CRITERIA FOR EXTREME FORMS

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1. A positive quadratic form $f(x) = \sum_{i,j=1}^{n} a_{ij}x_i x_j$ $(a_{ij} = a_{ji})$, of determinant $||a_{ij}|| = D$ and minimum M for integral $x \neq 0$, is said to be extreme if the ratio $M/D^{1/n}$ is a (local) maximum for small variations in the coefficients a_{ij}.

Minkowski [3] has given a criterion for extreme forms in terms of a fundamental region (polyhedral cone) in the coefficient space. This criterion, however, involves a complete knowledge of the edges of the region and is therefore of only theoretical value.

Voronoï [4] has given the only practical criterion in:

THEOREM 1. *A positive quadratic form is extreme if and only if it is perfect and eutactic.*

I have recently established, in [1], a criterion in terms of linear inequalities and shown how Theorem 1 may be simply deduced from it:

THEOREM 2. *If f has minimal vectors $\pm m_1, \cdots, \pm m_s$, then it is extreme if and only if there exists no non-trivial quadratic form $g(x) = \sum_{i,j=1}^{n} b_{ij}x_i x_j$ satisfying*

$$(1) \qquad g(m_k) \geqq 0 \quad (k = 1, \cdots, s), \quad \sum_{i,j=1}^{n} A_{ij}b_{ij} \leqq 0,$$

where $F(x) = \Sigma A_{ij}x_i x_j$ is the adjoint of $f(x)$.

I give here two further criteria, in Theorems 3 and 4. Theorem 3 amounts to a refinement of Theorem 1 in terms of a subset of the minimal vectors. It has the important practical consequences that, in general, (i) only a suitable subset of the minimal vectors need be specified or even known; and (ii) the calculations required to check that a form is eutactic are considerably simplified.

Theorem 4 shows further that the eutactic condition may sometimes be replaced by a simple condition on the group of automorphs of the form.

2. The minimal vectors of f are defined to be the integral solutions $x = \pm m_1, \cdots, \pm m_s$ of $f(x) = M$. Let H be any subset of the minimal vectors, say $\pm m_1, \cdots, \pm m_t$ $(t \leqq s)$. We shall say that f is H-perfect if

f is uniquely determined by H and its minimum M; i.e. if there exists no non-trivial quadratic form $g(x)$ satisfying

$$g(m_k) = 0 \quad (k = 1, \cdots, t). \tag{2}$$

If $F(x) = \Sigma A_{ij} x_i x_j$ is the adjoint of $f(x)$, we shall say that f is H-eutactic if $F(x)$ is expressible as

$$F(x) \equiv \sum_{k=1}^{t} \rho_k (m_k' x)^2 \text{ with } \rho_k > 0 \quad (k = 1, \cdots, t). \tag{3}$$

These definitions reduce to the accepted definitions of the terms perfect and eutactic if H is the set of all minimal vectors.

THEOREM 3. *f is extreme if and only if there exists a subset H of its minimal vectors such that f is H-perfect and H-eutactic.*

Proof. (i) The necessity of the condition is contained in Voronoï's Theorem 1, with H the set of all minimal vectors.

(ii) Suppose that f is H-perfect and H-eutactic, where $H = \{m_1, \cdots, m_t\}$. It then follows that a quadratic form $g(x) = \Sigma b_{ij} x_i x_j$ satisfying

$$g(m_k) \geqq 0 \quad (k = 1, \cdots, t), \quad \Sigma A_{ij} b_{ij} \leqq 0 \tag{4}$$

is necessarily trivial. For, choosing $\rho_k > 0$ to satisfy (3), we have

$$A_{ij} = \sum_{k=1}^{t} \rho_k m_{ki} m_{kj} \quad (i, j = 1, \cdots, n),$$

$$\Sigma A_{ij} b_{ij} = \sum_{k=1}^{t} \rho_k g(m_k);$$

since $\rho_k > 0$, the relations (4) show at once that

$$g(m_k) = 0 \quad (k = 1, \cdots, t),$$

whence $g(x) \equiv 0$, since f is H-perfect.

It follows that, a fortiori, the inequalities (1) have no non-trivial solution. Hence, by Theorem 2, f is extreme.

3. Let G be the group of automorphs of f, i.e. the set of integral unimodular transformations T satisfying $f(Tx) = f(x)$. If m is a minimal vector of f, then so also is Tm; thus G may be regarded as a permutation group on the minimal vectors.

THEOREM 4. *Suppose that there exists a subset H of the minimal vectors of f such that f is H-perfect and G is transitive on H. Then f is extreme.*

Proof. Since G is transitive on H, H is contained in a unique system of transitivity of G, say $K = \{m_1, \cdots, m_t\}$. Since f is H-perfect, it is K-perfect, and so the equations

$$\sum_{i,j=1}^{n} b_{ij} m_{ki} m_{kj} = 0 \quad (k = 1, \cdots, t), \quad (b_{ij} = b_{ji})$$

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have the unique solution $b_{ij} = 0$. The $t \times \frac{1}{2}n(n+1)$ matrix $(m_{ki}m_{kj})$ therefore has rank $\frac{1}{2}n(n+1)$, so that the equations

$$\sum_{k=1}^{t} \sigma_k m_{ki} m_{kj} = A_{ij} \quad (i,\ j = 1, \cdots, n)$$

certainly possess a solution $\sigma_1, \cdots, \sigma_t$. For any such solution, we have

$$F(x) = \Sigma A_{ij} x_i x_j = \sum_{k=1}^{t} \sigma_k (m'_k x)^2. \tag{5}$$

Let now $\boldsymbol{G}'$ be the group of automorphs of $F(x)$, so that $T \in \boldsymbol{G}'$ if and only if $T'^{-1} \in \boldsymbol{G}$. $\boldsymbol{G}'$ may be interpreted as a permutation group on the linear forms $m'_k x$, wherein the set $\{m'_1 x, \cdots, m'_t x\}$ now forms a system of transitivity. Hence, if $\boldsymbol{G}'$ has order g, there are precisely g/t elements of $\boldsymbol{G}'$ transforming any one form of this set into any other. Applying all the transformations of $\boldsymbol{G}'$ to (5), and adding, we therefore obtain

$$gF(x) = \frac{g}{t} \sum_{k=1}^{t} (\sigma_1 + \sigma_2 + \cdots + \sigma_t)(m'_k x)^2.$$

Thus

$$F(x) = \rho \sum_{k=1}^{t} (m'_k x)^2, \quad \rho = \frac{1}{t}(\sigma_1 + \cdots + \sigma_t),$$

where clearly $\rho > 0$ since F is positive definite.

f is therefore K-eutactic, and Theorem 3 shows now that f is extreme.

4. It is perhaps worth noting that Theorem 3 would become false if stated in the stronger form: 'If H is a subset of the minimal vectors of f such that f is H-perfect, then f is extreme if and only if it is H-eutactic.' A simple counter-example is the extreme form B_n (in the notation of [2]) defined by

$$f(x) = \sum_{1}^{n} x_i^2$$

with the lattice of integral x satisfying

$$\sum_{1}^{n} x_i \equiv 0 \pmod 2.$$

Here $D = 4$, $M = 2$, and the $n(n-1)$ pairs of minimal vectors are given by $m = e_i \pm e_j$ $(i < j)$ (where e_i is the i-th unit vector).

There are clearly proper subsets H for which f is H-perfect (and also proper subsets H for which f is H-eutactic). However, suppose that f is both H-perfect and H-eutactic, and consider any fixed pair of suffixes i, j $(i < j)$. H must contain at least one of $e_i \pm e_j$, else (2) could be satisfied by an arbitrary choice of b_{ij}. Also, in any relation of the type

$$F(x) = \sum_1^n x_i^2 = \Sigma \rho_{ij}(x_i + x_j)^2 + \Sigma \sigma_{ij}(x_i - x_j)^2$$

we have $\rho_{ij} - \sigma_{ij} = 0$; hence, since f is H-eutactic, H must contain neither or both of the vectors $e_i \pm e_j$. It follows that H contains both $e_i \pm e_j$, for all $i < j$, so that H is the complete set of minimal vectors.

It is not difficult to show also that the converse of Theorem 4 is false. The form defined by

$$f(x) = \sum^9 x_i^2$$

with the lattice of integral x satisfying

$$x_1 \equiv x_2 \equiv \cdots \equiv x_8 \pmod 2, \; \sum_1^9 x_i \equiv 0 \pmod 4,$$

has in fact no set H of minimal vectors satisfying the conditions of Theorem 4. However, it is easily seen to be extreme (with $M = 8$) by applying Theorem 3 to the subset H of minimal vectors $2e_i \pm 2e_j$ $(1 \leqq i < j \leqq 9)$.

5. I should like to take this opportunity of correcting an error of detail in [1] which was pointed out to me by Mr. A. L. Duquette of Illinois. The equation (7) of [1] implies that $A^{-1}B$ is symmetric, and this is not necessarily true. The proof as given becomes correct if we define $C = T'BT$, where T is chosen so that $T'AT = I$.

References

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