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On the C^r-closing lemma for flows on the torus T^2

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Abstract. Vector fields of $\mathfrak{X}^r(T^2)$, $1 \le r \le \infty$, with non-trivial recurrent points are classified in two types, one of which we call the constant type inspired by the terminology of continued fractions. Let $X \in \mathfrak{X}^r(T^2)$ have finitely many singularities and $p \in T^2$ be a non-wandering point of X. With the exception of the case when X is of constant type and, simultaneously, p is non-trivial recurrent, we prove that there exists $Y \in \mathfrak{X}^r(T^2)$ arbitrarily close to X (in the C^r-topology) having a periodic trajectory through p.

1. *Introduction*

Perhaps the most important generic property so far discovered – as Palis and de Melo say in [**Pa–Me**, p. 172] – is Pugh's C^1-general density theorem [**Pg.2**]. One way of extending this result to classes of differentiability $r \ge 2$, would be to give a positive answer to the open problem, known as the C^r-closing lemma, whose statement is:

"*Let M be a smooth compact manifold, $r \ge 2$ be an integer, $f \in \mathrm{Diff}^r(M)$ (resp. $X \in \mathfrak{X}^r(M)$) and p be a non-wandering point of f (resp. of X). There is $g \in \mathrm{Diff}^r(M)$ (resp. $Y \in \mathfrak{X}^r(M)$) arbitrarily close to f (resp. to X) in the C^r-topology so that p is a periodic point of g (resp. of Y)*".

C. Pugh proved the C^1-closing lemma [**Pg.2**] and put in doubt the validity of the C^2-closing lemma [**Pg.1**]. The only result known about the C^r-closing lemma problem, when $r \ge 2$, is that it is true for diffeomorphisms of the circle. In this paper we extend this result by proving a partial C^r-closing lemma for flows on the torus.

We wish to mention some C^r-closing lemma type results. One is the Peixoto's C^r-connecting lemma which was used to characterize structurally stable vector fields on two-manifolds [**Pe**]. Another is Mañé's C^1-ergodic closing lemma, which was utilized to characterize structurally stable diffeomorphisms of two-manifolds [**Ma**]. Moreover, we have the Takens C^1-connecting result which was used to prove generic properties in conservative systems [**Ta**]. Finally, the Pixton–Robinson C^r-connecting result for diffeomorphisms on the sphere S^2 which is a positive answer in the direction of the C^r-closing lemma [**Px**].

Now we proceed to explain our result. Let $r \ge 1$ be an integer, $X \in \mathfrak{X}^r(T^2)$, $p \in T^2$ and γ_p be the trajectory passing through the point p; we say that $x \in \{p, \gamma_p\}$ is *non-wandering* if there exist a sequence of points of the torus $p_n \mapsto p$ and a sequence of real numbers $t_n \mapsto \infty$ such that either $X_{t_n}(p_n) \mapsto p$ or $X_{(-t_n)}(p_n) \mapsto p$, where X_t $(t \in \mathbb{R})$ is the flow induced by X. If $p_n = p$ in this definition and moreover $X_t(p) \neq p$, for all $t \in \mathbb{R} - \{0\}$, the non-wandering $x \in \{p, \gamma_p\}$ will be called *non-trivial recurrent.* Let $\alpha \in \mathbb{R}$ and $R_\alpha : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ be the geometric rotation $x \to x + \alpha$ (mod 1). Let $\{a_i\}$ be the sequence of integers (with $a_i \in \mathbb{N}$ when $i \geq 1$) forming the partial quotients of α; i.e.

$$\alpha = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 \cdots}}}$$

We say that R_α is of *constant type* if $\sup\{a_i | i = 0, 1, 2, \ldots\} < \infty$.

As we shall state in theorem 4.1, when $X \in \mathfrak{X}^r(T^2)$, $r \geq 1$, has a non-trivial recurrent trajectory γ, there always exists a circle C transversal to X passing through γ. Moreover if $T: C \to C$ denotes the forward Poincaré map induced by X, there exists a monotone continuous map $h: C \to \mathbb{R}/\mathbb{Z}$ of degree one and a geometric rotation $R(C): \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ such that, for all $x \in \mathrm{Dom}(T)$ (the *domain of definition* of T) $h \circ T(x) = R(C) \circ h(x)$. We say that X, having a non-trivial recurrent trajectory, *is* (resp. *is not*) *of constant type* when $R(C)$ is (resp. is not). It will be proved that the property that $R(C)$ is (resp. is not) of constant type *depends* on the vector field X *only.*

The main result of this paper is:

THEOREM A. *Let $X \in \mathfrak{X}^r(T^2)$, $1 \leq r \leq \infty$, have finitely many singularities and $p \in T^2$ be a non-wandering point of X. Suppose that if p is non-trivial recurrent, X is not of constant type. Then, there exists $Y \in \mathfrak{X}^r(T^2)$ arbitrarily close to X (in the C^r-topology) having a periodic trajectory through p.*

The proof of theorem A will show us that the result is valid for a large set of vector fields on the torus.

We recall that the non-constant type real numbers of [0, 1] form a full Lebesgue measure subset of it [**Khi**].

We wish to observe that theorem A extends word-for-word to vector fields on the torus with a cross-cap. This can be done using [**Gu.1**, theorem 3.1] and [**Gu.2**, proposition 2].

All possible examples of smooth vector fields on T^2 with non-trivial recurrence (whether of constant type or not) can easily be constructed by using the existence theorem of [**Gu.1**].

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2. *Preliminaries*

We shall introduce some terminology and notation. Let M be a smooth compact two-manifold, $\varphi: \mathbb{R} \times M \to M$ be a continuous flow on M and p be a point of M.

The positive *semi-trajectory* (resp. *negative semi-trajectory*) of p is the set $\gamma_p^+ = \{\varphi(t, p)/t \in [0, \infty)\}$ (resp. $\gamma_p^- = \{\varphi(t, p)/t \in (-\infty, 0]\}$). The trajectory $\gamma_p^+ \cup \gamma_p^-$ of p will be denoted by γ_p. The point p is a *regular* point of φ if it is not a fixed point of φ. We say that $x \in \{p, \gamma_p\}$ is *periodic* if $\varphi(t, p) = p$ for some $t \geq 0$. A point $q \in M$ is an *ω-limit point* (resp. *α-limit point*) of $x \in \{p, \gamma_p\}$, if there is a sequence of real numbers $t_k \to \infty$ (resp. $t_k \to -\infty$) such that $\varphi(t_k; p) \to q$.

Let N be a submanifold of M disjoint from the fixed points of φ. We will say that N is a *flow box* (of φ) if there exists a rectangle $A = [a, b] \times [c, d] \subset \mathbb{R}^2$ and a homeomorphism $\theta: A \to N$ such that, for all $s \in [c, d]$, $\theta([a, b] \times \{s\})$ is an arc of trajectory of φ. Such a homeomorphism $\theta: A \to N$ will also be called a *flow box*. If p is a regular point of φ, there exists a neighbourhood of p which is a flow box (see [**B–S**, theorem 2.9, p. 50] and [**Wt**]). A *segment* or a *circle* C is said to be *transversal* to φ if for any $x \in C$ which is not an endpoint of it, there exists a flow box $\theta: [-1, 1] \times [-1, 1] \to N$ such that $\theta(0, 0) = x$ and $\theta(\{0\} \times [-1, 1]) = N \cap C$. Our definition of transversality *does not exclude* points of tangency for C^r flows.

Let C be a circle transversal to φ. The *forward Poincaré* map $T: C \to C$ is the one which takes $x \in C$ to the first intersection of $\gamma_x^+ - \{x\}$ with C. In most cases the domain of definition of T, Dom(T), is an open subset of C properly contained in it.

Let $q \in M$. We say that $x \in \{q, \gamma_q\}$ is *two-sided non-trivial recurrent* if for any open segment Σ passing through q, each semitrajectory starting at q meets both connected components of $\Sigma - \{q\}$ infinitely many times.

When $X \in \mathfrak{X}^r(M)$, $r \geq 1$, all the definitions of this work will be referred to the flow induced by X. The set of positive integers will be denoted by $\mathbb{N}$. T^2 will be the two-dimensional torus provided with a smooth riemannian structure.

Given $\alpha \in \mathbb{R}$, $R_\alpha: \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ will denote the geometric rotation $x \to x + \alpha$ (mod 1).

3. *The continued fraction of a real number*

The topics and the notation of this section are part of those of [**Her**, Chap. V]. The proofs can be found in [**Her**, Chap. V], [**La**] and [**Khi**].

Let $G: (0, 1] \to [0, 1)$ and $a: (0, \infty) \to \mathbb{N}$ be defined by

$$G(x) = \frac{1}{x} - \left[\frac{1}{x}\right] = \frac{1}{x} \pmod 1$$

$$a(x) = \left[\frac{1}{x}\right].$$

If $\alpha \in \mathbb{R} - \mathbb{Q}$ we have that $\alpha - [\alpha] \in (0, 1)$ and, for all $n \in \mathbb{N}$, $G^n(\alpha - [\alpha]) \neq 0$. Therefore we may define

$$a_0 = [\alpha]$$

and

$$a_n = a(G^{n-1}(\alpha - [\alpha]) \qquad \text{if } n \geq 1.$$

Thus, $a_0 \in \mathbb{Z}$ and, for all $n \in \mathbb{N}$, $a_n \in \mathbb{N}$. We shall write symbolically the infinite expression

$$\alpha = [a_0, a_1, a_2, \ldots, a_n, \ldots] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots}}$$

The a_n is called the nth *partial quotient* of α; any sequence $\{a_0, a_1, \ldots, a_n, \ldots\}$, where $a_0 \in \mathbb{Z}$ and $a_n \in \mathbb{N}$, for all $n \in \mathbb{N}$, is the sequence of partial quotients of a unique $\alpha \in \mathbb{R} - \mathbb{Q}$.

If $\alpha = [a_0, a_1, \ldots, a_i, \ldots]$ the nth *principal convergent* of α is the reduced fraction

$$\frac{p_n}{q_n} = [a_0, a_1, \ldots, a_n] = a_0 + 1/(a_1 + 1/(a_2 + \cdots + 1/(a_{n-1} + 1/a_n) \cdots)).$$

Given $x \in \mathbb{R}$ write $\|x\| = \operatorname{Inf}_{p\in\mathbb{Z}} |x+p|$. $\|x\|$ defines a metric on T^1.

In the following, α denotes an irrational number, $\{p_n/q_n\}$ is the sequence of principal convergents of α, $\{a_n\}$ is the sequence of partial quotients of α, $\widehat{q_n\alpha} = q_n\alpha - p_n$, and, for $\theta \in \mathbb{R}$, $I_n + \theta$ is the compact sub-interval of $\mathbb{R}$ with endpoints θ and $\widehat{q_n\alpha} + \theta$. The simplified notation $I_n + 0 = I_n$ will be used.

(3.1) PROPOSITION. *If $|q| > 0$ is an integer such that $|q| < q_{n+1}$, then $\|q\alpha\| \geq \|q_n\alpha\|$. Conversely, $q_0 = 1$, $q_1 = a_1$ and, if $n \geq 1$, q_{n+1} is the smallest positive integer such that $\|q_{n+1}\alpha\| < \|q_n\alpha\|$.*

(3.2) PROPOSITION. *p_n and q_n satisfy the following relations*

$$p_n = a_n p_{n-1} + p_{n-2} \quad \text{for } n \geq 2, \qquad p_0 = a_0, \quad p_1 = a_0 a_1 + 1$$
$$q_n = a_n q_{n-1} + q_{n-2} \quad \text{for } n \geq 2, \qquad q_0 = 1, \quad q_1 = a_1.$$

(3.3) PROPOSITION. *If $n \geq 1$, $|q_n\alpha - p_n| = \|q_n\alpha\|$, and, if $n \geq 3$:*

$$\|q_{n-2}\alpha\| = a_n \|q_{n-1}\alpha\| + \|q_n\alpha\|$$

and

$$a_n = \left[\frac{\|q_{n-2}\alpha\|}{\|q_{n-1}\alpha\|}\right].$$

(3.4) PROPOSITION. *Let $\theta \in \mathbb{R}$. For all $n \geq 2$ we have that:*

(a) *The intervals modulo 1 $\{R_{j\alpha}(I_n + \theta)\}_{0 \leq j \leq q_n}$, where j is an integer, are pairwise disjoint excepting for the fact that*

$$(I_n + \theta) \cap \{R_{q_n\alpha}(I_n + \theta)\} = \{\widehat{q_n\alpha} + \theta\};$$

(b) *the real numbers θ, $\widehat{q_n\alpha} + \theta$ and $\widehat{2q_n\alpha} + \theta$ (forming the set of endpoints of $I_n + \theta$ and $R_{q_n\alpha}(I_n + \theta)$) are ordered in the real line as follows:*

$$\theta < (-1)^n(\widehat{q_n\alpha}) + \theta < (-1)^n(\widehat{2q_n\alpha}) + \theta < (-1)^n(\widehat{q_{n-2}\alpha}) + \theta.$$

The proof of the following lemma is contained in [Sl]. See [**Her**, 8.4 of Chap. V].

(3.5) LEMMA. *For all $n \in \mathbb{N}$, $n \geq 2$, the elements of $\{A_{i,n} \mid i \in \{0, 1, 2, \ldots, [a_n/2]\}\}$ are pairwise disjoint sets, where*

$$A_{i,n} = \bigcup_{j=2iq_n}^{(2i+1)q_n} R_{j\alpha}(I_n) = \bigcup_{j=0}^{q_n} R_{j\alpha + 2i\widehat{q_n\alpha}}(I_n).$$

(3.6) PROPOSITION. *Let $\{i_k\}$ be a strictly increasing sequence of odd natural numbers. Suppose that* $\lim_k a_{i_k+1}=\infty$. *Given a finite subset* $\{x_1, x_2, \ldots, x_m\}$ *of* $\mathbb{R}/\mathbb{Z}$, *there exists* $n_0 \in \mathbb{N}$ *such that if* $i_k \geq n_0$ *then for some* $\theta \in (0, \|q_{i_k-2}\alpha\|)$, $\bigcup_{j=0}^{q_{i_k}} R_{j\alpha+\theta}(I_{i_k})$ *is disjoint from* $\{x_1, x_2, \ldots, x_m\}$.

Proof. We take $n_0 \in \mathbb{N}$ so that, for all $i_k \geq n_0$, $a_{i_k+1} > 2(m+1)$. It follows from lemma 3.5 that the elements of $\{A_{j,i_k} | j=1,2,\ldots,m+1\}$ are pairwise disjoint sets. Therefore, for some $\sigma \in \{1,2,\ldots,m+1\}$, $A_{\sigma,i_k}=\bigcup_{j=0}^{q_{i_k}} R_{j\alpha+\theta}(I_{i_k})$ is disjoint from $\{x_1, x_2, \ldots, x_m\}$ and so it satisfies the conditions of this proposition. □

4. *Flows of non-constant type*

In this section we shall define the flows of non-constant type. To give this definition we shall need theorem 4.1 and proposition 4.2.

The following result is contained in [**Gu.1**, structure theorem and (E3) of existence theorem].

(4.1) STRUCTURE THEOREM. *Let* $\varphi: \mathbb{R} \times T^2 \to T^2$ *be a continuous flow on the torus* T^2 *having non-trivial recurrent trajectories. Then there exists a compact* φ*-invariant subset* Ω *of* T^2 *such that any non-trivial recurrent trajectory of* φ *is dense in* Ω. *Moreover, the set* $\mathscr{C}(\Omega)$, *of the oriented circles which meet* Ω *and are transversal to* φ, *is not empty and the following are satisfied*:

(a) *Given* $C \in \mathscr{C}(\Omega)$, *there exists a unique non-periodic geometric rotation* $R(C): \mathbb{R}/\mathbb{Z} \circlearrowleft$ *such that if* $\mathbb{R}/\mathbb{Z}$ *is provided with the usual positive orientation and* $T: C \to C$ *denotes the forward Poincaré map induced by* φ, *there is an orientation preserving monotone continuous map* $h: C \to \mathbb{R}/\mathbb{Z}$ *of degree one semi-conjugating* T *with* $R(C)$ (*i.e. for all* $x \in \mathrm{Dom}\,(T)$, $h \circ T(x) = R(C) \circ h(x)$). *Moreover, if* $h_1, h_2: C \to \mathbb{R}/\mathbb{Z}$ *are maps as the map* h *above then, for some geometric rotation* $R_\alpha: \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$, $h_1 = R_\alpha \circ h_2$.

(b) *If the flow* φ *has* n *fixed points and* T *and* h *are as in* (a) *above, then there exists* $\{x_1, x_2, \ldots, x_m\} \subset \mathbb{R}/\mathbb{Z}$, *with* $m \leq n$, *such that*

$$h^{-1}(\mathbb{R}/\mathbb{Z} - \{x_1, x_2, \ldots, x_m\}) \subset \mathrm{Dom}\,(T).$$

The following proposition is an immediate consequence of lemma 4.7 below.

(4.2) PROPOSITION. *Let* φ *and* $\mathscr{C}(\Omega)$ *be as in the structure theorem. The property that* $R(C)$ *is* (*resp. is not*) *of constant type, where* $C \in \mathscr{C}(\Omega)$, *does not depend on the particular* C.

This proposition implies that the following definition is intrinsic.

(4.3) *Definition.* Let φ and $\mathscr{C}(\Omega)$ be as in the structure theorem. We say that φ is a *constant* (resp. *non-constant*) type flow if for some $C \in \mathscr{C}(\Omega)$, $R(C)$ is of constant (resp. non-constant) type.

Now we proceed to prove some results which will be needed in the proof of lemma 4.7.

(4.4) COROLLARY. *Assume the conditions and notation of the structure theorem. Given* $C \in \mathscr{C}(\Omega)$, *the subset of* C *of two-sided non-trivial recurrent points is dense in* $\Omega \cap C$.

Proof. Let $h: C \mapsto \mathbb{R}/\mathbb{Z}$, $T: C \mapsto C$ and $R(C): \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ be as in (a) of the structure theorem. Let $A_0 = \mathbb{R}/\mathbb{Z}$ and for $i \in \mathbb{Z} - \{0\}$ let $A_i = \{x \in \mathbb{R}/\mathbb{Z} \mid h^{-1}(x)$ is contained in $\text{Dom}(T^i)\}$. As, for all $i \in \mathbb{Z} - \{0\}$, $\text{Dom}(T^i)$ is an open subset of C, by (a) of theorem 4.1 we have that A_i is an open subset of $\mathbb{R}/\mathbb{Z}$.

Using the property that $C \cap \Omega$ contains a non-trivial recurrent point, say p, we shall prove that every A_i is dense in $\mathbb{R}/\mathbb{Z}$. In fact given $x_0 \in \mathbb{R}/\mathbb{Z}$, by the structure theorem, there exists a subsequence $\{x_n\}_{n \geq 1}$ of the $R(C)$-orbit of $h(p)$ such that $\lim_n x_n = x_0$ and $h^{-1}(x_n)$ contains exactly one element p_n of $\gamma_p \cap \text{Dom}(T^i)$. Denote by J_n the connected component of $\text{Dom}(T^i)$ containing p_n. Since J_n is open and p_n is non-trivial recurrent, the set $\gamma_p \cap J_n$ is an infinite set. Also h is injective in $\gamma_p \cap J_n$. Therefore $h(J_n)$ is an interval of $\mathbb{R}/\mathbb{Z}$. Because $\mathbb{R}/\mathbb{Z} - \mathfrak{A}$ is at most denumerable, where $\mathfrak{A} = \{x \in \mathbb{R}/\mathbb{Z} \mid h^{-1}(x)$ is a one point set$\}$, there exists $q_n \in \mathfrak{A} \cap h(J_n)$ very close to $x_n = h(p_n)$. As $h^{-1}(q_n)$ is unitary, $h^{-1}(q_n) \in J_n \subset \text{Dom}(T^i)$, i.e. $q_n \in A_i$. Certainly $\lim_n q_n = x_0$ which implies that A_i is dense. Therefore, $\beta = \mathfrak{A} \cap (\bigcap_{i \in \mathbb{Z}} A_i)$ is a residual subset of $\mathbb{R}/\mathbb{Z}$. Under these circumstances, using the structure theorem we may easily check that: Any $y \in h^{-1}(\beta)$ is a two-sided non-trivial recurrent point whose positive (resp. negative) T-orbit is dense in $C \cap \Omega$. This proves the corollary. □

The proof of the following lemma can be found in [**Gu.2**, lemma 2].

(4.5) Lemma. *Let Σ be an open interval containing a non-trivial recurrent point p of a continuous flow $\varphi: \mathbb{R} \times T^2 \to T^2$. If Σ is transversal to φ, there exists a circle C transversal to φ and such that $C \cap \Sigma$ contains an open interval passing through p.*

(4.6) *Definition.* Consider T^2 provided with an orientation and with a smooth riemannian metric $\langle\ ,\ \rangle$. Given $X \in \mathfrak{X}^r(T^2)$, $1 \leq r < \infty$, *we define* $X^\perp \in \mathfrak{X}^r(T^2)$ by the following two conditions:

(a) $\langle X, X \rangle = \langle X^\perp, X^\perp \rangle$; and

(b) when $p \in T^2$ is a regular point of X, the ordered pair $(X(p), X^\perp(p))$ is an orthogonal positive basis of $T_p(T^2)$ (according to the given orientation of T^2).

Let $\varphi: \mathbb{R} \times T^2 \to T^2$ be a continuous flow on T^2 and Σ be an oriented open segment transversal to φ. We say that Σ is *φ-positive* (resp. *φ-negative*) if – for increasing time – the flow φ crosses Σ as in figure 4.1 (resp. figure 4.2), where a positive basis is that of figure 4.3.

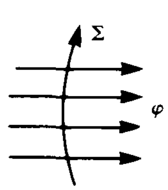

Figure 4.1

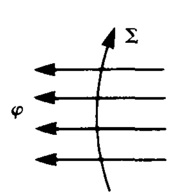

Figure 4.2

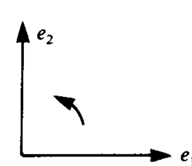

Figure 4.3. Positive basis $\{e_1, e_2\}$.

Let C be an oriented circle transversal to φ; we say that C is φ-positive if any sub-interval of it (with the induced orientation) is φ-positive.

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(4.7) LEMMA. *Assume the conditions and notation of the structure theorem* (4.1). *Given* $C, \tilde{C} \in \mathscr{C}(\Omega)$, *there exist* $m, n \in \mathbb{N}$ *such that, for all* $i \in \mathbb{N}$, $a_{i+n} = \tilde{a}_{i+m}$, *where* $\{a_k\}$ *and* $\{\tilde{a}_k\}$ *denote the sequence of partial quotients of the rotation numbers*, α *and* $\tilde{\alpha}$, *of* $R(C)$ *and* $R(\tilde{C})$, *respectively. Moreover we have that* $m-n$ *is an even integer if and only if both* C *and* $\tilde{C}$ *are simultaneously either* φ*-positive or* φ*-negative.*

Proof. When $p \in M$ and $q \in \varphi(\tau, p)$, for some $\tau > 0$, the arc of trajectory connecting p with q will be denoted by $\widehat{pq}$.

By corollary 4.4, there is a two-sided non-trivial recurrent point $p \in \Omega \cap C$. Denote by $T: C \mapsto C$ (resp. $\tilde{T}: \tilde{C} \mapsto \tilde{C}$) the forward Poincaré map induced by φ, and by Σ_n (resp. $\tilde{\Sigma}_n$), with $n \in \mathbb{N} - \{0\}$, the open sub-interval of C (resp. $\tilde{C}$) with endpoints $T^{-n}(p)$ and $T^n(p)$ (resp. $\tilde{T}^{-n}(p)$ and $\tilde{T}^n(p)$) and containing p. Let $\{q_n\}$ (resp. $\{\tilde{q}_n\}$) be the sequence of the denominators of the principal convergents of α (resp. $\tilde{\alpha}$). It follows from proposition 3.1 that:

(1.1) If $n \in \mathbb{N} - \{1, 2\}$, *then* $\widehat{pT^{q_{n+1}}(p)} \cap \Sigma_{q_n} = \{p, T^{q_{n+1}}(p)\}$.

(1.2) If $m \in \mathbb{N}$ is large enough and $\Sigma_m \cap \widehat{pT^m(p)} = \{p\}$, then m is equal to some q_n.

We observe that:

(2) If C and $\tilde{C}$ are homotopic and the homotopy carries the orientation of one to the given orientation of the other, then $R(C) = R(\tilde{C})$.

Let q be the first point where γ_p^+ meets $\tilde{C}$ and B be a closed flow box whose interior contains $\widehat{pq}$. We may use an homotopy - with support in B - to transform $\tilde{C}$ in a new circle still denoted by $\tilde{C}$ but now satisfying:

(3) $C \cap \tilde{C}$ contains an open interval Σ which contains p.

It follows from (2) that, without losing generality, we may proceed with the proof of this lemma under the assumption (3). So, by (1.1) and (1.2), we conclude that:

(4) There exist $n, m \in \mathbb{N}$ such that, for all $i \in \mathbb{N}$,

$$T^{q_{n+i}}(p) = \tilde{T}^{\tilde{q}_{m+i}}(p) \quad \text{and} \quad \Sigma_{q_{n+i}} = \tilde{\Sigma}_{\tilde{q}_{m+i}}.$$

Let $\mathfrak{A}_j(\Sigma_l)$ (resp. $\tilde{\mathfrak{A}}_j(\tilde{\Sigma}_l)$) be the cardinal number of the set $\widehat{pT^j(p)} \cap \Sigma_l$ (resp. $\widehat{p\tilde{T}^j(p)} \cap \tilde{\Sigma}_l$), where $j, l \in \mathbb{N}$. By (4), proposition 3.3 and because $R(C)$ and $R(\tilde{C})$ are uniquely ergodic we may use the Birkhoff's ergodic theorem [**Wa**] to conclude that if n, m are as in (4), for all $i \in \mathbb{N}$

$$a_{n+i} = \left[\frac{h(\Sigma_{q_{n+i-2}})}{h(\Sigma_{q_{n+i-1}})}\right] = \left[\lim_j \frac{\mathfrak{A}_{q_{n+i+j}}(\Sigma_{q_{n+i-2}})}{\mathfrak{A}_{q_{n+i+j}}(\Sigma_{q_{n+i-1}})}\right]$$
$$= \left[\lim_j \frac{\tilde{\mathfrak{A}}_{\tilde{q}_{m+i+j}}(\tilde{\Sigma}_{\tilde{q}_{m+i-2}})}{\tilde{\mathfrak{A}}_{\tilde{q}_{m+i+j}}(\tilde{\Sigma}_{\tilde{q}_{m+i-1}})}\right] = \tilde{a}_{m+i}.$$

Using (4) and (b) of proposition 3.4, we may easily conclude that $m-n$ is even if and only if the orientations of Σ, induced by those of C and $\tilde{C}$, are the same. The lemma is proved. □

5. *The main result*

(5.1) *Proof of theorem* A. As in Pugh's proof of the C^1-closing lemma, it is enough to prove that

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(1) Given any neighbourhoods $\mathcal{V}$ of X in $\mathfrak{X}^r(T^2)$ and V of p in T^2, there exists $Y \in \mathcal{V}$ which has a closed orbit meeting V.
In fact, let $\tilde{X} \in \mathfrak{X}^\infty(T^2)$ be such that $\tilde{X}(p) = X(p)$. Given a neighbourhood $\mathcal{W}$ of X in $\mathfrak{X}^r(T^2)$ we may find a neighbourhood $\mathcal{V} \subset \mathcal{W}$ of X and an $\varepsilon > 0$ so small that given any $(Y, t) \in \mathcal{V} \times (-\varepsilon, \varepsilon)$, $(\tilde{X}_t^\perp)_* Y \in \mathcal{W}$ and $V = \{X_s \circ \tilde{X}_t^\perp(p) | s, t \in (-\varepsilon, \varepsilon)\}$ is a flow box of X and a neighbourhood of p. Hence assuming that (1) is true, we may take $\hat{Y} \in \mathcal{V}$ having a closed orbit passing through the point $\tilde{X}_{-t_0}^\perp(p) = \hat{q}$ (with $t_0 \in (-\varepsilon, \varepsilon)$). Therefore $(\tilde{X}_{t_0}^\perp)_* \hat{Y} \in \mathcal{W}$ is the desired vector field to prove this theorem because it has a closed orbit passing through the point p.

We shall continue with the proof assuming that p is not periodic (and so it is not a singularity of X). Now, we shall prove this theorem when:

(2) p is regular and it is not non-trivial recurrent.
In this case γ_p connects (one or) two singularities of X [**S–T**, theorem 6.2]. Moreover there exists an open annulus U and a point $q \in U$ whose α- or ω-limit set contains γ_p [**Ne**, proposition 2.5]. See figure 5.1. In this case, it is very easy to produce a

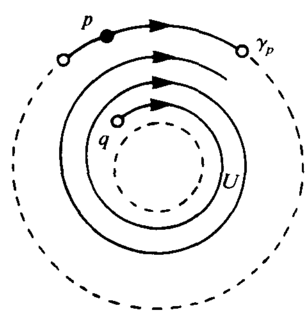

FIGURE 5.1

periodic orbit which meets any neighbourhood of p by arbitrarily small C^r-perturbations of X. Using (1), this theorem is proved under the assumption (2).

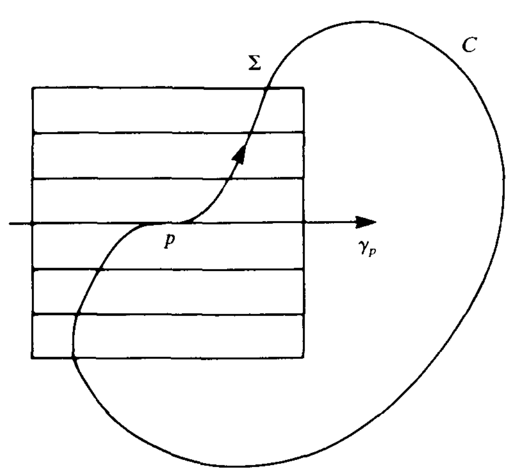

FIGURE 5.2

Therefore it follows from corollary 4.4 that, to prove (1) under the assumptions of theorem A, we may assume that

(3) The point p is two-sided non-trivial recurrent and X is not of constant type. Fix an orientation of T^2 and take a C^1 open interval Σ, X-positive, passing through p. See figure 5.2.

It follows from lemma 4.5 that we may assume that Σ is contained in a C^1 circle C transversal to X. Take in C the orientation induced by that of Σ. Let $T: C \to C$ be the forward Poincaré map induced by φ. Using lemma 4.7 and the fact that X is not of constant type we may assume that the orientation of T^2 has been chosen so that

(4) If h and $R(C)$ are maps as in (a) of the structure theorem and α is the rotation number of $R(C)$, then there exists a subsequence $\{a_{i_k}\}$ of the sequence $\{a_i\}$ of partial quotients of α such that every suffix i_k is an odd number and $\lim_k a_{i_k+1} = \infty$.

Let $\mathscr{V}$ be a neighbourhood of X in $\mathfrak{X}^r(T^2)$. Suppose that Σ has been taken to be tangent to γ_p at p and so that, for some $\varepsilon > 0$:

(5) There exists a continuous strictly increasing map $\sigma: [0, \varepsilon) \to C$, with $\sigma(0) = p$, and such that, for all $\mu \in (0, \varepsilon)$, $X + \mu X^\perp \in \mathscr{V}$ and the segments $\sigma((0, \mu))$ and $\sigma((\mu, \varepsilon))$, with the orientation induced by that of C, are $(X + \mu X^\perp)$-positive and $(X + \mu X^\perp)$-negative, respectively. See figure 5.3.

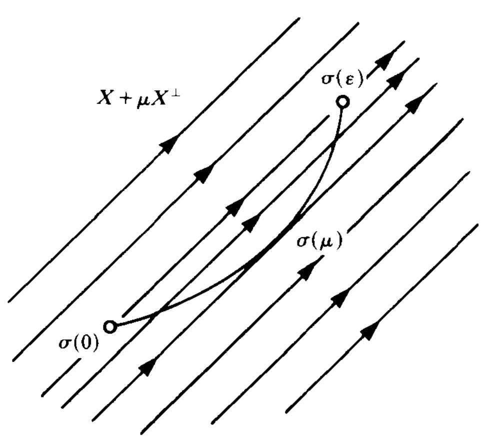

FIGURE 5.3

Since p is a two-sided non-trivial recurrent point $h(\sigma[0, \varepsilon))$ contains an open interval I. By corollary 4.4 we may take a two-sided non-trivial recurrent point $q \in h^{-1}(I) \cap \sigma(0, \varepsilon)$. By the structure theorem we may assume that $h(q) = 0$ and that there exists a finite subset $\{x_1, x_2, \ldots, x_m\} \subset \mathbb{R}$ such that

(6) $h^{-1}(\mathbb{R}/\mathbb{Z} - \{x_1, x_2, \ldots, x_m\}) \subset \mathrm{Dom}\,(T)$.

It follows from this, (4), and propositions 3.4 and 3.6, that there exist $n \in \mathbb{N}$ odd and $\theta \in \mathbb{R}$ such that

(7) $R_\theta([0, q_n\alpha])$ and $R_\theta([q_n\alpha, 2q_n\alpha])$ are contained in $h(\sigma(0, \varepsilon))$. Moreover $\bigcup_{j=0}^{q_n} R_{j\alpha+\theta}([0, q_n\alpha])$ is disjoint from $\{x_1, x_2, \ldots, x_m\}$.

Under these conditions, (b) of proposition 3.4, implies that if $p_1 \in h^{-1}(\theta)$, $p_2 \in h^{-1}(q_n\alpha+\theta)$ and $p_3 \in h^{-1}(2q_n\alpha+\theta)$, then

(8) $\sigma^{-1}(p_3) < \sigma^{-1}(p_2) < \sigma^{-1}(p_1)$.

Let us introduce the following notation: if $x, y \in \sigma([0, \varepsilon))$, $\overline{xy}$ will denote the sub-interval of $\sigma([0, \varepsilon))$ with the induced orientation. If $z \in T^2$ and $w = X_t(z)$, for some $t \in (0, \infty)$, $\widehat{zw}$ will be the oriented arc of trajectory of X starting at z and ending at w. By (7) and (8) we may assume that:

(9) $T^{q_n}(p_1) = p_2$, $T^{q_n}(p_2) = p_3$ and that $T^{q_n}(\overline{p_1 p_2}) = \overline{p_2 p_3}$.

The set $B = \bigcup \{\widehat{x T^{q_n}(x)} \mid x \in \overline{p_1 p_2}\}$ can be thought as a flow box with two of its corner points glued together at the point p_2. See figures 5.4 and 5.5.

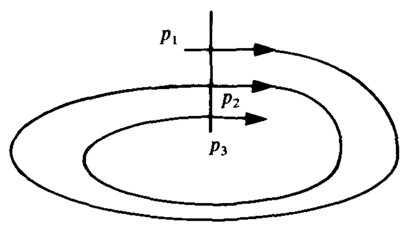

FIGURE 5.4

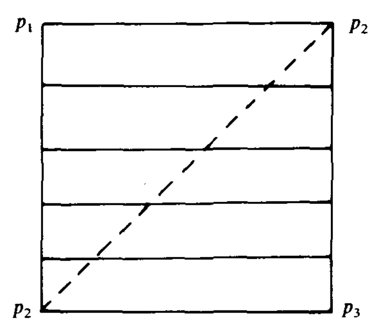

FIGURE 5.5

We shall see that there exists $\mu_0 \in (0, \sigma^{-1}(p_2))$ such that $X + \mu_0 X^{\perp}$ exhibit a closed orbit through the point p_2.

Given $\mu \in (0, \sigma^{-1}(p_2))$ denote by $A_\mu \subset \operatorname{int}(B)$ the oriented open arc of trajectory of $X + \mu X^{\perp}$ starting at p_2 and ending at the point $x(\mu)$ of the boundary of B. The following properties are satisfied:

(10.1) For all $\mu \in (0, \sigma^{-1}(p_2))$, $\overline{p_2 p_1}$ is an $(X + \mu X^{\perp})$-positive segment and $\widehat{p_2 p_3}$ and $\widehat{p_1 p_2}$ are $(X + \mu X^{\perp})$-negative segments.

(10.2) For all $\mu \in [0, \sigma^{-1}(p_3)]$, $\overline{p_3 p_2}$ is an $(X + \mu X^{\perp})$-positive segment.

(10.3) For all $\mu \in (\sigma^{-1}(p_3), \sigma^{-2}(p_2))$, $\overline{p_3 \sigma(\mu)}$ is an $(X + \mu X^{\perp})$-negative segment and $\overline{\sigma(\mu) p_2}$ is an $(X + \mu X^{\perp})$-positive segment.

It follows from these properties that A_μ is well defined, that $x(\mu)$ is a monotone

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strictly increasing continuous function of $\mu \in (0, \sigma^{-1}(p_2))$ and that for some $\mu_0 \in (0, \sigma^{-1}(p_2))$, $x(\mu_0) = p_2$. Theorem A is proved, (see figure 5.6). □

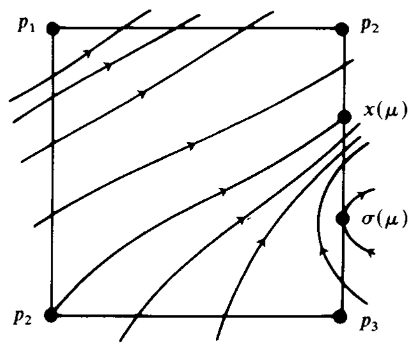

FIGURE 5.6

6. *Final remarks*

Let $k \in \mathbb{N}$ and $\mathfrak{A}_k$ be the set of geometric rotations R_α such that lim sup $a_i \geq k$, where $\{a_i\}$ is the sequence of partial quotients of α.

Under the conditions of theorem A, its proof can easily be improved so that:

(6.1) The result is valid for vector fields of constant type $R_\alpha \in \mathfrak{A}_k$ which have at most k singularities.

(6.2) If $X \in \mathfrak{X}^\omega(T^2)$ (i.e. X is analytic) there exist sequences $\{t_n\}$ and $\{v_n\}$ of real numbers such that $\lim(t_n) = 0 = \lim(v_n)$ and $(X^\perp_{v_n})_*(X + t_n X^\perp)$ has a periodic trajectory passing through p.

REFERENCES

[B–S] N. P. Bhatia & B. P. Szego. *Stability Theory of Dynamical Systems.* Springer-Verlag: New York (1970).

[Gu.1] C. Gutierrez. Smoothing continuous flows on two-manifolds and recurrences. *Ergod. Th. & Dynam. Sys.* **6** (1986), 17-44.

[Gu.2] C. Gutierrez. Structural stability for flows on the torus with a cross-cap. *Trans. Amer. Math. Soc.* **241** (1978), 311-320.

[Her] M. Herman. Sur la conjugaison différentiable des difféomorphismes du cercle a des rotations. *Publ. Math.* **49**, 5-234.

[Khi] A. Khintchine. Kettenbrüche, B. G. Teubner Verlagsgesellschaft, Leipzig, 1956.

[La] S. Lang. *Introduction to Diophantine Approximations.* Addison-Wesley Publishing Company, 1966.

[Ma] R. Mañé. An ergodic closing lemma. *Ann. of Math.* **116** (1982), 503-541.

[Ne] D. Neumann. Smoothing continuous flows on 2-manifolds. *Jour. Diff. Eq.* **28**, No. 3 (1978), 327-344.

[Pa–Me] J. Palis & W. de Melo. *Geometric Theory of Dynamical Systems.* Springer-Verlag: New York (1982).

[Pe] M. Peixoto. Structural stability on two-dimensional manifolds. *Topology* **1** (1962), 101-120.

[Px] D. Pixton. Planar homoclinic points. *J. Diff. Eq.* **44** (1982), No. 3, 365-382.

[Pg.1] C. Pugh. Against the C^2 closing lemma. *Jour. Diff. Eq.* **17** (1975), 435-443.

[Pg.2] C. Pugh. An improved closing lemma and general density theorem. *Amer. J. Math.* **89** (1967), 1010-1021.

[Sl] N. B. Slater. Gaps and steps for the sequence $n\theta$ mod 1, *Proc. Camb. Phil. Soc.* **63** (1967), 1115–1123.

[S–T] A. J. Schwartz & E. S. Thomas. The depth of the center of 2-manifolds. In *Global Anal. Proc. Symp. Pure Math.* **14**, Amer. Math. Soc.: Providence, R.I., pp. 253–264 (1970).

[Ta] F. Takens. Homoclinic points in conservative systems. *Invent. Math.* **18** (1972), 267–292.

[Wa] P. Walters. *Ergodic Theory - Introductory Lectures.* Springer-Verlag, 1975.

[Wt] H. Whitney. Regular families of curves. *Ann. of Math.* (2), **34** (1933), 244–270.