THE IMPLICIT FUNCTION THEOREM IN THE SCALAR CASE*

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1. Introduction. The implicit function theorem has applications at all levels of mathematics from elementary calculus (implicit differentiation) to finding periodic solutions of systems of differential equations ([1, Chapter 14] and [4], for example).

In 1961 W.S. Loud [3] studied the case of two equations in three unknowns. He considered only cases where up to third order derivatives were involved and only those cases where the derivative of the solutions at the critical point existed. Coddington and Levinson [1] consider a specific singular case involving n equations in $n + m$ unknowns. In general the number of distinct critical cases involving up to third derivatives for such a general system is not known.

In the present paper we are interested in the scalar case, i.e. one equation involving two unknowns. We wish to discuss as completely as possible when such an equation can be solved for one unknown in terms of the other in some interval, considering all possibilities where the equations start out with first or second order terms. To this end, we will always assume that our function is as differentiable as we need in order to resolve a particular case. However, we will not assume that the solution we are seeking need have a derivative at the critical point in question, nor will we restrict ourselves to derivatives of any order.

Suppose that

$$F(x, y) = 0 \tag{1}$$

is the equation in question. Without loss of generality, we may assume that $F(0, 0) = 0$. The problem is then to discover when equation (1) can be solved for y as a function of x, $y = \phi(x)$, such that $\phi(0) = 0$, $F(x, \phi(x)) = 0$ for sufficiently small x.

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We divide the possibilities we wish to consider into seven major cases.

2. The Non-critical Case.

Case I. This case involves the classical implicit function theorem which we state below.

THEOREM 1. *Let* $F(x,y) = 0$, $F(0,0) = 0$, $F_y(0,0) \neq 0$. *Then there is a unique* $\phi(x)$, $\phi(0) = 0$, *such that* $F(x,\phi(x)) = 0$ *for sufficiently small* x. *Furthermore* $\phi(x) = -F_x(0,0)\, F_y(0,0)^{-1}x + o(x)$.

For an early proof of this theorem see [2]. For a proof of the more general vector theorem see [2] or [5].

3. The First Critical Cases.

Case II. This case is the scalar equivalent of that discussed by Coddington and Levinson. Let $F(0,0) = F_x(0,0) = F_y(0,0) = 0$. Set $y = \alpha x$ and define

$$G(x,\alpha) = x^{-2}F(x,\alpha x).$$

By twice applying L'Hospital's rule, we easily get that

$$G(0,\alpha) = \tfrac{1}{2}F_{yy}(0,0)\alpha^2 + F_{xy}(0,0)\alpha + \tfrac{1}{2}F_{xx}(0,0)$$

and

$$G_\alpha(0,\alpha) = F_{yy}(0,0)\alpha + F_{xy}(0,0).$$

Let α_o be a real root of $G(0,\alpha) = 0$ if such exists. Then the basic assumption of this case is that $G_\alpha(0,\alpha_o) \neq 0$. Then, by Theorem 1, we can solve for α as a function of x,

$$\alpha = \alpha_o - G_x(0,\alpha_o)G_\alpha(0,\alpha_o)^{-1}\, x + o(x)$$

for sufficiently small x. This then gives

$$y = \phi(x) = \alpha_o x - G_x(0,\alpha_o)G_\alpha(0,\alpha_o)^{-1}x^2 + o(x^2) .$$

We remark that if $F_{yy}(0,0) \neq 0$ and no real α_o exists, then we cannot solve for $y = \phi(x)$, $\phi(0) = 0$. If $F_{yy}(0,0) = 0$, but $F_{xy}(0,0) \neq 0$, then real α_o always exist. $F_{yy}(0,0) = F_{xy}(0,0) = 0$ is discussed in other cases.

Case III. We use the results of Case II to resolve the following situation. Let $F(0,0) = F_y(0,0) = 0$, $F_x(0,0) \neq 0$, $F_{yy}(0,0) \neq 0$, $F_x(0,0)F_{yy}(0,0) < 0$. Let $x = t^2$ and define

$$E^{(2)}(t,y) \equiv F(t^2,y) .$$

Then since $E_t^{(2)}(t,y) = 2F_x(t^2,y)t$, $E^{(2)}(0,0) = E_t^{(2)}(0,0) = E_y^{(2)}(0,0) = 0$. Hence let $y = \theta t$ and define

$$D^{(2)}(t,\theta) \equiv t^{-2}E^{(2)}(t,\theta t) .$$

Then $D_\theta^{(2)}(0,\theta) = \frac{1}{2}E_{yy}^{(2)}(0,0)\theta^2 + E_{ty}^{(2)}(0,0)\theta + \frac{1}{2}E_{tt}(0,0)$ $= \frac{1}{2}F_{yy}(0,0)\theta^2 + F_x(0,0)$, whereas $D_\theta^{(2)}(0,\theta) = F_{yy}(0,0)$. Hence if θ_o is a real solution of $D^{(2)}(0,\theta) = 0$, we can take for θ_o either of $\pm\left(\frac{2F_x(0,0)}{F_{yy}(0,0)}\right)^{1/2}$ which is real, and for either choice $D_\theta^{(2)}(0,\theta_o) \neq 0$. Hence by Theorem 1 we can solve $D_\theta^{(2)}(t,\theta) = 0$ for θ as a function of t,

$$\theta = \theta_o - D_t^{(2)}(0,\theta_o)D_\theta^{(2)}(0,\theta_o)^{-1}t + o(t),$$

and get two solutions

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$$y = \phi(x) = \theta_o x^{\frac{1}{2}} - D_t^{(2)}(0, \theta_o) D_\theta^{(2)}(0, \theta_o)^{-1} x + o(x)$$

valid for sufficiently small positive x.

We remark that if $F_x(0,0)F_{yy}(0,0) > 0$, then letting $x = -t^2$ will give similar results valid for sufficiently small negative x.

4. Critical Cases where $F(x, y)$ begins with a Linear Term. Throughout the rest of this paper we will denote by $F_{x^p y^q}$ the partial derivative $\partial^{p+q} F / \partial x_p \, \partial y^q$.

Case IV. In order to discuss fully the situation when $F(x, y) = Ax +$ higher order terms, we must first discuss the equivalent of Case II when $F(x, y)$ begins with n^{th} order terms.

Let $F(0,0) = F_{x^p y^q}(0,0) = 0$, $p + q = 1, 2, \ldots, n-1$. Further let not all of $F_{x^p y^q}(0,0)$ be zero when $p + q = n$, $n \geq 3$. Let $y = \alpha x$ and define

$$H(x, \alpha) \equiv x^{-n} F(x, \alpha x) .$$

Then by repeated application of L'Hospital's rule, we get that

$$H(0, \alpha) = \frac{1}{n!} \left(\frac{\partial}{\partial y} \cdot \alpha + \frac{\partial}{\partial x} \right)^n F(x, y) \Bigg|_{\substack{x=o \\ y=o}} ,$$

$$H_\alpha(0, \alpha) = \frac{dH(0, \alpha)}{d\alpha} .$$

Let α_o be a real root of $H(0, \alpha) = 0$ if such exists and assume that $H_\alpha(0, \alpha_o) \neq 0$. Then under the above hypotheses we can solve $H(x, \alpha) = 0$ for α as a function of x (with as many solutions as there are distinct θ_o which satisfy the above),

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$$\alpha = \alpha_o - H_x(0,\alpha_o)H_\alpha(0,\alpha_o)^{-1}x + o(x)$$

and hence get

$$y = \phi(x) = \alpha_o x - H_x(0,\alpha_o)H_\alpha(0,\alpha_o)^{-1}x^2 + o(x^2) .$$

<u>Case V</u>. In this case we establish criteria under which $y = \phi(x)$, $\phi(0) = 0$, can and cannot be found.

(i) Let $F(0,0) = F_y(0,0) = \ldots = F_{y^{n-1}}(0,0) = 0$, $F_x(0,0) \neq 0$, $F_{y^n}(0,0) \neq 0$. Further, let either n be odd or if n is even, then $F_x(0,0)\, F_{y^n}(0,0) < 0$. Let $x = t^n$ and define

$$E^{(n)}(t,y) \equiv F(t^n, y) .$$

Then $E^{(n)}(0,0) = E^{(n)}_{t^p y^q}(0,0) = 0$ if $p+q = 1,2,\ldots, n-1$. Further, $E^{(n)}_{t^p y^q}(0,0) = 0$ if $p+q = n$ and $p \neq n$ or $q \neq n$, but $E^{(n)}_{t^n}(0,0) = n!\, F_x(0,0) \neq 0$ and $E^{(n)}_{y^n}(0,0) = F_{y^n}(0,0) \neq 0$. Since $E^{(n)}(t,y)$ satisfies the first hypotheses of Case IV, we let $y = \theta t$ and define

$$D^{(n)}(t,\theta) \equiv t^{-n}E^{(n)}(t,\theta t) .$$

Then $D^{(n)}(0,\theta) = \frac{1}{n!} F_{y^n}(0,0)\theta^n + F_x(0,0)$. Hence if θ_o is a real root of $D^{(n)}(0,\theta) = 0$, then

$$\theta_o = \left(\frac{-n!\,F_x(0,0)}{F_{y^n}(0,0)} \right)^{1/n}$$

if n is odd and plus or minus this value if n is even. Further, $D_\theta^{(n)}(0,\theta_o) \neq 0$. Hence we can solve $D^{(n)}(t,\theta) = 0$ for θ as a function of t (one solution if n is odd and two if n is even),

$$\theta = \theta_o - D_t^{(n)}(0,\theta_o) D_\theta^{(n)}(0,\theta_o)^{-1} t + o(t)$$

and get that

$$y = \phi(x) = \theta_o x^{1/n} - D_t^{(n)}(0,\theta_o) D^{(n)}(0,\theta_0)^{-1} x^{2/n} + o(x^{2/n})$$

for sufficiently small (positive only if n is even) x.

We note that if $F_x(0,0) F_{y^n}(0,0) > 0$, setting $x = -t^n$ gives analogous results for sufficiently small negative x only.

(ii) We now state and prove a theorem which takes care of the other possibility under this case.

THEOREM 2. _Let_ $F(0,0) = 0$, $F_x(0,0) \neq 0$, $F_{y^n}(0,0) = 0$, $n = 1, 2, \ldots$. _Then there does not exist a_ $\phi(x)$, $\phi(0) = 0$, _such that_ $F(x, \phi(x)) = 0$ _for sufficiently small_ x, _provided that_ $F(x,y)$ _is holomorphic in a neighbourhood of the origin_.

Proof. Define $U(x,y) \equiv x^{-1} F(x,y)$. Expanding $F(x,y)$ in a Taylor series in y we get

$$F(x,y) = \sum_{k=o}^{\infty} \frac{F_{y^k}(x,0) y^k}{k!} .$$

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Then since $F_{y^k}(0,0) = 0$ for all k, $F(0,y) = 0$. Hence

$$U(0,y) = \lim_{x \to o} x^{-1} F(x,y) = \lim_{x \to o} F_x(x,y) = F_x(0,y), \text{ and so}$$

$U(0,0) = F_x(0,0) \neq 0$. But $F(x,y) = 0$ can be solved for $y = \phi(x)$, $\phi(0) = 0$ only if $U(x,y) = 0$ can be solved for x, $y = \psi(x)$, with $\psi(0) = 0$, which clearly cannot happen since $U(0,0) \neq 0$.

5. Critical Cases where $F(x,y)$ begins with Quadratic Terms. We have already looked at such a situation in Case II. The situation we now wish to look at can be broken up into two cases, one occurring when the only quadratic term is x^2, and the other when $G_\alpha(0,\alpha_o) = 0$.

Case VI. This case can be split into three parts, which together exhaust all possibilities of the type considered when $F(x,y) = Ax^2 +$ higher order terms.

(i) Let $F(0,0) = F_x(0,0) = F_y(0,0) = F_{xy}(0,0) = F_{yy}(0,0) = 0$, $F_{xx}(0,0) \neq 0$. Further, let there exist $n \geq 3$ such that

$$F_{y^q}(0,0) = 0, \quad q = 1, 2, \ldots, n-1,$$

$F_{y^n}(0,0) \neq 0$, $F_{xy^q}(0,0) = 0$ for every $q < \frac{1}{2}n$.

Let $x = t^{\frac{1}{2}n}$ and define

$$A(t,y) \equiv F(t^{\frac{1}{2}n}, y).$$

Let $y = \alpha t$ and define

$$M(t,\alpha) \equiv t^{-n} A(t, \alpha t).$$

Consider each term in $F(x,y)$ of the form $x^p y^q$, $p + q \geq 2$, and if $p + q = 2$, then $p = 2$, $y = 0$, and there are no terms such that $p = 1$ if $q < \frac{1}{2}n$. The corresponding term of $A(t,y)$ has the form $t^{\frac{1}{2}pn} y^q$, and hence the corresponding term of $M(t,\alpha)$ has the form $\alpha^q t^{(\frac{1}{2}pn)+q-n}$. If $p + q = 2$, then the exponent of t is 0.

Now let $p + q \geq 3$. There are no exponents of t less than zero since $\frac{1}{2}pn + q - n = \frac{1}{2}(p-2)n + q < 0$, $p + q \geq 3$, only if $p = 0, 1$. But for $p = 0$, $\frac{1}{2}(p-2)n + q < 0$, only if $q < n$, and if $p = 1$, $\frac{1}{2}(p-2)n + q < 0$ only if $q < \frac{1}{2}n$, and there are no such terms by hypothesis.

We now look for those terms for which the exponent of t is zero. The term with $p = 2$, $y = 0$ is such a term. In general $\frac{1}{2}(p-2)n + q = 0$ if and only if $p = 0$, $q = n$; $p = 1$, $q = \frac{1}{2}n$; $p = 2$, $q = 0$. The term with $p = 1$, $q = \frac{1}{2}n$ is allowed only when n is even. For all other non-zero terms the exponent of t is positive.

We now consider separately the cases that n is odd and n is even.

(a) n odd. It is easy to see that $M(0,\alpha) = \frac{1}{n!} F_{y^n}(0,0)\alpha^n + \frac{1}{2}F_{xx}(0,0)$. Hence $M(0,\alpha)$ has the real root

$$\alpha_o = \left(\frac{-n!\,F_{xx}(0,0)}{2F_{y^n}(0,0)}\right)^{1/2},$$

and since $M_\alpha(0,\alpha_o) = \frac{1}{(n-1)!} F_{y^n}(0,0)\alpha_o^{n-1} \neq 0$, we can solve $M(t,\alpha) = 0$ for α as a function of t,

$$\alpha = \alpha_o - M_t(0,\alpha_o)M_\alpha(0,\alpha_o)^{-1}\,t + o(t)$$

and get for sufficiently small x that

$$y = \phi(x) = \alpha_o x^{2/n} - M_t(0,\alpha_o)M_\alpha(0,\alpha_o)x^{4/n} + o(x^{4/n}).$$

(b) n even. Here we get

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$$M(0,\alpha) = \frac{1}{n!} F_{y^n}(0,0)\alpha^n + \frac{1}{(\frac{1}{2}n)!} F_{xy^{\frac{1}{2}n}}(0,0)\alpha^{\frac{1}{2}n} + \tfrac{1}{2}F_{xx}(0,0),$$

$$M_\alpha(0,\alpha) = \frac{1}{n!} F_{y^n}(0,0)\alpha^{n-1} + \frac{1}{(\frac{1}{2}(n-2))!} F_{xy^{\frac{1}{2}n}}(0,0)\alpha^{\frac{1}{2}(n-2)} .$$

Viewing $M(0,\alpha) = 0$ as a quadratic equation in $\alpha^{\frac{1}{2}n}$ we can state that if there are no real roots to this quadratic equation, $\phi(x)$, $\phi(0) = 0$ does not exist. Suppose now that β_o is such a root. Then if $M_\alpha(0, \beta_o^{2/n}) = 0$, and if either $\frac{1}{2}n$ is odd, or $\frac{1}{2}n$ is even and $\beta_o > 0$, letting $\alpha_o = \beta_o^{2/n}$, we see that we can solve $M(t,\alpha) = 0$ for α as a function of t and hence for $y = \phi(x)$ as when n is odd. If $\frac{1}{2}n$ is even and $\beta_o < 0$ (for all choices of β_o) then no such solution exists. If $M_\alpha(0, \beta_o^{2/n}) = 0$, and either $\frac{1}{2}n$ is odd or $\frac{1}{2}n$ is even and $\beta_o > 0$, then for $M(t,\alpha)$ any of Cases I to VII can occur (where Case VI, (ii) and (iii), and Case VII are discussed below), and this analysis must be carried out before we know whether the required solution exists or not.

(ii) Let $F(0,0) = F_x(0,0) = F_y(0,0) = F_{xy}(0,0) = F_{yy}(0,0) = 0$, $F_{xx}(0,0) \neq 0$. Further, let there exist $n \geq 3$ such that $F_{xy^q}(0,0) = 0$, $q = 1, 2, \ldots, n-2$, $F_{xy^{n-1}}(0,0) \neq 0$, $F_{y^q}(0,0) = 0$, $q \leq 2(n-1)$. Let $x = t^{n-1}$ and define

$$B(t,y) \equiv F(t^{n-1}, y) .$$

Let $y = \alpha t$ and define

$$N(t,\alpha) = t^{-2(n-1)} B(t,\alpha t) .$$

The corresponding term of $x^p y^q$ in $B(t,y)$ has the form

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$t^{p(n-1)}y^q$. The corresponding term in $N(t,\alpha)$ has the form $\alpha^q t^{p(n-1)+q-2(n-1)}$, where, if $p+q=2$, then $p=2$, $q=0$, and if $p=0$, $q>2(n-1)$.

For the term $p=2$, $q=0$, the exponent of t is 0. For $p+q\geq 3$ we wish to show that the exponent of t is not less than zero. $(p-2)(n-1)+q<0$ only if $p=0$, $q<2(n-1)$ or $p=1$, $q=n-1$, and by hypothesis there are no such terms of this type.

The exponent of t is zero only when $p=0$, $q=2(n-1)$; $p=1$, $q=n-1$; $p=2$, $q=0$. By hypothesis the term involving $p=0$, $q=2(n-1)$ does not occur (this is taken care of by Case VI (i)). All other terms have involved t to a positive exponent.

From this we get that

$$N(0,\alpha) = \frac{1}{(n-1)!} F_{xy^{n-1}}(0,0)\alpha^{n-1} + \tfrac{1}{2}F_{xx}(0,0),$$

$$N_\alpha(0,\alpha) = \frac{1}{(n-2)!} F_{xy^{n-1}}(0,0)\alpha^{n-2} \neq 0 \quad \text{for} \quad \alpha \neq 0 .$$

Hence, if either n is even or if n is odd and $F_{xx}(0,0)F_{xy^{n-1}}(0,0) < 0$, we can solve $N(t,\alpha)=0$ for α as a function of t,

$$\alpha = \alpha_o - N_t(0,\alpha_o)N_\alpha(0,\alpha_o)^{-1}t + o(t)$$

and get

$$y = \phi(x) = \alpha_o x^{1/(n-1)} - N_t(0,\alpha_o)N_\alpha(0,\alpha_o)^{-1}x^{2/(n-1)} + o(x^{2/(n-1)}),$$

for sufficiently small x (positive x only if n is odd), where α_o is a real root of $N(0,\alpha)=0$.

Note that if n is odd, there are two solutions.

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If n is odd and $F_{xx}(0,0)F_{xy^{n-1}}(0,0) > 0$, letting $x = -t^{n-1}$ will give similar results for sufficiently small negative x.

(iii) THEOREM 3. Let $F(0,0) = F_x(0,0) = F_y(0,0) = F_{xy}(0,0) = F_{yy}(0,0) = 0$, $F_{xx}(0,0) \neq 0$, $F_{y^n}(0,0) = F_{xy^n}(0,0) = 0$, $n = 2, 3, \ldots$. Then there does not exist $\phi(x)$, $\phi(0) = 0$, such that $F(x, \phi(x)) = 0$ for sufficiently small x, providing that $F(x,y)$ is holomorphic near the origin.

Proof. The proof is similar to the proof of Theorem 2, letting $U(x,y) \equiv x^{-2}F(x,y)$.

Case VII. Let $F(0,0) = F_x(0,0) = F_y(0,0) = 0$, $F_{yy}(0,0) \neq 0$ (if $F_{yy}(0,0) = 0$, then we have Case II if $F_{xy}(0,0) \neq 0$, Case VI if $F_{xy}(0,0) = 0$ but $F_{xx}(0,0) \neq 0$, and Case IV if all quadratic terms are zero). Further, let the α_o of Case II exist and if $G(x,\alpha)$ is as defined in Case II, let $G_\alpha(0,\alpha_o) = 0$. Then for $G(x,\alpha)$ any of Cases I to VII may occur and this analysis must be carried out before we know whether the required solution exists or not.

6. Discussion. The preceding analysis takes care of all cases where $F(x,y)$ can be written as

$$F(x,y) = Ax + By + Cx^2 + Dxy + Ey^2 + \text{higher order terms}.$$

It is hoped that this will give some hints as to possible critical cases for the implicit function theorem for more general systems.

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