J. Aust. Math. Soc. **102** (2017), 20–33
doi:10.1017/S1446788715000385

THE CONTRIBUTION OF L. G. KOVÁCS TO THE THEORY OF PERMUTATION GROUPS

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(Received 2 March 2015; accepted 6 April 2015; first published online 5 November 2015)

Communicated by R. Stöhr

Dedicated to the memory of our friend Laci Kovács.

Abstract

The work of L. G. (Laci) Kovács (1936–2013) gave us a deeper understanding of permutation groups, especially in the O'Nan–Scott theory of primitive groups. We review his contribution to this field.

2010 *Mathematics subject classification*: primary 20E28; secondary 20B15, 20B99.

Keywords and phrases: L. G. Kovács, permutation groups, maximal subgroups, O'Nan–Scott theorem.

1. The context of Laci's work on permutation groups

The main contributions made by L. G. Kovács ('Laci' to his friends and colleagues) in the theory of permutation groups revolve around groups acting primitively and their point stabilizers, which are maximal subgroups.

The classification of finite simple groups was, undoubtedly, one of the greatest research projects ever undertaken in mathematics. It changed the face of finite group theory, and several important results on permutation groups grew from the classification. Laci's work contributed significantly to one such result, namely the O'Nan–Scott theorem for primitive permutation groups.

The origins of the O'Nan–Scott theorem can be traced back to Jordan's description of the maximal subgroups of the symmetric groups in his *Traité* [Jor70]. The modern form of the theorem dates back to 1978. By then it was clear that the announcement of the classification of the finite simple groups was imminent, and many mathematicians began to consider how this classification might influence work in other

The second author was supported by the research projects 302660/2013-5 (CNPq, Produtividade em Pesquisa), 475399/2013-7 (CNPq, Universal), and APQ-00452-13 (Fapemig, Universal). He also enjoyed the hospitality of the School of Mathematics and Statistics of the University of Western Australia, while a part of this paper was being written.

parts of mathematics, especially in finite group theory. With different applications in mind, Scott and O'Nan independently proved versions of this theorem, pin-pointing various roles for the simple groups in describing the possible kinds of finite primitive permutation groups. Both Scott and O'Nan brought papers containing their results to the Santa Cruz Conference of Finite Groups in 1979 [CM80], in which Laci also participated. As was kindly pointed out to us by the referee, a preliminary version of the conference proceedings included the papers by both Scott and O'Nan, while the final version only contained Scott's paper, which stated the theorem in an appendix [Sco80].

These early versions of the O'Nan–Scott theorem were presented as characterizations of the maximal subgroups of the finite alternating and symmetric groups, as well as structural descriptions for finite primitive permutation groups (see the theorems stated on [Sco80, pages 329 and 328], respectively). They were unfortunately not adequate for the latter purpose since the statement of the theorem on [Sco80, page 328] failed to identify primitive groups of twisted wreath type (that is, those with a unique minimal normal subgroup which is nonabelian and regular). In addition, the theorems on [Sco80, pages 328 and 329] erroneously claimed that the number of simple factors of the unique minimal normal subgroup of a primitive group of simple diagonal type had to be a prime number.

These oversights were quickly recognized. They were first rectified by Aschbacher and Scott [AS85, Appendix] and, independently of their work, by Laci [Kov86]. In particular, Laci identified the 'missing type' of primitive groups: these are groups, which, in his later work, were characterized as twisted wreath products. A short, self-contained proof of the O'Nan–Scott theorem was then published by Liebeck *et al.* [LPS88].

The O'Nan–Scott theorem has several interpretations depending on the context in which it is used. The theorem is most commonly interpreted, for instance in its applications in permutation group theory and algebraic combinatorics, as a structure theorem describing finite primitive permutation groups. It classifies such groups into several classes each of which is characterized by the structure of the socle viewed as a permutation group. The fact that maximal subgroups in finite groups are point stabilizers for primitive actions leads to a second interpretation of the O'Nan–Scott theorem: it can be considered as a characterization of (core-free) maximal subgroups of finite groups. Much of Laci's work related to the O'Nan–Scott theorem was concerned with giving a detailed description of maximal subgroups. His methodology relies, in large part, on the concept of induced extensions introduced in his 1984 paper [GK84] jointly written with Gross. Based on this concept, he gave in [Kov86] a general description of maximal subgroups of finite groups G having a normal subgroup that is a direct product of subgroups forming a conjugacy class of G.

In his papers on permutation groups, Laci was often concerned with counting a class of mathematical objects, for instance maximal subgroups (as in [Kov86]) or wreath decompositions (in [Kov89a]) of a group. His underlying philosophy is beautifully explained in the introduction of [Kov89a].

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> Counting the number of mathematical objects of a certain kind is often undertaken as a test problem ('if you can't count them, you don't really know them'): not so much because we want the answer, but because the attempt focuses attention on gaps in our understanding, and the eventual proof may embody insights beyond those which are capable of concise expression in displayed theorems. [Kov89a, page 255.]

We describe Laci's most important results in this field using modern terminology for the O'Nan–Scott theorem, namely we use a version of the theorem that classifies finite primitive permutation groups into eight types: holomorph of abelian (HA), almost simple (AS), simple diagonal (SD), compound diagonal (CD), holomorph of a simple group (HS), holomorph of a compound group (HC), product action (PA), and twisted wreath (TW)—with two-letter abbreviations as indicated. This 'type subdivision' was originally suggested by Laci and was first used in the first author's paper [Pra90], where a detailed description of each type is given. The paper [Pra90] analyses the possible inclusions between finite primitive groups in terms of these types, and was written in close consultation with Laci who at the time was working on the related article [Kov89b].

While Sections 2–4 focus on Laci's contributions around the O'Nan–Scott theorem, in the final section we mention several other papers by Laci on permutation groups, including his papers on minimal degree of a group and on abelian quotients of permutation groups.

2. Induced extensions

A central theme of Laci's papers in the 1980s was the study of groups that contain a normal subgroup M which is an unrestricted direct product $M = \prod_i M_i$ in such a way that the set of the M_i is closed under conjugation by G. It was his typical technique to consider, as building blocks in such direct products, the normal subgroups $\prod_{j\neq i} M_j$ of M instead of the direct factors M_i, and we will use this view here. Laci usually worked under the following assumption.

> (*) G is a group, M is a normal subgroup of G, $\{K_i\}$ is a G-conjugacy class containing normal subgroups of M in such a way that the natural homomorphism from M into $\prod_i M/K_i$ is an isomorphism. Let K be a fixed element of the set $\{K_i\}$ and let $N = \mathbb{N}_G(K)$.

Sometimes hypothesis (*) was relaxed to require only that the set $\{K_i\}$ was a G-invariant set of normal subgroups of M (that is, a union of G-conjugacy classes). Groups that satisfy (*) occur naturally among finite primitive permutation groups and Laci's results have indeed found several deep applications in permutation group theory. Many of the classes of the O'Nan–Scott theorem are characterized by the fact that their groups contain a nonabelian minimal normal subgroup $M = T^k$ which is the direct product of finite simple groups all isomorphic to a group T. Such groups clearly satisfy (*).

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The main tool in the investigation of the groups in which (*) holds is the induced extension, introduced in [GK84, Section 3]. The construction takes as input a group homomorphism $\alpha : A \to B$ and outputs a subgroup G of the wreath product $W = A \operatorname{wr} P$, where P is the permutation group induced by the multiplication action of B on the right coset space $I := [B : A\alpha]$. The group G is called the *induced extension defined by* α. To explain the construction, we use the following convention (often used by Laci himself). For sets X and Y, we denote by X^Y the set of functions $Y \to X$. If X is a group, then X^Y is also a group under pointwise multiplication, and is isomorphic to the (unrestricted) direct product of $|Y|$ copies of X.

Let α, I, P be as in the previous paragraph, and set $C = A\alpha$. It is well known that there exists an embedding λ of B into the wreath product $C \operatorname{wr} P = C^I \rtimes P$ (see the discussion on [GK84, page 136]). Further, there is an epimorphism $\bar{\alpha} : A \operatorname{wr} P \to C \operatorname{wr} P$ whose kernel is $M = (\ker \alpha)^I$. As λ is a monomorphism, there is a unique 'smallest' group G such that suitable homomorphisms make the following pull-back diagram commutative:

$$\begin{tikzcd} G \arrow[r] \arrow[d] & B \arrow[d, "\lambda"] \\ A \operatorname{wr} P \arrow[r, "\bar{\alpha}"] & C \operatorname{wr} P \end{tikzcd}$$

Further, as $M = (\ker \alpha)^I$ is the kernel of $\bar{\alpha}$, we obtain the short exact sequence

$$1 \longrightarrow M \longrightarrow A \operatorname{wr} P \xrightarrow{\bar{\alpha}} C \operatorname{wr} P \longrightarrow 1$$

Combining the pull-back diagram with this short exact sequence, we obtain the pull-back exact sequence

$$\begin{tikzcd} 1 \arrow[r] & M \arrow[r] \arrow[d, "\mathrm{id}"] & G \arrow[r] \arrow[d] & B \arrow[r] \arrow[d, "\lambda"] & 1 \\ 1 \arrow[r] & M \arrow[r] & A \operatorname{wr} P \arrow[r, "\bar{\alpha}"] & C \operatorname{wr} P \arrow[r] & 1 \end{tikzcd}$$

Since λ and the identity mapping of M represented by the first vertical arrow are monomorphisms, the short five lemma implies that the map $G \to A \operatorname{wr} P$ in the middle of the diagram is a monomorphism. Thus, we may consider G as a subgroup of $W = A \operatorname{wr} P$.

Induced extensions G satisfy hypothesis (*), since $\ker \bar{\alpha} = M$ is a normal subgroup of G and since M is the direct product $(\ker \alpha)^I$ such that the factors of $(\ker \alpha)^I$ are permuted transitively by G. One of the main results of [GK84] is that the converse is also true.

THEOREM 2.1 [GK84, Theorem 4.1]. *If (*) holds, then the map $\alpha : N/K \to G/M$ given by $Kx \mapsto Mx$ is a well-defined homomorphism and G is isomorphic to the induced extension defined by α.*

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Let us assume that G is the induced extension defined by a homomorphism $\alpha : A \to B$, so that (*) holds for G. One of the main objectives of [GK84] is to describe the subgroups H of G that satisfy

$$HM = G \quad \text{and} \quad H \cap M \cong \prod_{i \in I} (H \cap M)K_i/K_i. \tag{2.1}$$

Such subgroups are called *high subgroups* in [Kov86]. The second assertion of (2.1) implies that $H \cap M$ is not only a direct product, but its direct product decomposition is inherited from the given direct product decomposition of M. Condition (2.1) occurs often in the case of primitive permutation groups, since it holds for the point stabilizers of the finite primitive groups in several of the O'Nan–Scott classes, as was observed by Scott in [Sco80].

The paper [GK84] describes the subgroups of G that satisfy (2.1) in terms of subgroups of N/K. The following result plays a pivotal role in the description of maximal subgroups in the subsequent paper [Kov86].

Theorem 2.2 [GK84, Corollary 4.4]. *There is a bijection between the conjugacy classes in G of the subgroups H that satisfy condition (2.1) and the conjugacy classes in N/K of the subgroups L/K that satisfy $N/K = (M/K)(L/K)$. Under this bijection, complements correspond to complements.*

Twisted wreath products have played an important part in permutation group theory, since primitive groups with a nonabelian regular minimal normal subgroups (the missing case in the original O'Nan–Scott theorem) can be described as twisted wreath products, hence the name 'TW type'. Kovács and Gross noticed this connection by observing after [GK84, Corollary 3.3] that if $\alpha : A \to B$ is such that A splits over $\ker \alpha$, then the induced extension G can be written as a twisted wreath product $(\ker \alpha) \operatorname{twr} P$. In this case P is a complement of $M = (\ker \alpha)^I$ in G and M is a regular normal subgroup of G in its permutation representation on $\Omega = [G : P]$, a crucial observation in the study of TW-type primitive groups.

3. Maximal subgroups

Determining the subgroup structure of a given finite group has been one of the central problems in group theory. The subgroup lattice of a finite group has a rich structure and its full description is often beyond reach. Thus, one starts with the maximal subgroups.

The problem of describing maximal subgroups of an arbitrary (finite) group can be approached using a simple reduction argument. If G is a group with a normal subgroup M, then the class of maximal subgroups that contain M is in one-to-one correspondence with the class of maximal subgroups of the quotient G/M. Hence, one really needs to understand core-free maximal subgroups. Core-free maximal subgroups of G are precisely the point stabilizers in faithful primitive permutation representations of G. Hence, conjugacy classes of such maximal subgroups are in

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a bijective correspondence with equivalence classes of faithful primitive permutation representations of G. Thus, the problem of describing maximal subgroups of finite groups can be attacked using the O'Nan–Scott theorem. This problem was tackled in two papers independently, one by Aschbacher and Scott [AS85] and the second by Laci [Kov86]. There is significant overlap between the two papers. Laci only learned about the work of Aschbacher and Scott after his draft was finished. He explains in a footnote added to the second page of his paper:

> After the draft of this paper was completed, I learned that a forthcoming paper [AS85] by Aschbacher and Scott will address the same issues. [...] While the conclusions naturally have several common components, the approaches and expositions differ so much that detailed reconciliation [...] will not be attempted here. The difficulties involved strongly suggest that both versions of the story are worth telling. [Kov86, page 115.]

The common underlying strategy behind both papers is the one explained above. Laci's paper builds on the theory of induced extensions developed in his earlier paper [GK84] with Gross, which allows him to give a more approachable account.

Assuming that M is a minimal normal subgroup of G, we are interested in the maximal subgroups that do not contain M. In the case of finite groups, M is a direct product T^k of simple groups. If T is abelian, then so is M, and the problem is reduced to a problem in group cohomology. Hence, in the main part of the paper, $M = T^k$ such that T is a nonabelian simple group, and the k simple factors of M form a G-conjugacy class. In particular, condition (*) holds and so, by Theorem 2.1, G is an induced extension defined by a natural homomorphism $\alpha : N/K \to G/M$, where K is a maximal normal subgroup of M and N is its normalizer in G. In particular, there is a correspondence between a certain class of subgroups of G and the class of subgroups of N/K (see Theorem 2.2). Laci introduced two types of subgroups in G. The *high subgroups* H defined in (2.1) may be alternatively defined by requiring that $HM = G$, and that either $H \cap M = 1$, or (*) holds with H, $H \cap M$, $H \cap K$, and $H \cap N$ in place of G, M, K, and N. The second class of subgroups, the *full subgroups*, are those subgroups F such that $FK = G$ and (*) holds with F, $F \cap M$, $F \cap K$, and $\mathbb{N}_F(F \cap K)$ in place of G, M, K, and N.

The condition $FK = G$, in the definition of a full subgroup F, implies that $(F \cap M)K = M$ and hence that $F \cap M$ projects onto M/K; and then $FM = G$ yields that it projects onto each direct factor M/K_i. In other words, if F is a full subgroup, then $F \cap M$ is a subdirect subgroup with respect to the direct decomposition $M \cong \prod_i M/K_i$.

The main theorem of [Kov86] considers groups G that satisfy (*), with M/K nonabelian simple and the set $\{K_i\}$ finite, and gives a characterization of the maximal subgroups of G that do not contain M. By [Kov86, Lemma 4.2], each such maximal subgroup of G is either full or high. Laci split these maximal subgroups into three subfamilies:

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(A) full maximal subgroups;
(B) high maximal subgroups H that are not complements of M (that is, $H \cap M \neq 1$);
(C) high maximal subgroups H that are complements of M (that is, $H \cap M = 1$).

We shall see in a moment why it is necessary to distinguish between cases (B) and (C).

In each of these families the number of conjugacy classes of maximal subgroups is expressed using a smaller group. In families (B) and (C), this reduction essentially follows from Theorem 2.2, which states a one-to-one correspondence between the set of conjugacy classes of high subgroups of G and the set of conjugacy classes of supplements of M/K in N/K. In particular, maximal subgroups in family (B) correspond to:

(B1) maximal subgroups L/K of N/K that neither avoid nor contain M/K.

Describing maximal subgroups in family (C) is a bit more complicated. Let L/K be a *maximal complement* to M/K in N/K (that is, a maximal subgroup of N/K which complements M/K). The problem is that the corresponding subgroup in G, given by Theorem 2.2, may not be maximal, as it may be contained in a full subgroup of G. One of the main contributions of the paper [Kov86] is a sufficient and necessary condition on L/K that describes precisely when this situation occurs. As $N/K = M/K \rtimes L/K$, the group N/M acts, via the isomorphisms $L/K \cong (N/K)/(M/K) \cong N/M$, on M/K. Laci proved [Kov86, Theorem 4.3.c] that the complement H corresponding in G to L/K is maximal if and only if this action cannot be extended to a subgroup of G/M properly containing N/M. Hence, maximal subgroups of type (C) correspond to:

(C1) maximal complements of M/K in N/K such that the corresponding homomorphism $N/M \to \mathsf{Aut}\,(M/K)$ cannot be extended to a subgroup of G/M properly containing N/M.

The reduction for counting the full maximal subgroups is developed in [Kov86, Section 3], with the main result being Theorem 3.03. The number of conjugacy classes of full maximal subgroups is determined in terms of the following set:

(A1) the collection of all homomorphisms $\varphi : D \to \mathsf{Out}(M/K)$, where D is a subgroup of G/M minimally containing N/M and the restriction of φ to N/M is equal to the coupling determined by the short exact sequence

$$1 \to M/K \to N/K \to N/M \to 1$$

This leads to the main result.

Theorem 3.1 [Kov86, Theorem 4.3]. *Suppose that G satisfies (*) with M/K simple and the set $\{K_i\}$ finite. There is a bijection between the conjugacy classes of maximal subgroups of G in class (A) and the set of homomorphisms in (A1). Further, there are bijections between the conjugacy classes of maximal subgroups of G in types (B) and (C), and the conjugacy classes of maximal subgroups of N/M of types (B1) and (C1), respectively.*

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The families (A), (B), and (C) of maximal subgroups can be interpreted in terms of the O'Nan–Scott classes of the corresponding permutation representations. Suppose that G is finite with nonabelian, nonsimple, minimal normal subgroup M. Let H be a core-free maximal subgroup of G and consider the G-action on the right cosets of H. If H is in family (A), then G is a primitive group of SD or CD type depending on whether or not $H \cap M$ is simple. In these cases we also say that the maximal subgroup H is of SD or CD type. If H is in family (B), then G is primitive of PA type, while, if H is a maximal subgroup in family (C), then M is a regular normal subgroup of G and G can be embedded into the holomorph of M. Here G is either of TW type (when M is the unique minimal normal subgroup of G) or HC type (when G has a second minimal normal subgroup, distinct from M).

Full maximal subgroups of SD type in finite groups are further studied in the subsequent paper [Kov88]. The paper uses assumption (*) such that G is finite and M is a nonabelian, nonsimple minimal normal subgroup of G. A maximal subgroup of G with SD type corresponds to a homomorphism $G/M \to \mathsf{Out}(M/K)$ in (A1). The main results of [Kov88] are based on elegant counting arguments in the spirit of the quote at the end of Section 1 and describe full maximal subgroups of SD type in finite groups.

As observed in [GK84, after Corollary 3.3], finite primitive groups with a unique regular nonabelian minimal normal subgroup have a twisted wreath product structure, and they are most frequently studied using this structure. Suppose that G is a permutation group, that M is a nonabelian regular minimal normal subgroup, and that H is a point stabilizer. If T is a simple direct factor of M, then $M \cong T^k$ and G can be written as $G = T \,\mathsf{twr}_\varphi\, H$, where φ is the conjugation action of $Q = \mathbb{N}_H(T)$ on T. By Theorem 2.2, the subgroup H corresponds to a complement L/K of M/K in N/K. In the language of twisted wreath products, the subgroups N/K, M/K, and L/K can be identified with $T \rtimes Q$, T, and Q, respectively. Now the Kovács condition states that H is maximal or, equivalently, G is primitive if and only if Q is maximal in $T \rtimes Q$ and the conjugation action of Q cannot be extended to a larger subgroup of H. As T is a finite nonabelian simple group, the condition that Q is maximal in $T \rtimes Q$ is equivalent to the condition that Q does not normalize any nontrivial proper subgroup of T, which, in turn, is equivalent to the condition that conjugation by Q induces a group of automorphisms that contains $\mathsf{Inn}\, T$. The condition concerning $\mathsf{Inn}\, T$ is also obtained by Aschbacher and Scott in [AS85, Theorem 1(C)(1)], and conversely the 'nonextension' property in (C1) can be derived from it. So, the papers [AS85, Kov86] contain equivalent primitivity conditions for twisted wreath type groups.

Despite not having a published paper devoted to primitive groups of TW type, Laci's work has, perhaps, left its greatest impact on the theory of such groups. Laci championed the treatment of these groups as twisted wreath products, and today it would feel unnatural to treat them in any other way. He and Förster wrote an Australian National University Research Report [FK89] on TW groups and, as witnessed by the bibliographies of [FK90, Kov89b], he was working on more. His paper with Förster studies conditions under which the top group H in a twisted wreath product is maximal

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and they develop in [FK89, Theorem 1.1] conditions similar to the ones in (C2) above. It is an important and interesting consequence of their result [FK89, Corollary 1.2] that if H is such a maximal complement of M, then H must have a unique minimal normal subgroup N that is nonabelian, and a simple factor of M has to occur as a section in a simple factor of N. This implies the well-known result that H cannot have nontrivial solvable normal subgroups. The same research report contains a treatment of the inclusion problem for TW-type groups.

Laci's work on TW-type groups, albeit formally unpublished, had a huge influence on Baddeley's seminal work [Bad93] on this topic, as he acknowledged.

> It should be pointed out that a recent research report by Förster and Kovács [FK89] contains considerable overlap with our work in Sections 3 and 5, and indeed there may be even more in common between the material in this paper and their unpublished work. [Bad93, page 548]

Section 6 of Baddeley's paper treats the problem of permutational isomorphism between abstractly isomorphic twisted wreath products. He made Laci's contribution clear in a footnote.

> The ideas and results of this section are almost entirely due to Kovács. [Bad93, page 568]

Laci's work on maximal subgroups was also highly influential in the development of algorithms to determine the conjugacy classes of maximal subgroups of a finite group. In fact, a large part of the algorithm given in [EH01] relies on Theorem 3.1. Later, Cannon and Holt [CH04] presented essentially an algorithmic version of [Kov86] and in particular of Theorem 3.1, and this is now used in the MaximalSubroups function in Magma [BCP97].

4. Wreath products, blowups, and Wielandt's conjecture

A lot of work on primitive permutation groups concentrated on understanding links between primitive groups, nonabelian simple groups, and irreducible representations of finite groups. This information together with the wreath product construction led to the solution of many problems in algebra, number theory, and combinatorics.

The paper [Kov89a] presents methods for identifying if a primitive group can be written as a wreath product in product action and provides a means for counting the possible wreath decompositions. Laci introduced, as his main tool for tackling these counting problems, the concept of a *system of product imprimitivity*. A system of product imprimitivity can be used to detect embeddings of primitive groups into wreath products in product action in much the same way that a system of imprimitivity (or a block system) can be used to detect embeddings of transitive groups into wreath products in imprimitive actions.

Primitive wreath products usually have many primitive subgroups that are not themselves wreath products and, for some applications, detailed information is needed

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on precisely which subgroups of a wreath product $H \operatorname{wr} \mathsf{S}_k$ with H primitive on Γ are themselves primitive on Γ^k. In his seminal paper [Kov89b], Laci introduced the concept of a 'blow-up' of a primitive group and provided criteria for identifying such subgroups for almost all types of primitive groups H. Moreover, this led, in 1990, to a classification [Pra90] of embeddings of finite primitive groups into wreath products in product action.

Suppose that H is a primitive group acting on Γ with socle M. Then $W = H \operatorname{wr} \mathsf{S}_k$ can be considered as a permutation group on Γ^k in its product action. Let $\pi : W \to \mathsf{S}_k$ be the natural projection onto S_k, and consider π as a permutation representation of W. Let W_0 be the stabilizer of 1 under the representation π. Then W_0 can be written as a direct product $H \times (H \operatorname{wr} \mathsf{S}_{k-1})$. Let $\pi_0 : W_0 \to H$ denote the projection onto the first direct factor.

In the language of [Kov89b], a subgroup B of W is *large* if $B\pi$ is transitive, and $(B \cap W_0)\pi_0 = H$. Input to the *blow-up construction* consists of a primitive group G on Γ with socle M and a large subgroup B of $(G/M) \operatorname{wr} \mathsf{S}_k$ with $k \geqslant 1$. The output $G \uparrow B$, called the blow-up of G by B, is a permutation group on Γ^k, namely the full inverse image of B under the natural homomorphism $G \operatorname{wr} \mathsf{S}_k \to (G/M) \operatorname{wr} \mathsf{S}_k$.

THEOREM 4.1 [Kov89b, Theorems 1 and 2]. *All blow-ups of a finite primitive permutation group G are primitive if and only if the socle of G is not regular. If a primitive group G with nonregular socle is a blow-up, then it is a blow-up $G_0 \uparrow B$ of a unique G_0 which is not itself a blow-up.*

To identify which primitive groups are blow-ups, Laci introduced a corresponding decomposition concept. A *blow-up decomposition* of a primitive group G with nonregular socle M and point stabilizer H is a direct decomposition of M such that:

(1) the direct factors form a G-conjugacy class;
(2) $H \cap M$ is the direct product of its intersections with these direct factors.

If G is a primitive group with nonregular socle acting on Ω, then a blow-up decomposition of G with k factors leads to a bijection $\Omega \to \Gamma^k$ and a permutational isomorphism $G \to G_0 \uparrow B$, where G_0 is a primitive group acting on Γ. In other words, such a primitive group that admits a blow-up decomposition is itself a blow-up of a smaller group. The primitive groups that admit blow-up decompositions are, in modern terminology, the groups of PA, CD, and HC types and they are blow-ups of groups of AS, SD, and HS types, respectively.

Laci's approach that led to the blow-up concept also appears in the version of the O'Nan–Scott theorem presented in Cameron's book [Cam99]. Cameron divides the finite primitive groups into two large families: the groups in the first family are called *basic groups* and the other family is formed by the *nonbasic primitive groups*. In Cameron's terminology, a primitive group with a nonregular socle is basic if and only if it is not a blow-up of a smaller group, while blow-ups are nonbasic.

The philosophy of the blow-up construction is used in the paper [FK90] jointly written with Förster. This paper presents an application to Wielandt's problem on primitive permutation groups. The problem can be stated as follows.

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> Suppose that π_1 and π_2 are permutation representations of a finite group G with the same character. Given that $G\pi_1$ is primitive, does it follow that $G\pi_2$ is primitive?

By obvious reduction, we may assume that π_1 and π_2 are faithful. Suppose that G, π_1, and π_2 is a counterexample to Wielandt's conjecture and let A and B be point stabilizers for the representations π_1 and π_2, respectively. Then A is maximal while B is not, and we may assume that A and B are core-free. If $\operatorname{Soc} G$ is not simple, the argument of the paper shows that G, considered as a transitive group with point stabilizer A, is a primitive group of PA type. As discussed above, this implies that G is a blow-up of an almost simple group G_0 and it is shown in the paper that the almost simple group G_0 is also a counterexample to the conjecture. In fact, the paper shows how to construct the set of all faithful counterexamples, given the set of faithful almost simple counterexamples. Hence, the conjecture is reduced to the case of almost simple groups.

The Wielandt conjecture was proven to be false by Guralnick and Saxl [GS92], who presented the first almost simple counterexamples. Later, more counterexamples emerged in the work of Breuer [Bre95].

5. Other work on permutation groups

The *minimal faithful degree* $\mu(G)$ of a finite group G is the size of the smallest set on which G can be represented faithfully as a permutation group. Certainly, by Cayley's theorem, $\mu(G) \leqslant |G|$, but it can be considerably smaller. It is interesting to know (for example, in computational applications) about the relation, if any, between $\mu(G)$ and $\mu(G/N)$, where N is a normal subgroup of G. In general, $\mu(G/N)$ can be much larger than $\mu(G)$. In fact, Neumann [Neu86] gave examples for which $\mu(G/N) > c^{\mu(G)}$ with $c = 2^{0.25}$, while Holt and Walton [HW02] showed that $\mu(G)$, for arbitrary finite groups G, satisfies $\mu(G/N) \leqslant c^{\mu(G)-1}$ with $c = 4.5$. On the other hand, if G/N has no nontrivial abelian normal subgroup (for example, if N is the soluble radical of G), then Laci and the first author [KP00] showed that $\mu(G/N) \leqslant \mu(G)$. They proved the same inequality if G/N is elementary abelian [KP89, page 284] and conjectured that $\mu(G/N) \leqslant \mu(G)$ should hold whenever G/N is abelian.

Although this conjecture remains unresolved, [KP00, Theorem 2] shows the following (where the last assertion is by Franchi [Fra11]).

THEOREM 5.1. *For a potential counterexample* (G, N) *with both* $\mu(G)$ *and* $|G|$ *minimal,* G *must be a directly indecomposable* p*-group for some prime* p, N *must be the derived subgroup* G', $\mu(G/N) = \mu(G) + p$, *and* G *has no abelian maximal subgroup.*

The problem above, for G/N abelian, motivated the study [KP89] of permutation groups with nontrivial abelian quotients. Suppose that G is a permutation group on n points and that, for some prime divisor p of $|G|$, a Sylow p-subgroup moves exactly kp points. Then (see [KP89, Theorem]) the largest abelian p-quotient of G has order at most p^k, with equality if and only if G is the direct product of its largest p'-constituent

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and its transitive non-p'-constituents, and the possibilities for the latter are explicitly listed. As a corollary: if G is not perfect, that is, if $G \neq G'$, then for some prime divisor p of $|G/G'|$, the order $|G/G'| \leqslant p^{n/p}$, and the groups G for which equality holds are described. The cyclic group of order 30 represented as a permutation group on 10 points demonstrates that this upper bound may hold for more than one but not all prime divisors of $|G/G'|$. The result about $\mu(G/N)$ for G/N elementary abelian is then deduced. Another immediate consequence is that $|G/G'| \leqslant 3^{n/3}$ for all permutation groups G on n points, and so $|G| \leqslant 3^{n/3}$ if G is an abelian permutation group on n points. This upper bound is markedly different from the situation for primitive groups G, where it was proved in [AG89] that $|G/G'| \leqslant n$ and that n must be prime if equality holds.

There are several other papers by Laci on permutation groups to be noted. In the first [CKNP85] from 1985, the authors show that a transitive permutation group P of order a power of a prime p has at least $(p|P| - 1)/(p + 1)$ fixed-point-free elements, and analyses the possible structures of groups attaining the bound. A second paper [KN88], written in collaboration with Newman, concerns the number of generators for finite nilpotent transitive groups. They prove that there is a constant c such that each nilpotent transitive permutation group on n points can be generated by a set of $cn(\log n)^{-1/2}$ elements, and on the other hand they show that, for each prime p, there is a constant c_p such that, for each p-power n, there is a transitive p-group on n points which cannot be generated by $c_p n(\log n)^{-1/2}$ elements. The upper bound is now known to hold for all transitive permutation groups [BKR95, LMM00].

Laci, with Robinson [KR93], obtained an exponential upper bound 5^{n-1} on the number of conjugacy classes of a permutation group of degree n. Despite many improvements over the years, the best currently known general upper bound is still of this form with $5^{1/3}$ in place of 5 [MG14].

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https://doi.org/10.1017/S1446788715000385 Published online by Cambridge University Press

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