknown set, or some of its characteristic quantities, from a random sample of points is a much-investigated classical problem that arises naturally in various fields, like stereology [1], computational geometry [13], statistical quality control [8], etc. Estimating the shape, volume, surface area, and other characteristic quantities of sets is of interest both in geometry and statistics, although the aspects investigated are in many cases different in the respective fields. For an overview of set estimation see, for example, [7]. The set may be quite arbitrary, but often various restrictions are imposed on it. One common such restriction that has received much attention is when the set is required to be convex. In such a setting polytopes spanned by random samples of points from the set form a natural estimator. The theory of random polytopes is a rich and lively field with numerous applications. For a recent review and further references see, for example, [31]. The convex hull is an optimal estimator if no further restrictions are imposed on the set K other than convexity. However, in this paper we study another estimator under further assumptions on K, namely that the degree of smoothness of the boundary of K is prescribed to be C^{k+1}, and it also assumed that the curvature is positive everywhere. Under these circumstances, using congruent circles to form the hull of the sample yields better performance than the classical convex hull.

Since the case when the number of random points is fixed is notoriously difficult, it has become common to investigate the asymptotic behaviour of functionals associated with random polytopes as the number of points in the sample tends to infinity. The investigations of the asymptotic behaviour of random polytopes started with the classical papers [26, 27] in

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the 1960s. They studied the following particular model in the plane. Let K be a convex body (a compact convex set with nonempty interior) in d-dimensional Euclidean space $\mathbb{R}^d$, and let $x_1, \ldots, x_n$ be independent random points from K selected according to the uniform probability distribution.

The convex hull $K_n = [x_1, \ldots, x_n]$ of $x_1, \ldots, x_n$ is called a (uniform) random polytope in K. Asymptotic formulas in the plane were proved in [26, 27] for the expected number $f_0(K_n)$ of vertices of K_n and the expectation of the missed area $A(K \setminus K_n)$ under the assumption that the boundary ∂K of K is sufficiently smooth, and also in the case when K itself is a convex polygon. This was extended in [35] to the d-dimensional ball B^d, and in [2] for d-dimensional convex bodies with at least a C^3_+-smooth boundary (three times continuously differentiable with everywhere positive Gauss–Kronecker curvature). All smoothness conditions were removed in [33]. The results were extended in [6] for nonuniform distributions and weighted volume difference.

Let $V_i(\cdot)$, $i = 1, \ldots, d$, denote the ith intrinsic volume of a convex body. A power series expansion of the quantity $\mathbb{E}(V_i(K) - V_i(K_n))$ for all $i = 1, \ldots, d$ as $n \to \infty$ was established in [24] under stronger smoothness conditions on the boundary of K.

Theorem 1. ([24].) *Let K be a convex body in $\mathbb{R}^d$ with $V_d(K) = 1$ whose boundary ∂K is C^{k+1}_+ for some integer $k \geq 2$. Then*

$$\begin{aligned}
&\mathbb{E}(V_i(K) - V_i(K_n)) \\
&= c_2^{(i,d)}(K) n^{-2/(d+1)} + c_3^{(i,d)}(K) n^{-3/(d+1)} + \cdots + c_k^{(i,d)}(K) n^{-k/(d+1)} + O(n^{-(k+1)/(d+1)}) \quad (1)
\end{aligned}$$

as $n \to \infty$. Moreover, $c_{2m+1}^{(i,d)} = 0$ for all $m \leq d/2$ if d is even, and $c_{2m+1}^{(i,d)} = 0$ for all m if d is odd.

Under the same conditions as in Theorem 1, we can obtain from (1) a series expansion for the number of vertices $\mathbb{E}(f_0(K_n))$ via Efron's identity [10]:

$$\begin{aligned}
&\mathbb{E}(f_0(K_n)) \\
&= d_2(K) n^{(d-1)/(d+1)} + d_3(K) n^{(d-2)/(d+1)} + \cdots + d_k(K) n^{(d-k+1)/(d+1)} + O(n^{(d-k+2)/(d+1)})
\end{aligned}$$

as $n \to \infty$, where the coefficients $d_i(K)$ also depend on the dimension d.

Theorem 1 was proved in [14] when $i = 1$. Using properties of the convex floating body, the planar case of Theorem 1 was established for the area ($d = 2$, $i = 2$) in [23]. In particular, it was proved that

$$d_4(K) = c_4^{(2,2)}(K) = -\Gamma\left(\frac{7}{3}\right) \frac{1}{5} \sqrt[3]{\frac{3}{2}} \int_{\partial K} k(x) \kappa^{1/3}(x)\, \mathrm{d}x,$$

where $\Gamma(\cdot)$ is Euler's gamma function, $k(x)$ is the affine curvature (for information about the affine curvature see, for example, [5, pp. 12–15] or [15, Section 7.3]), $\kappa(x)$ is the curvature of ∂K at x, and integration on the boundary ∂K of K is with respect to arc length.

For more information about approximations of convex bodies by classical random polytopes we refer to [3, 25, 31, 32, 34].

When estimating a planar convex body under curvature restrictions, it may naturally be more advantageous to use suitably curved arcs to form the boundary of the approximating set that fit K better than line segments. One of the simplest such constructions uses radius-R circular arcs and the resulting (convex) hull is called, among other names, the R-spindle convex hull; for precise definitions, see below. The radius should be chosen in such a way that the (generalised) random polygon is still contained in K. This imposes the condition on R that it should be at least as large as the maximum radius of curvature of ∂K. However, similarly to the classical convex case, difficulties arise when R is equal to the maximal radius of curvature, so this case usually needs separate treatment using different methods.

In this paper, we study the R-spindle convex variant of the above probability model in the Euclidean plane $\mathbb{R}^2$. Let $R > 0$ be fixed, and let $x, y \in \mathbb{R}^2$ be such that their distance is at most $2R$. We call the intersection $[x, y]_R$ of all (closed circular) discs of radius R that contain both x and y the R-spindle of x and y. A set $X \subseteq \mathbb{R}^2$ is called *R-spindle convex* if from $x, y \in X$ it follows that $[x, y]_R \subseteq X$. Spindle convex sets are also convex in the usual linear sense. In this paper we restrict our attention to compact spindle-convex sets. One can show (cf. [4, Corollary 3.4, p. 205]) that a convex body in $\mathbb{R}^2$ is R-spindle convex if it is the intersection of (not necessarily finitely many) closed discs of radius R. The intersection of finitely many closed discs of radius R is called a convex R-disc-polygon. Let X be a compact set which is contained in a closed disc of radius R. The intersection of all planar R-spindle-convex bodies containing X is called the R-spindle-convex hull of X, and it is denoted by $[X]_R$. Perhaps it is easier to grasp this notion if we point out the similarity with the classical convex hull. In the R-spindle-convex case the radius-R discs play a similar role to what closed half-spaces do for classical convex hulls. Thus, in a heuristic way, we can consider the classical convex hull as a limiting case as $R \to \infty$. If $X \subset K$ for an R-spindle-convex body K in $\mathbb{R}^2$, then $[X]_R \subset K$. A prominent class of R-spindle-convex sets in $\mathbb{R}^2$ that are directly relevant in this paper is provided by convex bodies whose boundary is C^2_+-smooth with curvature $\kappa(x) \geq 1/R$ for all boundary points $x \in \partial K$ [30, Sections 2.5 and 3.2]. For more detailed information about spindle convexity we refer to [4, 19].

We note that there exist further generalisations of spindle convexity, most notably the concept of L-convexity in which the translates of a fixed convex body L play the role of the radius-R closed disc; for more information, see, for example, [17]. Another further generalisation is H-convexity as introduced in [16], where the hull of a set is generated by intersections of transformed copies of a fixed convex set C by a set H of affine transformations. A similar concept (see, for example, [18]) to R-spindle convexity, called α convexity, also exists, where the α-convex hull of a set is defined as the complement of the union of all radius-r open balls disjoint from the set. The α-convex hull of a finite sample is different from its R-spindle-convex hull as it is nonconvex while the R-convex hull is always convex. We note that the α-convex hull can be used to estimate not necessarily convex sets as well; see [21, 22, 28], where several such results are proved about random samples chosen from the set according to an absolute continuous probability distribution.

A convex R-disc-polygon is clearly R-spindle convex. We also consider a single radius-R disc and a single point as R-disc-polygons, albeit trivial ones. The nonsmooth points of the boundary of a nontrivial convex R-disc-polygon are called vertices. The vertices divide the boundary into a union of radius-R circular arcs of positive arc length that we call edges. Thus, a nontrivial convex R-disc-polygon has an equal number of edges and vertices, just like a classical convex polygon, except the sides are radius-R circular arcs. The radius-R disc has one edge and no vertex, and a single point has one vertex and no side.

Our probability model is the following. Let K be a convex body in $\mathbb{R}^2$ with an at least C^2_+-smooth boundary, and let R be such that $\kappa(x) > 1/R$ for all $x \in \partial K$. Let $x_1, \ldots, x_n$ be independent random points in K chosen according to the uniform probability distribution. The

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R-spindle-convex hull $K_n^R = [x_1, \dots, x_n]_R$ is called a *uniform random R-disc-polygon* in K, and is a convex R-disc-polygon. It is clear that K_n^R has an equal number of vertices and sides with probability 1, and its vertex set is formed by some of the random points $x_1, \dots, x_n$. Let $f_0(K_n^R)$ denote the number of vertices of K_n^R. We note that in [21] the radius r_n of the discs used in the estimation of an α-convex set tends to zero as $n \to \infty$. In our model, we use suitable fixed-radius discs in order to guarantee that the R-spindle-convex hull of the random sample is contained in K. However, after the statements of our main results, we briefly discuss what happens to the quality of the approximation when the radius R tends to the limits of its possible range.

It was proved in [11, Theorem 1.1, p. 901] that under the above conditions, as $n \to \infty$,

$$\mathbb{E}(f_0(K_n^R)) = z_1(K)n^{1/3} + o(n^{1/3}), \tag{2}$$

$$\mathbb{E}(A(K \setminus K_n^R)) = A(K)z_1(K)n^{-2/3} + o(n^{-2/3}), \tag{3}$$

where

$$z_1(K) = \sqrt[3]{\frac{2}{3A(K)}} \cdot \Gamma\left(\frac{5}{3}\right) \int_{\partial K} \left(\kappa(x) - \frac{1}{R}\right)^{1/3} \mathrm{d}x,$$

and $A(K)$ denotes the area of K.

We note that (2) and (3) are connected by an Efron-type [10] identity [11, (5.10), p. 910], which states that

$$\mathbb{E}(f_0(K_n^R)) = n\frac{\mathbb{E}(A(K \setminus K_{n-1}^R))}{A(K)}.$$

In this paper we prove the following theorems that provide power-series expansions of $\mathbb{E}(f_0(K_n^R))$ and $\mathbb{E}(A(K \setminus K_n^R))$ in the case when ∂K satisfies stronger differentiability conditions.

Theorem 2. *Let $k \geq 2$ be an integer, and let K be a convex body in $\mathbb{R}^2$ with a C_+^{k+1}-smooth boundary. Then, for all $R > \max_{x \in \partial K} 1/\kappa(x)$,*

$$\mathbb{E}(f_0(K_n^R)) = z_1(K)n^{1/3} + \cdots + z_{k-1}(K)n^{-(k-3)/3} + O(n^{-(k-2)/3})$$

as $n \to \infty$. All the coefficients $z_1, \dots, z_k$ can be determined explicitly. In particular,

$$z_1(K) = \sqrt[3]{\frac{2}{3A(K)}}\Gamma\left(\frac{5}{3}\right) \int_{\partial K} \left(\kappa(x) - \frac{1}{R}\right)^{1/3} \mathrm{d}x,$$

$$z_2(K) = 0,$$

$$z_3(K) = -\Gamma\left(\frac{7}{3}\right)\frac{1}{5}\sqrt[3]{\frac{3A(K)}{2}} \int_{\partial K} \left(\frac{\kappa''(x)}{3(\kappa(x) - 1/R)^{4/3}} + \frac{2R^2\kappa^2(x) + 7R\kappa(x) - 1}{2R^2(\kappa(x) - 1/R)^{1/3}} - \frac{5(\kappa'(x))^2}{9(\kappa(x) - 1/R)^{7/3}}\right) \mathrm{d}x.$$

By the spindle-convex version of Efron's identity we obtain the following corollary.

Theorem 3. *Let $k \geq 2$ be an integer, and let K be a convex body in $\mathbb{R}^2$ with a C_+^{k+1}-smooth boundary. Then, for all $R > \max_{x \in \partial K} 1/\kappa(x)$,*

$$\mathbb{E}(A(K \setminus K_n^R)) = z_1'(K)n^{-2/3} + \cdots + z_{k-1}'(K)n^{-k/3} + O(n^{-(k+1)/3})$$

as $n \to \infty$, where ${z_i}'(K) = A(K)z_i(K)$ for $i = 1, \dots, k$.

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We note that we only evaluate $z_i(K)$, $i=1, 2, 3$, explicitly in this paper because the calculation, although possible, becomes more complicated as i increases, even when K is a closed disc. The coefficients $z_i(K)$ depend only on R, the area of K, and on the power-series expansion of the local representation of the boundary of K, see (6); in particular, on the derivatives of κ up to order $i-1$.

Although Theorems 2 and 3 are only valid for $R>R_{\mathrm{M}}=\max_{x\in\partial K} 1/\kappa(x)$, it may also be interesting to look at the behaviour of the coefficients $z_i(K)$ at the limits of the range of R. When $R\to\infty$, the integral in $z_1(K)$ tends to the affine arc length of ∂K [11]. For $z_3(K)$, direct calculation yields

$$\lim_{R\to\infty} \frac{\kappa''(x)}{3(\kappa(x)-1/R)^{4/3}}+\frac{2R^2\kappa^2(x)+7R\kappa(x)-1}{2R^2(\kappa(x)-1/R)^{1/3}}-\frac{5(\kappa'(x))^2}{9(\kappa(x)-1/R)^{7/3}}=k(x)\kappa^{1/3}(x),$$

where $k(x)$ is the affine curvature of ∂K at x, cf. also (1).

On the other hand, when $R\to R_{\mathrm{M}}^+$, then

$$\lim_{R\to R_M^+} z_1(K)=\sqrt[3]{\frac{2}{3A(K)}}\Gamma\left(\frac{5}{3}\right)\int_{\partial K}\left(\kappa(x)-\frac{1}{R_{\mathrm{M}}}\right)^{1/3}\mathrm{d}x,$$

where the integrand is bounded, nonnegative, and zero in exactly those points where $\kappa(x)=1/R_{\mathrm{M}}$. We conjecture that the right-hand side is equal to $\lim_{n\to\infty}\mathbb{E}f_0(K_n^R))n^{-1/3}$ when $R=R_{\mathrm{M}}$ and K is not a closed disc. However, this asymptotic expectation is not known. We also note that $z_1(K)$ is a monotonically decreasing function of R, which shows that it is indeed more advantageous to use circular arcs to form the hull of the random sample of n points in order to approximate K better. Although the order of magnitude in n of the approximation is the same as in the linearly convex case, the main coefficient is smaller.

Furthermore, we note that in the particular case when $K=B^2$ and $R>1$,

$$z_1(B)=\sqrt[3]{\frac{2}{3\pi}}\Gamma\left(\frac{5}{3}\right)2\pi\left(1-\frac{1}{R}\right)^{1/3},$$

$$z_2(B)=0,$$

$$z_3(B)=-\Gamma\left(\frac{7}{3}\right)\frac{1}{5}\sqrt[3]{\frac{3\pi}{2}}2\pi\frac{2R^2+7R-1}{2R^2(1-1/R)^{1/3}}.$$

If $R\to 1^+$ then $z_1(B)\to 0$ and $z_3(B)\to-\infty$, and both are monotonically increasing functions showing that the quality of approximation improves as R tends to 1. This behaviour comes as no surprise as the expected number of vertices behaves fundamentally differently from the previously discussed situation when $K\neq B$; the order of magnitude in n is different if $K=B$, as we will see below. Finally, we note that we also suspect that $z_3(K)$ behaves similarly to $z_3(B)$ when $R\to R_{\mathrm{M}}^+$, but this is not clear from its current form.

It was proved in [11] that

$$\mathbb{E}(f_0(B(R)_n^R))=\frac{\pi^2}{2}+o(1),\qquad \mathbb{E}(A(B(R)\setminus B(R)_n^R))=\frac{R^2\pi^3}{2}\frac{1}{n}+o\left(\frac{1}{n}\right)$$

as $n\to\infty$. The unusual behaviour of $\mathbb{E}(f_0(B(R)_n^R))$, i.e. that it tends to a finite constant, was explained in [20], which proved, in the much wider context of L-convexity (see also [12]), that $\mathbb{E}(f_0(B(R)_n^R))$ tends to the expectation of the number of vertices of the polar of the zero cell of a Poisson line process whose intensity measure on $\mathbb{R}$ is $A(B(R))^{-1} = 1/(R^2\pi)$ times the Lebesgue measure, and whose directional distribution is uniform on S^1 [20, (6.1), p. 29]. In Section 4, we calculate (the first three terms of) the power-series expansion of $\mathbb{E}(f_0(B(R)_n^R))$ for the sake of completeness. This gives the speed of convergence of $\mathbb{E}(f_0(B(R)_n^R))$ to $\pi^2/2$. We note that here we only quoted the result from [20] in the plane; however, it was proved in $\mathbb{R}^d$.

The rest of the paper is organised as follows. In Section 2, we briefly recall from [11] the necessary background and describe how $\mathbb{E}(f_0(B(R)_n^R))$ can be calculated. In Section 3, we provide the power-series expansions of the involved geometric quantities. In Section 4, we quote a power-series expansion of the incomplete beta function from [14]. We prove Theorem 2 in Section 5. Finally, in Section 6, we treat the case when $K = B(R)$.

2. Expectation of the number of vertices of K_n^R

Our arguments are based on the methods of [14, 26]. We also note that, compared to those of [21], our methods essentially depend on the higher regularity and smoothness of the boundary of K and the explicit local power-series expansion of ∂K. Notice that it is enough to prove the theorem for $R = 1$; from that, the statement for general R follows by a scaling argument.

Due to the C_+^{k+1} condition, K is both smooth, i.e. has a unique supporting line at each boundary point, and strictly convex. Let $u_x \in S^1$ denote the unique outer unit normal vector to K at x, and for $u \in S^1$ let x_u be the (again) unique boundary point where the outer unit normal is equal to u.

We use B° to denote the interior of B. A subset D of K is a *disc-cap of* K if $D = K \setminus (B^\circ + p)$ for some point $p \in \mathbb{R}^2$. It was proved in [11] that for a disc-cap of K, $D = K \setminus (B^\circ + p)$, there exists a unique point $x_0 \in \partial K \cap D$ and $t \geq 0$ such that $B + p = B + x_0 - (1 + t)u_{x_0}$. We call x_0 the vertex and t the height of D.

We may assume that $o \in \operatorname{int} K$. Let $A = A(K) = V_2(K)$. Let $X_n = \{x_1, \dots, x_n\}$ be a sample of independent and identically distributed uniform random points from K. For $x_i, x_j \in X_n$, we denote by x_ix_j the shorter unit circular arc connecting x_i and x_j with the property that x_i and x_j are in counterclockwise order on the arc. Let

$$\mathcal{E}(K_n^1) = \{x_ix_j \colon x_i, x_j \in X_n \text{ and } x_ix_j \text{ is an edge of } K_n^1\}$$

be the set of directed edges of K_n^1. For $x_i, x_j \in X_n$, let C_{ij} be the disc-cap of K determined by the disc of x_ix_j, and $A_{ij} = A(C_{ij})$. Note that $x_ix_j \in \mathcal{E}(K_n^1)$ exactly when all the other $n - 2$ random points of X_n are in $K \setminus C_{ij}$. Thus, due to the independence of the random points,

$$\begin{aligned}\mathbb{E}(f_0(K_n^1)) &= \sum \frac{1}{A^n} \int_K \cdots \int_K \mathbf{1}\{x_ix_j \in \mathcal{E}(K_n^1)\}\, \mathrm{d}x_1 \cdots \mathrm{d}x_n \\ &= \binom{n}{2} \frac{1}{A^2} \int_K \int_K \left(1 - \frac{A_{12}}{A}\right)^{n-2} + \left(1 - \frac{A_{21}}{A}\right)^{n-2} \mathrm{d}x_1\, \mathrm{d}x_2, \end{aligned} \tag{4}$$

where in the first line the summation extends over all ordered pairs of distinct points from X_n. Now, we use the same reparametrization for the pair (x_1, x_2) as in [11]. Let $(x_1, x_2) = \Phi(u, t, u_1, u_2)$, where $u, u_1, u_2 \in S^1$ and $0 \leq t \leq t_0(u)$ are chosen such that $C(u, t) = C_{12}$, where $C(u, t)$ is the unique disc-cap of K with vertex x_u and height t, and

$$(x_1, x_2) = (x_u - (1 + t)u + u_1, x_u - (1 + t)u + u_2).$$

The vectors u_1 and u_2 are the unique outer unit normals of $\partial B + x_u - (1+t)u$ at x_1 and x_2, respectively. For fixed u and t, both u_1 and u_2 are contained in the same arc $L(u, t)$ of S^1, whose length is denoted by $\ell(u, t)$. The uniqueness of the vertex and height of disc-caps guarantees that the map Φ is well defined, bijective, and differentiable on a suitable domain of (u, t, u_1, u_2). The Jacobian of Φ is

$$|J\Phi| = \left(1 + t - \frac{1}{\kappa(x_u)}\right)|u_1 \times u_2|.$$

Let $A(u, t)$ denote the area of the disc-cap with vertex x_u and height t. For each $u \in S^1$, let $t_0(u)$ be maximal such that $A(u, t_0(u)) \geq 0$. Then, after the change of variables, from (4) we get

$$\begin{aligned}\mathbb{E}(f_0(K_n^1)) &= \binom{n}{2}\frac{1}{A^2}\int_{S^1}\int_0^{t_0(u)}\int_{L(u,t)}\int_{L(u,t)}\left(1 - \frac{A(u, t)}{A}\right)^{n-2} \\ &\qquad\qquad \times \left(1 + t - \frac{1}{\kappa(x_u)}\right)|u_1 \times u_2| \,\mathrm{d}u_1 \,\mathrm{d}u_2 \,\mathrm{d}t \,\mathrm{d}u \\ &= \binom{n}{2}\frac{1}{A^2}\int_{S^1}\int_0^{t_0(u)}\left(1 - \frac{A(u, t)}{A}\right)^{n-2} J(u, t) \,\mathrm{d}t \,\mathrm{d}u,\end{aligned}$$

where

$$\begin{aligned}J(u, t) &= \left(1 + t - \frac{1}{\kappa(x_u)}\right)\int_{L(u,t)}\int_{L(u,t)} |u_1 \times u_2| \,\mathrm{d}u_1 \,\mathrm{d}u_2 \\ &= 2\left(1 + t - \frac{1}{\kappa(x_u)}\right)(\ell(u, t) - \sin \ell(u, t)).\end{aligned}$$

We note that due to the C^2_+ property of ∂K, $J(u, t) \leq C$ for some $0 < C \leq 6(2\pi + 1)$ that depends only on K.

Let $0 < \delta < A$ be an arbitrary but fixed small number. Let $0 < t_1$ be such that, for arbitrary $t \in [t_1, t_0(u)]$ and $u \in S^1$, $A(u, t) \geq \delta$. Then

$$\begin{aligned}\int_{S^1}\int_{t_1}^{t_0(u)}\left(1 - \frac{A(u, t)}{A}\right)^{n-2} J(u, t) \,\mathrm{d}t \,\mathrm{d}u &\leq C\int_{S^1}\int_{t_1}^{t_0(u)}\left(1 - \frac{A(u, t)}{A}\right)^{n-2} \mathrm{d}t \,\mathrm{d}u \\ &\leq 2\pi C\int_{t_1}^{2}\left(1 - \frac{\delta}{A}\right)^{n-2} \mathrm{d}t \\ &\leq 4\pi C\left(1 - \frac{\delta}{A}\right)^{n-2},\end{aligned}$$

and thus, in particular, with a suitably small choice of δ,

$$\mathbb{E}(f_0(K_n^1)) = \binom{n}{2}\frac{1}{A^2}\int_{S^1}\int_0^{t_1}\left(1 - \frac{A(u, t)}{A}\right)^{n-2} J(u, t) \,\mathrm{d}t \,\mathrm{d}u + O(n^{-k}). \tag{5}$$

In the following sections we evaluate the integral (5) under different smoothness assumptions on ∂K.

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3. Power-series expansions

Let $k \geq 2$ be an integer and $K \subset \mathbb{R}^2$ a convex body with a C_+^{k+1} boundary ($(k+1)$ times continuously differentiable with everywhere positive curvature). We will use the following statement from [14] (see also [29]). We state it in the form used in [24], but only for $d = 2$.

Lemma 1. *Let K be a convex body in $\mathbb{R}^2$ with a C_+^{k+1} boundary for some integer $k \geq 2$. Then there exist constants $\alpha, \beta > 0$ depending only on K such that the following holds for every boundary point x of K. If $x = 0$ and the (unique) tangent line of K at x is $\mathbb{R}$, then there is an α-neighbourhood of x in which the boundary of K can be represented by a convex function $f(\sigma)$ of differentiability class C^{k+1} in $\mathbb{R}$. Moreover, all derivatives of f up to order $k+1$ are uniformly bounded by β.*

Let $u \in S^1$ and let $x = x_u \in \partial K$. Assume that K is in the position described in Lemma 1. Let f be the function that represents the boundary of K in an α-neighbourhood of x. Then f is of the form $f(\sigma) = b_2(u)\sigma^2 + \cdots + b_k(u)\sigma^k + O(\sigma^{k+1})$, where the coefficients $b_i = b_i(u)$, $i = 2, \ldots, k$, depend on u. We will suppress the dependence of coefficients on u (and thus on x) when we work with a fixed u. We will only indicate dependence when u is used in the argument.

We recall the following facts from the differential geometry of plane curves. Let $r(s)$ be the arc-length parametrization of ∂K with $r(0) = x$ in the neighbourhood of x such that the following hold. With the above assumptions on K, let the vector $r'(0)$ and the unit normal vector $r''(0)/\kappa(0) = -u$ form the basis of a Cartesian coordinate system, in which we denote the coordinate along the r'-axis by σ, and the r''-axis by η. Then

$$\begin{aligned} \sigma &= \sigma(s) = s - \frac{\kappa^2(0)}{3!}s^3 - 3\kappa(0)\kappa'(0)\frac{s^4}{4!} + O(s^5), \\ \eta &= \eta(s) = \kappa(0)\frac{s^2}{2} + \kappa'(0)\frac{s^3}{3!} + (\kappa''(0) - \kappa^3(0))\frac{s^4}{4!} + O(s^5); \end{aligned} \tag{6}$$

see, for example, [9, Section 1.6]. From the equality $f(\sigma(s)) = \eta(s)$ we can identify the coefficients $b_2, \ldots, b_k$. In particular,

$$b_2 = \frac{\kappa(0)}{2}, \qquad b_3 = \frac{\kappa'(0)}{6}, \qquad b_4 = \frac{\kappa''(0) + 3\kappa^3(0)}{24}.$$

With a slight abuse of notation, in the above formulas we use κ to denote the curvature as a function of s, which is different from the previous usage. Later, we will also use the same letter when the curvature is a function of the outer unit normal u. Moreover, when u (s or x) is fixed, we suppress the dependence of κ on u (s or x). It will always be clear from the context which function we consider.

We will also use the following statement due to [14], see also [24]. (We state it again only for $d = 2$, so this is a simpler version of the original theorem.)

Lemma 2. *Let $\eta = \eta(\sigma) = b_m\sigma^m + \cdots + b_k\sigma^k + O(\sigma^{k+1})$ for $0 \leq \sigma \leq \alpha$, $2 \leq m \leq k$, be a strictly increasing function. Then there are coefficients $c_1, \ldots, c_{k-m+1}$ and a constant $\gamma > 0$ such that the inverse function $\sigma = \sigma(\eta)$ has the representation*

$$\sigma = \sigma(\eta) = c_1\eta^{1/m} + \cdots + c_{k-m+1}\eta^{(k-m+1)/m} + O(\eta^{(k-m+2)/m})$$

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for $0 \le \eta \le \gamma$. *The coefficients* $c_1, \ldots, c_{k-m+1}$ *can be determined explicitly in terms of* $b_m, \ldots, b_k$. *In particular,*

$$c_1 = \frac{1}{b_m^{1/m}}, \qquad c_2 = -\frac{b_{m+1}}{m b_m^{(m+2)/m}}, \qquad c_3 = -\frac{b_{m+2}}{m b_m^{(m+3)/m}} + \frac{(m+3)b_{m+1}^2}{2m^2 b_m^{(2m+3)/m}}.$$

For $t \ge 0$, let the unit-radius lower semicircle with centre $(0, 1+t)$ be represented by the function

$$g_t(\sigma) = t + 1 - \sqrt{1-\sigma^2} = t + 1 - \sum_{i=0}^{\infty} (-1)^i \binom{\frac{1}{2}}{i} \sigma^{2i} = t + g_2\sigma^2 + \cdots + g_{2i}\sigma^{2i} + \cdots$$

for $\sigma \in [-1, 1]$, where $g_2 = \frac{1}{2}$, $g_3 = 0$, and $g_4 = \frac{1}{8}$.

Let $\sigma_+ = \sigma_+(t) > 0$ and $\sigma_- = \sigma_-(t) < 0$ such that $f(\sigma_+) = g_t(\sigma_+)$ and $f(\sigma_-) = g_t(\sigma_-)$. For sufficiently small $\sigma > 0$,

$$t = t(\sigma) = f(\sigma) - 1 + \sqrt{1-\sigma^2} = u_2\sigma^2 + \cdots + u_k\sigma^k + O(\sigma^{k+1}),$$

where, in particular, $u_2 = b_2 - g_2$, $u_3 = b_3$, and $u_4 = b_4 - g_4$.

Note that we subsequently express coefficients in terms of the u_i (as long as it does not become too complicated) as they carry all the information about ∂K and the circle. We will only substitute their values when we determine our final answer.

Since $u_2 > 0$ by the conditions on ∂K, Lemma 2 yields

$$\sigma_+ = \sigma_+(t) = c_1 t^{1/2} + \cdots + c_{k-1} t^{(k-1)/2} + O(t^{k/2}), \tag{7}$$

where

$$c_1 = u_2^{-1/2}, \qquad c_2 = -\frac{u_3}{2u_2^2}, \qquad c_3 = \frac{5u_3^2 - 4u_2u_4}{8u_2^{7/2}}.$$

Similarly, we obtain that

$$\sigma_- = \sigma_-(t) = \tilde{c}_1 t^{1/2} + \cdots + \tilde{c}_{k-1} t^{(k-1)/2} + O(t^{k/2}), \tag{8}$$

where the coefficients $\tilde{c}_1, \ldots, \tilde{c}_{k-1}$ can be determined explicitly. In particular, $\tilde{c}_1 = -c_1$, $\tilde{c}_2 = c_2$, and $\tilde{c}_3 = -c_3$. Thus, using (7) and (8), the area of the disc cap $C(u, t)$ is

$$\begin{aligned} A(u, t) &= \int_{\sigma_-}^{\sigma_+} g_t(\sigma) - f(\sigma)\, \mathrm{d}\sigma \\ &= \int_{\sigma_-}^{\sigma_+} t - u_2\sigma^2 - \cdots - u_k\sigma^k + O(\sigma^{k+1})\, \mathrm{d}\sigma \\ &= \left[t\sigma - \frac{u_2}{3}\sigma^3 - \cdots - \frac{u_k}{k+1}\sigma^{k+1} + O(\sigma^{k+2}) \right]_{\sigma_-}^{\sigma_+} \\ &= a_1 t^{3/2} + a_2 t^2 + \cdots + a_{k-1} t^{(k+1)/2} + O(t^{(k+2)/2}), \end{aligned} \tag{9}$$

where the coefficients $a_1, \ldots, a_{k-1}$ can be expressed explicitly. In particular,

$$a_1 = \frac{4}{3} u_2^{-1/2}, \qquad a_2 = 0, \qquad a_3 = \frac{5u_3^2 - 4u_2u_4}{10u_2^{7/2}}.$$

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Note that, for sufficiently small t, $\partial A(u, t)/\partial t = \sigma_+(t) - \sigma_-(t)$.

Now we turn to expressing the Jacobian $J(u, t)$ in the form of a series expansion in t. Using (7) and (8), we get

$$\begin{aligned} \ell(u, t) &= \int_{\sigma_-}^{\sigma^+} \sqrt{1 + (g_t'(\sigma))^2}\, \mathrm{d}\sigma = \int_{\sigma_-}^{\sigma^+} \sqrt{\frac{1}{1-\sigma^2}}\, \mathrm{d}\sigma = \big[\arcsin \sigma\big]_{\sigma_-}^{\sigma_+} \\ &= h_1 t^{1/2} + h_2 t + \cdots + h_{k-1} t^{(k-1)/2} + O(t^{k/2}), \end{aligned} \tag{10}$$

where the coefficients $h_1, \ldots, h_{k-1}$ can be expressed explicitly. In particular,

$$h_1 = 2u_2^{-1/2}, \qquad h_2 = 0, \qquad h_3 = \frac{15u_3^2 + 4u_2(u_2 - 3u_4)}{12u_2^{7/2}}.$$

Note that the coefficients c_1, c_2, c_3 (also $\tilde{c}_1, \tilde{c}_2, \tilde{c}_3$), a_1, a_2, a_3, and h_1, h_2, h_3 were calculated in [11, pp. 911–912] with a different notation.

Now, using (10), we get

$$\ell(u, t) - \sin \ell(u, t) = \sum_{i=0}^{\infty} (-1)^i \frac{\ell^{2i+1}(u, t)}{(2i+1)!} = l_1 t^{3/2} + \cdots + l_{k-1} t^{(k+1)/2} + O(t^{(k+2)/2}),$$

where the coefficients $l_1, \ldots, l_{k-1}$ can be calculated explicitly. In particular,

$$l_1 = \frac{4}{3} u_2^{-3/2}, \qquad l_2 = 0, \qquad l_3 = \frac{25u_3^2 + 4u_2(u_2 - 5u_4)}{10u_2^{9/2}}.$$

Then,

$$J(u, t) = 2\left(1 + t - \frac{1}{\kappa(x_u)}\right)(\ell(u, t) - \sin \ell(u, t)) = j_1 t^{3/2} + \cdots + j_{k-1} t^{(k+1)/2} + O(t^{(k+2)/2}), \tag{11}$$

where the coefficients $j_1, \ldots, j_{k-1}$ can be calculated explicitly. In particular,

$$j_1 = \frac{8u_2^{-3/2}(\kappa - 1)}{3\kappa}, \qquad j_2 = 0, \qquad j_3 = \frac{8u_2^{-3/2}}{3} + \frac{25u_3^2 + 4u_2(u_2 - 5u_4)}{5u_2^{9/2}} \frac{(\kappa - 1)}{\kappa}.$$

For a fixed n, let $y = y(u, t)$ be defined by

$$\frac{y}{n-2} = \frac{A(u, t)}{A}.$$

Then, by (9) and using Lemma 2 for $\sqrt{t}$ and then squaring, we obtain

$$t = p_1 \left(\frac{y}{n-2}\right)^{2/3} + \cdots + p_{k-1} \left(\frac{y}{n-2}\right)^{k/3} + O\left(\left(\frac{y}{n-2}\right)^{(k+1)/3}\right), \tag{12}$$

where the coefficients $p_1, \ldots, p_{k-1}$ can be calculated explicitly. In particular,

$$p_1 = \left(\frac{3A}{4}\right)^{2/3} u_2^{1/3}, \qquad p_2 = 0, \qquad p_3 = \frac{9A(-5u_3^2 + 4u_2 u_4)}{320u_2^2}.$$

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Then, substituting (12) into (11), we obtain

$$J\left(u, \frac{y}{n-2}\right) = q_1\left(\frac{y}{n-2}\right) + \cdots + q_{k-1}\left(\frac{y}{n-2}\right)^{(k+1)/3} + O\left(\left(\frac{y}{n-2}\right)^{(k+2)/3}\right), \quad (13)$$

where the coefficients $q_1, \ldots, q_{k-1}$ can be calculated explicitly. In particular,

$$q_1 = j_1 p_1^{3/2}, \qquad q_2 = 0, \qquad q_3 = j_3 p_1^{5/2} + \frac{3 j_1 p_3 p_1^{1/2}}{2}.$$

In the coefficients q_1, q_3 we used j_1, j_3 and p_1, p_3 instead of the u_i in order to simplify the notation.

4. The incomplete beta function

In evaluating the integral (5), we use the following expansion of the incomplete beta function from [14].

Lemma 3 ([14].) *Let $\beta \in \mathbb{R}$. There are coefficients $\gamma_1, \gamma_2, \ldots \in \mathbb{R}$ depending on β that can be determined explicitly such that, for a fixed $l = 1, 2, \ldots$ and $0 < \alpha \le 1$,*

$$\int_0^{\alpha n} \left(1 - \frac{t}{n}\right)^n t^\beta \, \mathrm{d}t = \Gamma(\beta+1) + \frac{\gamma_1}{n} + \cdots + \frac{\gamma_l}{n^l} + O\left(\frac{1}{n^{l+1}}\right) \quad \text{as } n \to \infty.$$

In particular,

$$\gamma_1 = -\frac{\Gamma(\beta+3)}{2}, \qquad \gamma_2 = -\frac{\Gamma(\beta+4)}{3} + -\frac{\Gamma(\beta+5)}{8}.$$

If α is chosen from a closed subinterval of (0,1], then the constant in $O(\cdot)$ can be chosen independent of α.

In our calculations, we need the following corollary of Lemma 3.

Lemma 4. *Under the assumptions of Lemma 3, there are coefficients $\gamma_1', \gamma_2', \ldots \in \mathbb{R}$ such that*

$$\int_0^{\alpha(n-2)} \left(1 - \frac{t}{n-2}\right)^{n-2} t^\beta \, \mathrm{d}t = \Gamma(\beta+1) + \frac{\gamma_1'}{n} + \cdots + \frac{\gamma_l'}{n^l} + O\left(\frac{1}{n^{l+1}}\right) \quad \text{as } n \to \infty.$$

In particular,

$$\gamma_1' = -\frac{\Gamma(\beta+3)}{2}, \qquad \gamma_2' = -\frac{\Gamma(\beta+4)}{3} - 2\Gamma(\beta+3).$$

If α is chosen from a closed subinterval of (0,1], then the constant in $O(\cdot)$ can be chosen independent of α.

Proof. Using Lemma 3 and

$$\frac{n}{n-2} = \frac{1}{1-(2/n)} = 1 + \frac{2}{n} + \frac{4}{n^2} + \cdots,$$

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we obtain

$$\begin{aligned}\int_0^{\alpha(n-2)} \left(1-\frac{t}{n-2}\right)^{n-2} t^\beta \,\mathrm{d}t \\ &= \Gamma(\beta+1) + \frac{\gamma_1}{n}\frac{n}{n-2} + \cdots + \frac{\gamma_l}{n^l}\frac{n^l}{(n-2)^l} + O\left(\frac{1}{n^{l+1}}\frac{n^{l+1}}{(n-2)^{l+1}}\right) \\ &= \Gamma(\beta+1) + \frac{\gamma_1'}{n} + \cdots + \frac{\gamma_l'}{n^l} + O\left(\frac{1}{n^{l+1}}\right),\end{aligned}$$

from which we can get the coefficients $\gamma_1', \ldots, \gamma_l'$ by simple calculation. □

5. Proof of Theorem 2

Proof of Theorem 2. Substituting (13) in the integral (5) and using (12), we obtain

$$\begin{aligned}\mathbb{E}(f_0(K_n^1)) &= \binom{n}{2}\frac{1}{A^2}\int_{S^1}\int_0^{t_1}\left(1-\frac{A(u,t)}{A}\right)^{n-2} J(u,t)\,\mathrm{d}t\mathrm{d}u + O(n^{-k}) \\ &= \binom{n}{2}\frac{1}{A^2}\frac{1}{n-2}\int_{S^1}\int_0^{\tau(n-2)}\left(1-\frac{y}{n-2}\right)^{n-2} J\left(u,\frac{y}{n-2}\right)t'\left(\frac{y}{n-2}\right)\mathrm{d}y\mathrm{d}u \\ &\quad + O(n^{-k}).\end{aligned}$$

We evaluate the inner integral as follows. Collecting the terms according to the exponent of $y/(n-2)$ and also the error term yields

$$\begin{aligned}\binom{n}{2}\frac{1}{A^2}\frac{1}{n-2}\int_0^{\tau(n-2)}\left(1-\frac{y}{n-2}\right)^{n-2} J\left(u,\frac{y}{n-2}\right)t'\left(\frac{y}{n-2}\right)\mathrm{d}y \\ &= v_1\binom{n}{2}\frac{1}{A^2}\frac{1}{(n-2)^{5/3}}\int_0^{\tau(n-2)}\left(1-\frac{y}{n-2}\right)^{n-2} y^{2/3}\,\mathrm{d}y + \cdots \\ &\quad + v_{k-1}\binom{n}{2}\frac{1}{A^2}\frac{1}{(n-2)^{(k+3)/3}}\int_0^{\tau(n-2)}\left(1-\frac{y}{n-2}\right)^{n-2} y^{k/3}\,\mathrm{d}y \\ &\quad + O\left(\frac{1}{(n-2)^{(k-2)/3}}\int_0^{\tau(n-2)}\left(1-\frac{y}{n-2}\right)^{n-2} y^{(k+1)/3}\,\mathrm{d}y\right)\end{aligned} \tag{14}$$

as $n \to \infty$. The coefficients $v_1, \ldots, v_{k-1}$ can be determined explicitly. In particular,

$$v_1 = \frac{2}{3}p_1q_1, \qquad v_2 = 0, \qquad v_3 = \frac{4}{3}q_1p_3 + \frac{2}{3}p_1q_3.$$

Here we use p_1, p_3 and q_1, q_3 to express v_1, v_3 for the sake of brevity. Of course, they can also be expressed explicitly in terms of the u_i.

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We evaluate the above integrals one by one using Lemma 4. In particular, the first integral is as follows:

$$
\begin{aligned}
v_1 \binom{n}{2} \frac{1}{A^2} \frac{1}{(n-2)^{5/3}} \int_0^{\tau(n-2)} \left(1 - \frac{y}{n-2}\right)^{n-2} y^{2/3}\,\mathrm{d}y
&= \sqrt[3]{\frac{2}{3A}} \frac{(\kappa-1)^{1/3}}{\kappa} \frac{n(n-1)}{(n-2)^{5/3}} \left(\Gamma\left(\frac{5}{3}\right) - \frac{\Gamma(10/3)}{2}\frac{1}{n} + \cdots\right) \\
&= \sqrt[3]{\frac{2}{3A}} \frac{(\kappa-1)^{1/3}}{\kappa} \left(\Gamma\left(\frac{5}{3}\right) n^{1/3} + \left(\frac{7}{3}\Gamma\left(\frac{5}{3}\right) - \frac{\Gamma(10/3)}{2}\right)\frac{1}{n^{2/3}} + \cdots\right),
\end{aligned}
$$

where in the last line we used the binomial series expansion

$$
\frac{n(n-1)}{(n-2)^{5/3}} = n^{1/3} + \frac{7}{3} n^{-2/3} + \cdots
$$

The second (nonzero) integral is the following:

$$
\begin{aligned}
v_3 \binom{n}{2} \frac{1}{A^2} \frac{1}{(n-2)^{7/3}} \int_0^{\tau(n-2)} \left(1 - \frac{y}{n-2}\right)^{n-2} y^{4/3}\,\mathrm{d}y
&= \frac{v_3}{2A^2} \frac{n(n-1)}{(n-2)^{7/3}} \left(\Gamma\left(\frac{7}{3}\right) - \frac{\Gamma(13/3)}{2}\frac{1}{n} + \cdots\right) \\
&= \frac{v_3}{2A^2} \left(\Gamma\left(\frac{7}{3}\right) n^{-1/3} + \left(\frac{11\Gamma(7/3)}{3} - \frac{\Gamma(13/3)}{2}\right) n^{-4/3} + \cdots\right),
\end{aligned}
$$

where we used the binomial series expansion

$$
\frac{n(n-1)}{(n-2)^{7/3}} = n^{-1/3} + \frac{11}{3} n^{-4/3} + \cdots
$$

Evaluating the $k-1$ integrals in (14) and collecting the terms, including the error term, we obtain

$$
\begin{aligned}
\binom{n}{2} \frac{1}{A^2} \int_0^{t_1} \left(1 - \frac{A(u,t)}{A}\right)^{n-2} J(u,t)\,\mathrm{d}t = w_1 n^{1/3} + w_2 n^0 + \cdots + w_{k-1} n^{-(k-3)/3} \\
+ O(n^{-(k-2)/3}),
\end{aligned}
$$

where, in principle, all the coefficients $w_1, \ldots, w_{k-1}$ can be calculated explicitly. In particular,

$$
\begin{aligned}
w_1(u) &= \sqrt[3]{\frac{2}{3A}} \Gamma\left(\frac{5}{3}\right) \frac{(\kappa(u)-1)^{1/3}}{\kappa(u)}, \\
w_2(u) &= 0, \\
w_3(u) &= -\Gamma\left(\frac{7}{3}\right) \frac{1}{5} \sqrt[3]{\frac{3A}{2}} \left(\frac{\kappa''(u)}{3(\kappa(u)-1)^{4/3}\kappa(u)} + \frac{2\kappa^2(u) + 7\kappa(u) - 1}{2(\kappa(u)-1)^{1/3}\kappa(u)} - \frac{5(\kappa'(u))^2}{9(\kappa(u)-1)^{7/3}\kappa(u)}\right),
\end{aligned}
$$

where we recall that κ is a function of u.

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We note here that, when calculating further coefficients, we must also take into account some of the lower-order terms from previous integrals. This does not yet affect the evaluation of w_3, as the second-largest term in the first integral is $n^{-2/3}$. However, this would have to be added when calculating w_4, and so on.

Finally, integration with respect to u yields

$$\begin{aligned}\mathbb{E}(f_0(K_n^1)) &= \int_{S^1} w_1(u)n^{1/3} + w_2(u)n^0 + \cdots + w_{k-1}(u)n^{-(k-3)/3} + O(n^{-(k-2)/3})\,\mathrm{d}u \\ &= z_1(K)n^{1/3} + z_2(K)n^0 + \cdots + z_{k-1}(K)n^{-(k-3)/3} + O(n^{-(k-2)/3}),\end{aligned}$$

where, again, all coefficient can be found explicitly. In particular,

$$\begin{aligned}z_1(K) &= \int_{S^1} w_1(u)\,\mathrm{d}u = \sqrt[3]{\frac{2}{3A}}\,\Gamma\left(\frac{5}{3}\right)\int_{\partial K}(\kappa(x)-1)^{1/3}\,\mathrm{d}x, \\ z_2(K) &= 0, \\ z_3(K) &= \int_{S^1} w_3(u)\,\mathrm{d}u \\ &= -\Gamma\left(\frac{7}{3}\right)\frac{1}{5}\sqrt[3]{\frac{3A}{2}}\int_{\partial K}\frac{\kappa''(x)}{3(\kappa(x)-1)^{4/3}} + \frac{2\kappa(x)^2+7\kappa(x)-1}{2(\kappa(x)-1)^{1/3}} - \frac{5(\kappa'(x))^2}{9(\kappa(x)-1)^{7/3}}\,\mathrm{d}x,\end{aligned}$$

where we use that if ∂K is C^2_+-smooth and $f(u)$ is a measurable function on S^1, then $\int_{S^1} f(u)\,\mathrm{d}u = \int_{\partial K} f(u_x)\kappa(x)\,\mathrm{d}x$ [30, (2.62)]. This completes the proof of Theorem 2. □

6. The case of the unit circle

For the sake of completeness, we consider the case when $K = B(R)$. Since $\mathbb{E}(f_0(B(R)_n^R))$ is independent of R, we may assume that $R = 1$. We will use the simpler notation $B_n^1 = B(1)_n^1$. In [11, p. 916] it was proved that

$$\mathbb{E}(f_0(B_n^1)) = \binom{n}{2}\,4\int_0^{\pi}\sin(\sigma)\left(1 - \frac{\sin(\sigma)+\sigma}{\pi}\right)^{n-1}\mathrm{d}\sigma.$$

Let

$$\frac{y}{n-1} = \frac{\sin(\sigma)+\sigma}{\pi}.$$

Since $\sin(\sigma)+\sigma$ is a strictly monotonically increasing analytic function on $[0, \pi]$, its inverse is also a strictly monotonically increasing analytic function by the Lagrange inversion theorem. Then σ has a power-series expansion in terms of $y/(n-1)$ around $y = 0$ as follows:

$$\sigma = c_1\left(\frac{y}{n-1}\right) + c_3\left(\frac{y}{n-1}\right)^3 + \cdots + c_{2k+1}\left(\frac{y}{n-1}\right)^{2k+1} + \cdots,$$

where all the coefficients can be calculated explicitly. In particular,

$$c_1 = \frac{\pi}{2}, \qquad c_3 = \frac{\pi^3}{96}, \qquad c_5 = \frac{\pi^5}{1920}.$$

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Thus,

$$\sin(\sigma) = e_1\left(\frac{y}{n-1}\right) + e_3\left(\frac{y}{n-1}\right)^3 + \cdots + e_{2k+1}\left(\frac{y}{n-1}\right)^{2k+1} + \cdots,$$

where the coefficients can be calculated explicitly. In particular,

$$e_1 = \frac{\pi}{2}, \qquad e_3 = -\frac{\pi^3}{96}, \qquad e_5 = -\frac{\pi^5}{1920}.$$

Therefore,

$$\begin{aligned}
\mathbb{E}(f_0(B_n^1)) = &\binom{n}{2}\frac{4}{n-1}\int_0^{n-1}\left(1-\frac{y}{n-1}\right)^{n-1}\sin\left(\sigma\left(\frac{y}{n-1}\right)\right)\sigma'\left(\frac{y}{n-1}\right)\mathrm{d}y \\
= &f_1\binom{n}{2}\frac{4}{(n-1)^2}\int_0^{n-1}\left(1-\frac{y}{n-1}\right)^{n-1} y\,\mathrm{d}y \\
&+ f_3\binom{n}{2}\frac{4}{(n-1)^4}\int_0^{n-1}\left(1-\frac{y}{n-1}\right)^{n-1} y^3\,\mathrm{d}y \\
&+ \cdots + f_{2k+1}\binom{n}{2}\frac{4}{(n-1)^{2k+2}}\int_0^{n-1}\left(1-\frac{y}{n-1}\right)^{n-1} y^{2k+1}\,\mathrm{d}y + \cdots,
\end{aligned}$$

where all the coefficients $f_1, \ldots, f_{2k+1}, \ldots$ can be evaluated explicitly using Lemma 3 and the binomial series expansion of $n/(n-1)^{2k+1}$. In particular,

$$f_1 = \frac{\pi^2}{4}, \qquad f_3 = \frac{\pi^4}{96}, \qquad f_5 = \frac{11\pi^6}{15\,360}.$$

Thus, by Lemma 4, the first integral yields

$$\begin{aligned}
&\binom{n}{2}\frac{\pi^2}{(n-1)^2}\int_0^{n-1}\left(1-\frac{y}{n-1}\right)^{n-1} y\,\mathrm{d}y \\
&\qquad = \frac{\pi^2}{2}\frac{n}{n-1}\left(\Gamma(2) - \frac{\Gamma(4)}{2}\frac{1}{n-1} + \left(\frac{-\Gamma(5)}{3} + \frac{\Gamma(6)}{8}\right)\frac{1}{(n-1)^2} + \cdots\right) \\
&\qquad = \frac{\pi^2}{2}\left(1 - \frac{2}{n} + \frac{2}{n^2} + \cdots\right)
\end{aligned}$$

The second integral yields

$$\begin{aligned}
&f_3\binom{n}{2}\frac{4}{(n-1)^4}\int_0^{n-1}\left(1-\frac{y}{n-1}\right)^{n-1} y^3\,\mathrm{d}y \\
&\qquad = \frac{\pi^4}{48}\frac{n}{(n-1)^3}\left(\Gamma(4) - \frac{\Gamma(6)}{3}\frac{1}{n-1} + \left(-\frac{\Gamma(7)}{3} + \frac{\Gamma(8)}{8}\right)\frac{1}{(n-1)^2} + \cdots\right)
\end{aligned}$$

Thus, for any k, $\mathbb{E}(f_0(B_n^1)) = w_0 n^0 + w_1 n^{-1} + w_2 n^{-2} + \cdots + w_k n^{-k} + O(n^{-k-1})$, where all the coefficients $w_1, \ldots w_1, \ldots, w_k$ can be calculated explicitly. In particular,

$$w_0 = \frac{\pi^2}{2}, \qquad w_1 = -\pi^2, \qquad w_2 = \frac{\pi^4 + 8\pi^2}{8}, \qquad w_3 = \frac{13\pi^2}{3} - \frac{11\pi^4}{24}.$$

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Competing interests

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