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ASYMPTOTIC BEHAVIOR OF NORMAL MAPPINGS OF SEVERAL COMPLEX VARIABLES

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1. Introduction. Let M and N be connected Hermitian manifolds of dimensions m and n with Hermitian metrics h_M and h_N, respectively. Then the space $\mathscr{C}(M, N)$ of continuous mappings between M and N endowed with the compact-open topology is second countable so that a metric can be furnished in $\mathscr{C}(M, N)$ which induces the compact-open topology. A sequence $\{f_n\}$ in $\mathscr{C}(M, N)$ converges to an f in $\mathscr{C}(M, N)$ in this topology if and only if f_n converges to f uniformly on compact subsets of M. It is then an easy consequence of the Cauchy integral formula to show that the space $\mathscr{H}(M, N)$ of holomorphic mappings $f:M \to N$ is a closed subspace of $\mathscr{C}(M, N)$.

In this paper, generalizing the classical notions of normal functions, Bloch functions, regular sequences and P-point sequences of one complex variable to the mappings in $\mathscr{H}(M, N)$, see also **[25]**, we obtain various relations which exist between these notions. These relationships are then used to draw some interesting conclusions on the asymptotic behavior of Bloch mappings, normal mappings and, more generally, non-normal holomorphic mappings along P-point sequences, regular sequences and certain asymptotic path to the boundary of M.

2. Definitions and preliminary lemmas.

2.1. *Definition.* A sequence $\{f_n\}$ in $\mathscr{C}(M, N)$ is called *compactly divergent* if for each compact K in M and compact K' in N there exists an $n_0 > 0$ such that $f_n(K) \cap K' = \emptyset$ for all $n \geqq n_0$. In particular, a sequence $\{p_n\}$ of points in N is *compactly divergent* if for each compact K in N there exists an $n_0 > 0$ such that $p_n \notin K$ for all $n \geqq n_0$.

We observe that if the metric h_N under consideration is complete in N, then the sequence $\{p_n\}$ being compactly divergent is equivalent to the fact That

$$\overline{\lim_{n\to\infty}} d_N(p_0, p_n) = \infty,$$

where p_0 is a fixed point in N and d_N denotes the distance function on N generated by h_N.

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2.2. *Definition*. A family $\mathscr{F} \subset \mathscr{C}(M, N)$ is called *normal* if every sequence of $\mathscr{F}$ contains a subsequence which is either relatively compact in $\mathscr{C}(M, N)$ or compactly divergent, and *equicontinuous* if for every $\epsilon > 0$ and $p \in M$ there exists a $\delta > 0$ such that $d_M(p, q) < \delta$ implies

$$d_N(f(p), f(q)) < \epsilon \quad \text{for all } f \in \mathscr{F}.$$

The normality of $\mathscr{F}$ need not imply the equicontinuity, while the equicontinuity of $\mathscr{F}$ together with the completeness of N implies the normality. If N is compact, then $\mathscr{F}$ is normal if and only if it is equicontinuous. See [**26**] for details.

2.3. *Definition*. Assume that M is homogeneous, i.e., the group $\mathrm{Aut}(M)$ of holomorphic automorphisms of M is transitive. A mapping $f \in \mathscr{H}(M, N)$ is called *normal* if the family $\{f \circ \varphi | \varphi \in \mathrm{Aut}(M)\}$ forms a normal family. By $\mathscr{N}(M, N)$ we shall denote the set of all normal mappings $f: M \to N$.

A complex manifold M is called *hyperbolic* if the Kobayashi pseudometric k_M is a metric [**17**]. Denote the infinitesimal Kobayashi metric by K_M. The Kobayashi metric is then the integrated form of K_M. See [**20**] for details.

2.4. *Definition*. Assume that M is hyperbolic. A mapping $f \in \mathscr{H}(M, N)$ is called *Bloch* if

(1a) $\quad Q_f \equiv \sup\{Q_f(p) \,|\, p \in M\} < \infty,$

where

(1b) $\quad Q_f(p) = \sup\limits_{|\xi|=1} \dfrac{h_N(f(p), df(p)\xi)}{K_M(p, \xi)}$

is the "maximal derivative" of f with respect to K_M at P.

We shall call the constant Q_f *the order of normality* of f and denote by $\mathscr{B}_\Omega$ the set of Bloch maps whose orders are less than or equal to Ω. Then

(2) $\quad \mathscr{B} = \mathscr{B}(M, N) = \bigcup\limits_{\Omega > 0} \mathscr{B}_\Omega(M, N)$

is the set of all Bloch maps $f: M \to N$. See [**12**].

Notice that the notion of maximal derivative is invariant under $\varphi \in \mathrm{Aut}(M)$ in the sense that for any $\varphi \in \mathrm{Aut}(M)$,

(3) $\quad Q_{f \circ \varphi}(p) = Q_f(\varphi(p)), \; p \in M,$

which follows from the invariant property of K_M.

Assume that M is hyperbolic and complete with respect to the Kobayashi metric. It is again a consequence of the invariant property of the Kobayashi metric that a hyperbolic manifold with the transitive group $\mathrm{Aut}(M)$ is complete. Under this assumption, M has a compact exhaustion:

(4a) $$M = \bigcup_{n=1}^{\infty} M_n$$

(4b) $$M_n = \{p \in M : k_M(p_0, p) \leqq n\}, \; n = 1, 2, \ldots$$

where p_0 is a fixed point of M.

2.5. *Definition*. The mappings $f \in \mathscr{B}(M, N)$ for which

(5) $$\lim_{n\to\infty} \sup_{p \in M\setminus M_n} Q_f(p) = 0$$

constitute an interesting subclass of $\mathscr{B}(M, N)$. This subclass will be denoted by $\mathscr{B}_0(M, N)$. See [**25**].

2.6. LEMMA. *Let M and N be complex manifolds. For each $p \in M$, $\xi \in \mathbf{C}^m$ and $f \in \mathscr{H}(M, N)$, define*

(6a) $$R_M(p) = \sup_{|\xi|=1} R_M(p, \xi),$$

(6b) $$r_M(p) = \inf_{|\xi|=1} R_M(p, \xi),$$

where

(6c) $$R_M(p, \xi) = \sup\{\,|\varphi'(0)|: \exists \varphi \in \mathscr{H}(\Delta, M), \quad \varphi(0) = p, \; \varphi'(0)a = \xi \text{ for some } a > 0\},$$

$$\Delta = \{z \in \mathbf{C} : |z| < 1\}.$$

Further, define

(7a) $$\Lambda_f(p) = \sup_{|\xi|=1} h_N(f(p), df(p)\xi)$$

(7b) $$\lambda_f(p) = \inf_{|\xi|=1} h_N(f(p), df(p)\xi).$$

Then

(8) $$\frac{|\xi|}{R_M(p)} \leqq K_M(p, \xi) \leqq \frac{|\xi|}{r_M(p)}$$

and

(9) $$\lambda_f(p) r_M(p) \leqq Q_f(p) \leqq \Lambda_f(p) R_M(p).$$

Furthermore, the following hold.

(a) *If M is hyperbolic, for each $p \in M$ there exists a neighborhood W of p and a number $R_M(W) \in (0, \infty)$ such that*

$$(10) \quad \frac{|\xi|}{R_M(W)} \leqq K_M(q, \xi) \quad \textit{for } q \in W.$$

(b) *Let M be hyperbolic. If M is complete, then*

$$\lim_{n\to\infty} r_M(p_n) = 0$$

for every compactly divergent sequence $\{p_n\}$ in M. Conversely, if

$$\overline{\lim_{n\to\infty}}\, R_M(p_n) = 0$$

for any compactly divergent sequence $\{p_n\}$ in M, then M is complete with respect to k_M.

(c) *If M is taut, i.e., $\mathscr{H}(\Delta, M)$ is normal, then Q_f is a continuous function in M. In particular, Q_f is continuous on a complete hyperbolic manifold M.*

Proof. Inequalities (8) follow immediately from the definition of the infinitesimal Kobayashi pseudometric [**20**]:

$$(11) \quad K_M(p, \xi) = \inf \{a > 0 : \exists \varphi \in \mathscr{H}(\Delta, M), \varphi(0) = p, \varphi'(0)a = \xi\}$$
$$= |\xi|/R_M(p, \xi), p \in M, \xi \in \mathbf{C}^m.$$

Thus, (9) follows from (8) and the definition of Q_f. (a) is just a result of H. Royden [**20**] which states that M is hyperbolic if and only if for each $p \in M$ there exists a neighborhood W of p and a number $R_M(W) > 0$ such that (10) holds. (b) is an immediate consequence of (8). The standard argument of normal families yields that the infinitesimal metric K_M is a continuous function of $(p, \xi) \in M \times \mathbf{C}^m$ whenever M is taut [**2**]. Therefore, the first statement of (c) is immediate. The second statement of (c) also follows, since a complete hyperbolic manifold is always taut [**16**].

2.7. LEMMA. *Let M be hyperbolic and N compact. A family $\mathscr{F} \subset \mathscr{H}(M, N)$ is normal if and only if for each compact $E \subset M$ there exists a constant $C(E) > 0$ such that*

$$(12) \quad \sup\{Q_f(p) : f \in \mathscr{F}\} \leqq C(E)$$

for all $p \in E$.

Proof. Suppose that (12) holds. Then $\mathscr{F}$ is equicontinuous and, hence, normal from the compactness of N. Conversely, suppose that $\mathscr{F}$ is normal but (12) does not hold. Then there must be a compact subset, E, of M, a sequence of points $\{p_n\}$ in E with $p_n \to p_0 \in E$, a sequence of unit vectors $\{\xi_n\}$ in $\mathbf{C}^m$ for with $\xi_n \to \xi_0$, $|\xi_0| = 1$, and a sequence of functions $\{f_n\}$ in $\mathscr{F}$ such that $n = 1, 2, \dots$

$$(13) \quad h_N(f_n(p_n), df_n(p_n)\xi_n) > nK_M(p_n, \xi_n).$$

Since $\mathscr{F}$ is normal, by extracting a subsequence $\{f_\nu\}$, we may assume that

$p_\nu \to p_0 \in E$, $\xi_\nu \to \xi_0 \in \mathbf{C}^m$ and $\{f_\nu\}$ converges to $f \in \mathscr{H}(M, N)$ uniformly on compact subsets of M and satisfies (13). Since N is compact, the left hand side of (13) tends to a finite number

$$h_N(f_0(p_0), df_0(p_0)\xi_0),$$

while the right hand side, because of inequality (10), can be made as large as we please as $\nu \to \infty$. This is a contradiction.

3. Some remarks on special cases. Let M be the open unit ball

$$B = \{z \in \mathbf{C}^m: |z| < 1\},$$

where

$$|z|^2 = (z, z) = \sum_{\nu=1}^{m} z_\nu \bar{z}_\nu.$$

Then Aut(B) acts transitively on B and consists of the transformations of the form:

$$(1) \quad \varphi_a(z) = \Gamma(a) \frac{z - a}{1 - (z, a)} (a \in B, z \in B)$$

where $\Gamma(a) = (1 - aa^*)^{1/2}(I - a^*a)^{-1/2}$, $(z, a) = za^*$, a^* is the complex conjugate transposed of a and I the identity operator in $\mathbf{C}^m$. See [**14**].

The infinitesimal Kobayashi metric coincides (modulo constant multiple) with the Poincaré-Bergman metric and is given by

$$(2) \quad K_B(z, \xi) = [(1 - |z|^2)|\xi|^2 + |(z, \xi)|^2]^{1/2}/(1 - |z|^2); \quad (z \in B, \xi \in \mathbf{C}^m),$$

which clearly satisfies the inequalities:

$$(3) \quad \frac{|\xi|}{\sqrt{1 - |z|^2}} \leqq K_B(z, \xi) \leqq \frac{|\xi|}{1 - |z|^2} (z \in B, \xi \in \mathbf{C}^m).$$

Integrating both sides of (2) along the line segment $z = at$, $a \in B$, $t \in [0, 1]$, we have

$$(4a) \quad k_B(0, a) = \frac{1}{2} \log \frac{1 + |a|}{1 - |a|} = \tan h^{-1}|a|$$

and for any two points a and b in B,

$$(4b) \quad k_B(a, b) = k_B(0, \varphi_a(b)) = \tan h^{-1}|\varphi_a(b)|,$$

where

$$(4c) \quad |\varphi_a(b)| = \frac{[|(a, b)|^2 - |a|^2|b|^2 + |a - b|^2]^{1/2}}{|1 - (a, b)|}.$$

Let N be the complex euclidean space $\mathbf{C}^n$ with the standard metric. Then the space $\mathscr{B}(B, \mathbf{C}^n)$ is a Banach space. More precisely,

3.1. PROPOSITION. *The space $\mathscr{B}(B, \mathbf{C}^n)$ is a Banach space with respect to the norm*:

(5a) $$\|f\|_{\mathscr{B}} = |f(0)| + \sup_{z \in B} Q_f(z),$$

where

(5b) $$Q_f(z) = \sup_{|\xi|=1} \frac{|df(z)\xi|}{K_B(z, \xi)}.$$

Moreover, $\mathscr{B}_0(B, \mathbf{C}^n)$ is a nowhere dense closed subspace of $\mathscr{B}(B, \mathbf{C}^n)$ which is the closure of the space of complex polynomial maps from $\mathbf{C}^m$ to $\mathbf{C}^n$.

This proposition is a generalization of a result in [**1**] for one variable and remains valid when B is replaced by a bounded symmetric domain [**23**] and $\mathbf{C}^n$ by any complex Hilbert space [**24**]. The corresponding result is not true for the space $\mathscr{N}(B, \mathbf{C}^n)$. It is not even a linear space. The sum of two normal mappings is not normal in general [**18**]. Actually, $\mathscr{B}(B, \mathbf{C}^n)$ is a proper subspace of $\mathscr{N}(B, \mathbf{C}^n)$. More generally, we have

3.2. PROPOSITION. *If M is hyperbolic and N is complete then $\mathscr{B}(M, N)$ is a proper subspace of $\mathscr{N}(M, N)$.*

Proof. By definition, $\mathscr{B}(M, N)$ is equicontinuous. Since N is complete, every closed bounded set of N is compact, by the Rinow-Hopf theorem. Therefore $\mathscr{B}(M, N)$ forms a normal family as remarked earlier in Section 2. The fact that $\mathscr{B}(B^m, \mathbf{C}^n)$ is proper in $\mathscr{N}(B, \mathbf{C}^n)$ follows from the example given in [**12**].

On the other hand, every bounded holomorphic mapping on B belongs to $\mathscr{B}(B, \mathbf{C}^n)$. See [**12**], for example. This result can be extended to any strongly pseudoconvex domain Ω.

3.3. PROPOSITION. *The class of all bounded holomorphic mappings on a strongly pseudoconvex domain Ω is a proper subspace of $\mathscr{B}(\Omega, \mathbf{C}^n)$.*

Proof. This follows from the fact that the Kobayashi metric K_Ω is uniformly equivalent to the Bergman metric B_Ω on every strongly pseudoconvex domain Ω (see [**6**] or [**11**]) and the fact that any bounded holomorphic mapping $f:\Omega \to \mathbf{C}^n$ satisfies:

(6) $$\sup_{|\xi|=1} \frac{|df(z)\xi|}{B_\Omega(z, \xi)} \leqq A$$

where A is an upper bound of $|f|$ on Ω. See [**13**].

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Let N be the complex projective space $P^n(\mathbf{C})$ with the standard hermitian metric:

$$(7) \quad \chi(z, \xi) = \frac{[(1 - |z|^2)|\xi|^2 - |(z, \xi)|^2]^{1/2}}{1 + |z|^2} (z \in P^n, \xi \in \mathbf{C}^n).$$

Then the space $\mathscr{B}(B, P^n)$ ceases to be a linear space. In fact, $\mathscr{B}(B, P^n) = \mathscr{N}(B, P^n)$ in this case. See Section 5 for a more general result.

4. Some basic properties of normal mappings.

4.1. *Definition.* A sequence $\{p_n\}$ of points in M is called *regular* for $f \in \mathscr{H}(M, N)$ if there exists a positive δ such that for any sequence $\{q_n\}$ in M with $k_M(p_n, q_n) < \delta$ for all n,

$$\lim_{n \to \infty} d_N(f(p_n), f(q_n)) = 0.$$

A sequence $\{p_n\}$ that is not regular is called *irregular*.

4.2. *Definition.* A sequence $\{p_n\}$ of points in M is called a *P-point sequence* of $f \in \mathscr{H}(M, N)$ if there exists a sequence $\{q_n\}$ in M and a number $\epsilon > 0$ such that

$$\lim_{n \to \infty} k_M(p_n, q_n) = 0$$

but

$$\lim_{n \to \infty} \sup d_N(f(p_n), f(q_n)) \geqq \epsilon.$$

The notion of regular sequence was introduced in [**21**], while the notion of P-point sequence was originally defined by V. Gavrilov [**9**] for meromorphic functions in the unit disc Δ. The present version of P-point sequence was introduced by P. Gauthier [**8**] in which he proved that the two notions are equivalent for meromorphic functions in Δ.

In [**25**] F. Wicker generalized these notions to higher dimensional case. In this paper, we adopt Gauthier's definition which is advantageous for higher dimensional generalization.

4.3. Proposition. *Every P-point sequence $\{p_n\}$ of a holomorphic mapping $f \in \mathscr{H}(M, N)$ is compactly divergent.*

Proof. Suppose that $\{p_n\}$ is not compactly divergent. Then there is a subsequence $\{p_\nu\}$ which converges to some point $p_0 \in M$. If $\{q_n\}$ is a sequence in M such that

$$\lim_{n \to \infty} k_M(p_n, q_n) = 0$$

then

$$\lim_{\nu\to\infty} k_M(q_\nu, p_0) = 0$$

by the triangle inequality. If $f \in \mathscr{H}(M, N)$ then

$$d_N(f(p_\nu), f(q_\nu)) \leqq d_N(f(p_\nu), f(p_0)) + d_N(f(p_0), f(q_\nu))$$

implies that

$$\lim_{\nu\to\infty} d_N(f(p_\nu), f(q_\nu)) = 0$$

which is a contradiction.

It is clear that a regular sequence of $f \in \mathscr{H}(M, N)$ cannot be a P-point sequence. Therefore, every P-point sequence is irregular. The converse is in general false. See Example 9.2.

The following proposition is essentially due to F. Wicker [**25**] for slightly smaller class.

4.4. PROPOSITION. *Let M be a homogeneous hyperbolic manifold and N any complex manifold. Let* $\{p_n\}$ and $\{q_n\}$ *be sequences in M such that*

$$\lim_{n\to\infty} k_M(p_n, q_n) = 0.$$

The following statements hold for $f \in \mathscr{N}(M, N)$.

(a) *If there is a subsequence $\{p_\nu\}$ of $\{p_n\}$ such that*

$$\lim_{\nu\to\infty} f(p_\nu) = l \in N,$$

then $\lim_{\nu\to\infty} f(q_\nu) = l$.

(b) *If $\{f(p_n)\}$ is compactly divergent then $\{f(q_n)\}$ is also compactly divergent.*

(c) *If $\{p_n\} \subset M$ is a P-point sequence of $f \in \mathscr{N}(M, N)$, then $\{f(p_n)\}$ is compactly divergent.*

Remark that if N is compact, then N contains no compactly divergent sequence. Therefore, both (b) and (c) hold trivially.

Proof. Let $p_0 \in M$ be a fixed point. Since $\mathrm{Aut}(M)$ is transitive, for each n there exists $\varphi_n \in \mathrm{Aut}(M)$ such that $\varphi_n(p_0) = p_n$. Let $w_n = \varphi_n^{-1}(q_n)$. Then

$$\lim_{n\to\infty} k_M(p_0, w_n) = \lim_{n\to\infty} k_M(p_n, q_n) = 0.$$

Therefore, for any $\epsilon > 0$ there exists an n_0 such that for all $n \geqq n_0$, $w_n \in B_k(p_0, \epsilon)$, the ball of radius ϵ, centered at p_0 and measured by k_M. To show (a), it is enough to show:

$$\lim_{n\to\infty} d_N(l, g_n(w_n)) = \lim_{n\to\infty} d_N(l, f(q_n)) = 0,$$

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where $g_n = f \circ \varphi_n$. By the normality of f, there is a subsequence $\{g_\nu\}$ of $\{g_n\}$ which converges to $g \in \mathscr{H}(M, N)$ with

$$\lim_{\nu\to\infty} g_\nu(p_0) = \lim_{\nu\to\infty} f(p_\nu) = l = g(p_0).$$

Therefore,

$$(1) \quad d_N(l, g_\nu(w_\nu)) = d_N(l, f(q_\nu)) \leqq d_N(g(p_0), g(w_\nu)) + d_N(g(w_\nu), g_\nu(w_\nu)).$$

The first term on the right hand side of (1) can be made arbitrarily small since g is continuous and

$$\lim_{\nu\to\infty} w_\nu = p_0,$$

and the second term can be made arbitrarily small since $\{g_\nu\}$ converges to g uniformly on the set

$$\{w_\nu\} \subset \overline{B_k(p_0, \epsilon)}$$

which is compact.

For the proofs of (b) and (c), we may assume that N is noncompact.

(b) Since $\{g_n(p_0)\}$, $g_n(p_0) = f(p_n)$, is compactly divergent in N, no subsequence of $\{g_n\}$ can converge to a holomorphic map on $B_k(p_0, \epsilon)$. Therefore, $\{g_n(w_n)\}$, $g_n(w_n) = f(q_n)$, is compactly divergent.

(c) By the definition of a P-point sequence there exists a sequence $\{q_n\}$ in M such that

$$\lim_{n\to\infty} k_M(p_n, q_n) = 0$$

but

$$(2) \quad \limsup_{n\to\infty} d_N(f(p_n), f(q_n)) \geqq \epsilon, \quad \text{for some } \epsilon > 0.$$

Suppose that there is a subsequence $\{p_\nu\}$ for which $\{f(p_\nu)\}$ converges to $l \in N$, say. Then, by (a),

$$\lim_{\nu\to\infty} f(q_\nu) = l,$$

contradicting (2). Therefore, $\{f(p_n)\}$ is compactly divergent.

Remark that Proposition 4.4 is false in general for non-normal holomorphic mappings. Examples are given in Example 9.3.

5. Normal mappings into compact manifolds. In this section we assume that M is hyperbolic and homogeneous, and N is compact. The following theorems characterize normal (or Bloch) mappings.

5.1. THEOREM. *The following statements are equivalent for* $f \in \mathscr{H}(M, N)$:

(a) $f \in \mathscr{N}(M, N)$.

(b) $f \in \mathscr{B}(M, N)$.

(c) $f \in \mathscr{H}(M, N)$ *is uniformly continuous, i.e., for each* $\epsilon > 0$ *there exists a* $\delta > 0$ *such that for all* $p, q \in M$ *with* $k_M(p, q) < \delta$ *we have*

$$d_N(f(p), f(q)) < \epsilon.$$

(d) $f \in \mathscr{H}(M, N)$ *has no P-point sequence in M, i.e., for any two sequences* $\{p_n\}$ *and* $\{q_n\}$ *in M with*

$$\lim_{n\to\infty} k_M(p_n, q_n) = 0,$$

we have

$$\overline{\lim_{n\to\infty}}\, d_N(f(p_n), f(q_n)) = 0.$$

Proof. (a) $\Rightarrow$ (b): Let $f \in \mathscr{H}(M, N)$. If f is not in $\mathscr{B}(M, N)$, there exists a sequence $\{p_n\}$ in M and unit vectors $\xi_n \in \mathbf{C}^m$ such that

$$(1) \qquad h_N(f(p_n), df(p_n)\xi_n) > nK_M(p_n, \xi_n) \quad \text{for all } n.$$

Let $p_0 \in M$ and $\eta \in \mathbf{C}^m$, $|\eta| = 1$, be fixed. Since Aut(M) is transitive, there exists $\varphi_n \in \mathrm{Aut}(M)$, for each n, such that

$$\varphi_n(p_0) = p_n, \ d\varphi_n(p_0)\eta = \xi_n.$$

Then (1) becomes

$$(2) \qquad h_N(g_n(p_0), dg_n(p_0)\eta) > nK_M(\varphi_n(p_0), d\varphi_n(p_0)\eta)$$

where $g_n = f \circ \varphi_n$. Since f is normal, $\{g_n\}$ has a subsequence $\{g_\nu\}$ which converges to a mapping $g \in \mathscr{H}(M, N)$ and

$$\lim_{\nu\to\infty} g_\nu(p_0) = \lim_{\nu\to\infty} f(p_\nu) = g(p_0).$$

By the invariant property of the Kobayashi metric under Aut(M)

$$(3) \qquad K_M(\varphi(p_0), d\varphi(p_0)\eta) = K_M(p_0, \eta), \quad \varphi \in \mathrm{Aut}(M).$$

Since M is hyperbolic, there exists a positive constant $C > 0$ such that

$$(4) \qquad K_M(p, \eta) \geqq C|\eta|$$

for all p in some neighborhood U of p_0.

Combining (2), (3) and (4), we have

$$(5) \qquad h_N(g_\nu(p_0), dg_\nu(p_0)\eta) > \nu C|\eta|$$

hold for all large ν. It is clearly impossible because, as $\nu \to \infty$, the right hand side of (5) increases to ∞ while the left hand side tends to

$$h_N(g(p_0), dg(p_0)\eta)$$

which is finite.

(b) $\Rightarrow$ (c). If (b) holds, then there exists a constant $\Omega > 0$ such that for each $p \in M$ and $\xi \in \mathbf{C}^m$

$$h_N(f(p), df(p)\xi) \leqq \Omega K_M(p, \xi).$$

By integrating along any C^1 curve γ connecting any two points p_1 and p_2 in M, we obtain

$$d_N(f(p_1), f(p_2)) \leqq \Omega k_M(p_1, p_2),$$

from which (c) follows.

(c) $\Leftrightarrow$ (d). If (d) fails to hold, then there exist sequences $\{p_n\}$ and $\{q_n\}$ in M with

$$\lim_{n\to\infty} k_M(p_n, q_n) = 0$$

but $d_N(f(p_n), f(q_n)) \geqq \epsilon$ for some constant $\epsilon > 0$ which contradicts (c). If (c) fails to hold, then there exists an $\epsilon > 0$ and sequences $\{p_n\}$ and $\{q_n\}$ in M such that $k_M(p_n, q_n) < 1/n$ but

$$d_N(f(p_n), f(q_n)) \geqq \epsilon$$

which contradicts (d).

(c) $\Rightarrow$ (a). Since $f \in \mathscr{H}(M, N)$ is uniformly continuous, for each $\epsilon > 0$ there exists $\delta > 0$ such that $d_N(f(\varphi(p)), f(\varphi(q))) < \epsilon$ whenever

$$k_M(\varphi(p), \varphi(q)) < \delta,$$

where $p, q \in M$ and $\varphi \in \mathrm{Aut}(M)$. But

$$k_M(\varphi(p), \varphi(q)) = k_M(p, q) \quad \text{for all } \varphi \in \mathrm{Aut}(M).$$

Therefore, $\{f \circ \varphi | \varphi \in \mathrm{Aut}(M)\}$ is an equicontinuous family. Since N is compact, it is also a normal family. See **[25]** for a similar proof.

The following Corollary is an immediate consequence of Theorem 5.1.

5.2. COROLLARY. *The following statements are equivalent.*

(a) *The mapping $f \in \mathscr{H}(M, N)$ has a P-point sequence in M.*

(b) $\sup_{z\in M} Q_f(z) = \infty$.

(c) $f \notin \mathscr{N}(M, N)$.

5.3. *Remark.* It is clear that a Bloch mapping $f \in \mathscr{H}(M, N)$ can not possess a P-point sequence, regardless of N being compact or not. However, a P-point sequence can be possessed by a normal mapping $f \in \mathscr{H}(M, N)$ for a non-compact N, as the following example illustrates: Let

$$f(z) = (5 + w + e^{1-w})^4, \; w = \frac{1+z}{1-z}, \; z \in \Delta.$$

After some computation, we find that

$$Q_f(z_n) = |f'(z_n)|(1 - |z_n|^2) \to \infty$$

for

$$z_n = \frac{1 - 2n^2 + 2n^3\pi i}{1 + 2n^3\pi i}.$$

It is readily seen that the mapping f omits a neighborhood of the origin on the unit disc Δ, i.e., $f \in \mathcal{N}(\Delta, \mathbf{C})$ by the classical theorem of Montel. See [**21**].

5.4. THEOREM. ([**25**]). *All sequences* $\{p_n\}$ *in* M *are regular for* $f \in \mathcal{N}(M, N)$ *if and only if* $f \in \mathcal{B}_0(M, N)$.

Proof. Assuming all sequences of points in M are regular for $f \in \mathcal{N}(M, N)$, define

$$M_n = \{p \in M | k_M(p_0, p) < n\},$$

where p_0 is a fixed point in M. There exists a sequence $\{p_n\}$ in M such that

$$(6) \qquad Q_f(p_n) = \sup\{Q_f(p) | p \in \overline{M_n \setminus M_{n-1}}\}.$$

By the regularity of $\{p_n\}$ for f, for every $\epsilon > 0$ there is a $\delta > 0$ such that for any sequence $\{q_n\}$ in M with $k_M(p_n, q_n) < \delta$, we have

$$\lim_{n \to \infty} d_N(f(p_n), f(q_n)) = 0.$$

Now for each n there exists $\varphi_n \in \mathrm{Aut}(M)$ such that $p_n = \varphi_n(p_0)$. Since f is normal, $\{f \circ \varphi_n\}$ has a subsequence $\{f \circ \varphi_\nu\}$ which converges uniformly to $g \in \mathcal{H}(M, N)$ on $\overline{B_k(p_0, \delta)}$. In particular,

$$\lim g_\nu(p_0) = \lim f \circ \varphi_\nu(p_0) = g(p_0) = l.$$

For any $z \in B_k(p_0, \delta)$,

$$(7) \qquad d_N(l, g(z)) \leqq d_N(g(p_0), g_\nu(p_0)) + d_N(g_\nu(p_0), g_\nu(z)) + d_N(g_\nu(z), g(z)).$$

On the right hand side of inequality (7), the first and third terms converge to 0 by the uniform convergence of g_ν on $B_k(p_0, \delta)$. The second term also converges to 0 by the regularity of $\{p_n\}$. Therefore, $g(z) \equiv l$ for all $z \in B_k(p_0, \delta)$. By the uniqueness theorem for holomorphic functions, $g(z) \equiv l$ on M since M is connected. Therefore, g_ν converges to a constant function l uniformly on compact subsets of M. Therefore,

$$\lim_{\nu \to \infty} Q_f(p_\nu) = \lim_{\nu \to \infty} Q_{g_\nu}(p_0) = 0.$$

Since

$$M \backslash M_\nu = \bigcup_{\alpha=\nu}^{\infty} \overline{M_{\alpha+1} \backslash M_\alpha},$$

$$\sup\{Q_f(p) | p \in \overline{M \backslash M_\nu}\} = \sup_{\alpha \geqq \nu+1} Q_f(p_\alpha).$$

Thus, $f \in \mathscr{B}_0(M, N)$ because

$$\lim_{\nu \to \infty} Q_f(p_\nu) = 0.$$

To prove the converse, let $f \in \mathscr{B}_0(M, N)$. Since any convergent sequence $\{p_n\}$ in M is regular for any $f \in \mathscr{H}(M, N)$, we only consider sequences which are compactly divergent. Let p_0 be a fixed point in M and let $s_n = k_M(p_0, p_n)$. Then

$$\overline{\lim_{n \to \infty}} s_n = \infty.$$

From $\{s_n\}$ we select a strictly increasing sequence and call this sequence $\{s_n\}$ again. Let

$$\delta = \inf\{k_M(p_n, p_{n+1})/3 | n = 1, 2, \dots\}.$$

Then $\delta > 0$. Let $\{q_n\}$ be any sequence in M with $k_M(p_n, q_n) < \delta$. If $f \in \mathscr{B}_0(M, N)$,

$$\Omega_n = \sup\{Q_f(p) | p \in M \backslash M_n\}$$

exists and is finite for each n and $\Omega_n \to 0$ as $n \to \infty$. Therefore,

$$d_N(f(p_n), f(q_n)) \leqq k_M(p_n, q_n)\Omega_{n-1} \leqq \delta\Omega_{n-1}$$

and, hence,

$$\lim_{n \to \infty} d_N(f(p_n), f(q_n)) = 0.$$

Thus, by definition $\{p_n\}$ is a regular sequence.

6. Further properties of normal mappings.

6.1. THEOREM. *Let Ω be any bounded domain in $\mathbf{C}^m$ and N a complex manifold. Suppose that there exist sequences $\{p_n\}$ in Ω and $\{r_n\}$, $r_n > 0$, with the property:*

$$(1) \quad \lim_{n \to \infty} \frac{r_n}{\delta_\Omega(p_n)} = 0,$$

where $\delta_\Omega(p) = \rho(p, \partial\Omega)$, the euclidean distance from p to $\partial\Omega$, such that $\{f(p_n + r_n\zeta)\}$, $f \in \mathscr{H}(\Omega, N)$, converges locally uniformly to a nonconstant *holomorphic mapping* $g \in \mathscr{H}(\mathbf{C}^m, N)$. *Then* $\{p_n\}$ *is a P-point sequence for* f. *In particular, if* Ω *is a bounded homogeneous domain in* $\mathbf{C}^m$ *and* N *is compact, then*

$$\sup\{Q_f(p):p \in \Omega\} = \infty$$

and f *is non-normal.*

Proof. Since g is nonconstant in $\mathbf{C}^m$, there are two distinct points ζ^1 and ζ^2 in $\mathbf{C}^m$ such that $g(\zeta^1) \neq g(\zeta^2)$. Let

$$\delta' = d_N(g(\zeta^1), g(\zeta^2)) > 0, \quad q_n^i = p_n + r_n\zeta^i \quad (i = 1, 2).$$

The triangle inequality implies that

$$d_N(f(q_n^1), f(q_n^2)) \geqq \delta \quad \text{for some } \delta > 0.$$

It is therefore enough to show that

$$k_\Omega(q_n^1, q_n^2) \to 0 \quad \text{as } n \to \infty.$$

Let $R > \max\{|\zeta^1|, |\zeta^2|\}$. Then

$$|p_n - q_n^i| = r_n|\zeta^i| \leqq r_nR.$$

Thus,

$$q_n^i \in \overline{B(p_n, r_nR)} \quad (i = 1, 2).$$

We observe that if $B(z, r) \subset \Omega$, then

$$(2) \qquad k_\Omega(z, w) \leqq \tan h^{-1}\frac{|z - w|}{r} \quad \text{for } w \in B(z, r).$$

To see this, let $w \neq z$ and define

$$f(\lambda) = z + \lambda v,$$

where $v = (w - z)/s$, $s = |w - z|/r$. Then by the distance decreasing property of the Kobayashi metric under $f \in \mathscr{H}(\Delta, \Omega)$,

$$k_\Omega(z, w) = k_\Omega(f(0), f(s)) \leqq k_\Delta(0, s) = \tan h^{-1}\frac{|w - z|}{r}$$

which is to be proved. Using (2) on $B(p_n, r_nR)$,

$$k_\Omega(p_n, q_n^i) \leqq \tan h^{-1}\frac{|q_n^i - p_n|}{\delta_\Omega(p_n)} = \tan h^{-1}\frac{r_nR}{\delta_\Omega(p_n)},$$

from which we have

$$\lim_{n\to\infty} k_\Omega(p_n, q_n^i) = 0 \quad (i = 1, 2).$$

The triangle inequality now yields that

$$k_\Omega(q_n^1, q_n^2) \to 0 \quad \text{as } n \to \infty.$$

The second claim follows from Corollary 5.2.

6.2. COROLLARY. *Let* Ω *be a bounded domain in* $\mathbf{C}^m$ *and* N *a compact manifold. If* $f \in \mathcal{N}(\Omega, N)$, *then for every choice of sequences* $\{p_n\}$ *in* Ω *and* $\{r_n\}$, $r_n > 0$, *with the property* (1), *the sequence* $\{f(p_n + r_n\zeta)\}$ *converges to a constant mapping in* $\mathbf{C}^m$ *and, therefore, to a single point in* N.

The converse to Theorem 6.1, as it stands does not seem to hold even on the open unit ball in $\mathbf{C}^m$ if $m \geqq 2$. However, we can still prove the following slightly weak converse.

6.3. THEOREM. *Let* Ω *be a bounded domain in* $\mathbf{C}^m$ *and* N *a compact manifold. If* $f \in \mathcal{H}(\Omega, N)$ *satisfies*:

$$(3) \qquad \sup_{p \in \Omega} \Lambda_f(p)\delta_\Omega(p) = \infty$$

then there are sequences $\{p_n\}$ *in* Ω *and* $\{r_n\}$, $r_n > 0$, *satisfying condition* (1) *such that* $\{f(p_n + r_n\zeta)\}$ *converges locally uniformly in* $\mathbf{C}^m$ *to a nonconstant holomorphic mapping* $g \in \mathcal{H}(\mathbf{C}^m, N)$.

Proof. Condition (3) implies that there exists a sequence $\{q_n\}$ in Ω such that

$$\lim_{n \to \infty} \Lambda_f(q_n)\delta_\Omega(q_n) = \infty.$$

It is clear that q_n tends to the boundary of Ω. Therefore, there exists a sequence $\{\delta_n\}$, $\delta_n > 0$, $\delta_n \to 0$, such that $\delta_\Omega(q_n) > \delta_n$ and

$$(4) \qquad \lim_{n \to \infty} \Lambda_f(q_n)\delta_{\Omega_n}(q_n) = \infty,$$

where

$$\Omega_n = \{p \in \Omega : \delta_\Omega(p) > \delta_n\}.$$

Define

$$(5) \qquad M_n = \max\{\Lambda_f(p)\delta_{\Omega_n}(p) : p \in \bar{\Omega}_n\}.$$

Since Λ_f is continuous on $\bar{\Omega}_n$, there exists a sequence $\{p_n\}$ in $\bar{\Omega}_n$ such that

$$(6) \qquad M_n = \Lambda_f(p_n)\delta_{\Omega_n}(p_n).$$

Since $q_n \in \bar{\Omega}_n$, it follows from (4) that $M_n \to \infty$ and, hence, $\Lambda_f(p_n) \to \infty$ as $n \to \infty$. Set

$$(7) \qquad r_n = \frac{1}{\Lambda_f(p_n)} = \frac{\delta_{\Omega_n}(p_n)}{M_n}.$$

Then

$$(8)\qquad \frac{r_n}{\delta_\Omega(p_n)} \leqq \frac{r_n}{\delta_{\Omega_n}(p_n)} = \frac{1}{M_n} \to 0$$

as $n \to \infty$ and

$$(9)\qquad r_n \Lambda_f(p_n) = 1 \quad \text{for all } n.$$

Since

$$R_n = \delta_{\Omega_n}(p_n)/r_n \to \infty,$$

for any $R > 0$, $R \leqq R_n$ for sufficiently large n. Let $|\zeta| \leqq R$. Then

$$p_n + r_n\zeta \in \bar{\Omega}_n.$$

Therefore, the mappings

$$(10)\qquad g_n(\zeta) = f(p_n + r_n\zeta)$$

are well-defined and holomorphic for all $|\zeta| < R$. Since $p_n + r_n\zeta \in \bar{\Omega}_n$, (5) implies

$$(11)\qquad M_n \geqq \Lambda_f(p_n + r_n\zeta)\delta_{\Omega_n}(p_n + r_n\zeta).$$

Therefore, from (10), (11) and (7),

$$(12)\qquad \Lambda g_n(\zeta) = r_n\Lambda_f(p_n + r_n\zeta) \leqq \frac{r_n M_n}{\delta_{\Omega_n}(p_n + r_n\zeta)} = \frac{\delta_{\Omega_n}(p_n)}{\delta_{\Omega_n}(p_n + r_n\zeta)}.$$

Since the right hand side of (12) tends to 1 for all ζ lying in a compact subset of $\mathbf{C}^m$, $\{g_n\}$ is equicontinuous and, hence, normal in $\mathbf{C}^m$ by Lemma 2.7 since N is compact. By passing to a subsequence, if necessary, which will be denoted again by $\{g_n\}$, we may suppose that there exist a sequence $\{p_n\}$ in Ω and $\{r_n\}$, $r_n > 0$, with

$$\frac{r_n}{\delta_\Omega(p_n)} \to 0 \quad \text{as } n \to \infty$$

such that $\{g_n(\zeta)\}$ converges locally uniformly to a nonconstant holomorphic mapping $g \in \mathscr{H}(\mathbf{C}^m, N)$. That g is nonconstant follows from the first relation of (12) by setting $\zeta = 0$ and (9).

For the case where $m = 1$, let Ω be the open unit disc Δ in $\mathbf{C}$ and N as before. Then for any $f \in \mathscr{H}(\Delta, N)$,

$$(13)\qquad Q_f(z) = (1 - |z|^2)\Lambda_f(z),$$

where

$$\Lambda_f(z) = \sup_{|\xi|=1} h_N(f(z), df(z)\xi).$$

Since $\delta_\Delta(z) \leqq 1 - |z|^2 \leqq 2\delta_\Delta(z)$, Condition (1) is equivalent to the condition:

$$(14)\quad \lim_{n\to\infty} \frac{r_n}{1 - |p_n|} = 0$$

and condition (3) is equivalent to the condition:

$$\sup_{z\in\Delta} Q_f(z) = \infty.$$

Therefore, combining Theorems 6.1 and 6.3, we obtain the following result which is an extension of Theorem 1 of [**19**].

6.4. THEOREM. *Let Δ be the open unit disc in $\mathbf{C}$ and N a compact manifold. The mapping $f \in \mathscr{H}(\Delta, N)$ is not normal if and only if there exist sequences $\{z_n\}$ in Δ, $\{r_n\}$, $r_n > 0$, with property* (14) *such that $\{f(z_n + r_n\zeta)\}$ converges locally uniformly in $\mathbf{C}$ to a nonconstant holomorphic mapping $g \in \mathscr{H}(\mathbf{C}, N)$.*

Theorem 6.4 could have been proved directly by following the method of [**19**]. In the same manner we can also prove the following:

6.5. THEOREM. *Let Δ be the open unit disc and N a compact manifold. A family $\mathscr{F}$ in $\mathscr{H}(\Delta, N)$ is not normal if and only if there exist an $r \in (0, 1)$ and sequences $\{z_n\}$, $|z_n| \leqq r$, $\{r_n\}$, $r_n \downarrow 0$, and $\{f_n\}$ in $\mathscr{F}$ such that $\{f_n(z_n + r_n\zeta)\}$ converges locally uniformly to a nonconstant holomorphic mapping $g \in \mathscr{H}(\mathbf{C}, N)$.*

Let X be a complex submanifold of N. Then, by definition, $\mathscr{H}(\Delta, X)$ is a normal family if and only if X is taut [**2**]. Therefore, we have the following criterion for tautness of X.

6.6. COROLLARY. *A closed complex submanifold X of compact manifold N is taut if and only if for every choice of sequences $\{z_n\}$ in Δ, $\{f_n\}$ in $\mathscr{H}(\Delta, X)$ and $\{r_n\}$ with $r_n \downarrow 0$, $\{f_n(z_n + r_n\zeta)\}$ converges locally uniformly to a constant map in $\mathbf{C}$.*

In particular, we have the following result of R. Brody [**3**] as a corollary.

6.7. COROLLARY. *A compact manifold N is hyperbolic or, equivalently, taut if and only if there is no nonconstant holomorphic mapping $f:\mathbf{C} \to N$.*

7. Characterization of P-point sequences for meromorphic functions. In this section we consider the case where M is a bounded homogeneous domain Ω in $\mathbf{C}^m$ and N is the Riemann sphere Σ with the chordal metric

$$(1)\quad d\chi(w) = \frac{|dw|}{1 + |w|^2}(w \in \Sigma).$$

Then the space $\mathscr{H}(\Omega, \Sigma)$ consists of all meromorphic functions on Ω and the space $\mathcal{N}(\Omega, \Sigma)$ coincides with the space $\mathcal{B}(\Omega, \Sigma)$ of Bloch meromorphic functions. In this case, the following characterizations of a P-point sequence hold true as in the classical one dimensional case.

7.1. THEOREM. *Let $\{p_n\}$ be a sequence of points in Ω such that $p_n \to p \in \partial\Omega$. Let $f \in \mathcal{H}(\Omega, \Sigma)$. The following statements are equivalent.*

(a) *There exists a sequence $\{q_n\}$ in Ω and a positive ϵ such that $k_\Omega(p_n, q_n) \to 0$ but*

$$\limsup_{n\to\infty} \chi(f(p_n), f(q_n)) \geqq \epsilon.$$

(b) *For each $\delta > 0$ and subsequence $\{p_\nu\}$ of $\{p_n\}$, f assumes every value of Σ, with at most two exceptions, infinitely often in the union*

$$\bigcup_{\nu=1}^{\infty} B_k(p_\nu, \delta).$$

(c) *For every value $w \in \Sigma$ there exists a sequence of points $\{q_n\}$ with $k_\Omega(p_n, q_n) \to 0$ such that $\chi(f(q_n), w) \to 0$.*

Proof. (a) $\Rightarrow$ (b). Let $\{p_n\}$ and $\{q_n\}$ be two sequences of Ω such that

$$k_\Omega(p_n, q_n) \to 0 \quad \text{and}$$

$$\chi(f(p_n), f(q_n)) \geqq \epsilon \quad \text{for some } \epsilon > 0 \text{ and all } n.$$

Let p_0 be a fixed point in Ω. Since $\mathrm{Aut}(\Omega)$ is transitive, for each n there exists $\varphi_n \in \mathrm{Aut}(\Omega)$ such that $\varphi_n(p_0) = p_n$. Let $w_n = \varphi_n^{-1}(q_n)$. Then

$$\lim_{n\to\infty} k_\Omega(p_0, w_n) = \lim_{n\to\infty} k_\Omega(p_n, q_n) = 0.$$

Therefore, for any $\delta > 0$ there exists an $n_0 > 0$ such that for $n \geqq n_0$,

$$k_\Omega(p_0, w_n) < \delta \quad \text{or} \quad w_n \in B_k(p_0, \delta) \quad \text{and}$$

$$\chi(g_n(p_0), g_n(w_n)) \geqq \epsilon, \quad g_n = f \circ \varphi_n.$$

Therefore, no subsequence of $\{g_n\}$ can ever converge to a function meromorphic at p_0. Consequently, $\{g_n\}$ is not a normal family in $B_k(p_0, \delta)$ for any $\delta > 0$. Since Montel's theorem can still be applied to this case, we can conclude that for any $\delta > 0$ $\{g_n\}$ must assume each value of Σ infinitely often with at most two exceptions in $B_k(p_0, \delta)$. Namely, for any $\delta > 0$ f assumes each value of Σ with at most two exceptions infinitely often on the union

$$\bigcup_{n=1}^{\infty} B_k(p_n, \delta).$$

The same argument holds for any subsequence of $\{p_n\}$ which proves (b).

(b) $\Rightarrow$ (c). Let $\{p_n\}$ satisfy the hypothesis of (b). It is to show that for each $w \in \Sigma$ there exists a sequence of points $\{q_n\}$ with $k_\Omega(p_n, q_n) \to 0$ such that $f(q_n) \to w$. For each positive integer k there exists an integer n_k such that for each $n > n_k$ there is a point q_n such that

$$k_\Omega(p_n, q_n) < 1/k \quad \text{and} \quad \chi(f(q_n), w) < 1/k.$$

Let $q_n = p_n$ for $n \leqq n_1$ and, for $n_k < n \leqq n_{k+1}$, let q_n be chosen as

$$k_\Omega(p_n, q_n) < 1/k \quad \text{and} \quad \chi(f(q_n), w) < 1/k.$$

Then the sequence $\{q_n\}$ is as desired.

(c) $\Rightarrow$ (a). Assume the contrary to the claim. Then for every sequence $\{q_n\}$ with $k_\Omega(p_n, q_n) \to 0$ we must have

$$\chi(f(p_n), f(q_n)) \to 0.$$

This means that f assumes a unique value of Σ for each subsequence of $\{q_n\}$ with

$$k_\Omega(p_n, q_n) \to 0.$$

This is a contradiction.

7.2. THEOREM. *Let Ω be a homogeneous bounded domain in $\mathbf{C}^m$ and let $\{p_n\}$ be a sequence of points in Ω. Suppose that there exists a sequence of positive numbers $\{\epsilon_n\}$ with $\epsilon_n \to 0$ such that*

$$\lim_{n\to\infty} \sup_{z \in B_k(p_n,\epsilon_n)} Q_f(z) = \infty$$

for some $f \in \mathscr{H}(\Omega, \Sigma)$. Then $\{p_n\}$ is a P-point sequence for f.

Proof. Since $\mathrm{Aut}(\Omega)$ is transitive, for each n there exists $\varphi_n \in \mathrm{Aut}(\Omega)$ such that $\varphi_n(p_0) = p_n$ for some fixed $p_0 \in \Omega$. Set $g_n = f \circ \varphi_n$. Let $q_n \in B_k(p_n, \epsilon_n)$ be a sequence in Ω such that $Q_f(q_n) \to \infty$. Let $w_n = \varphi_n^{-1}(q_n)$. Then by (3) of Section 2

$$Q_{g_n}(w_n) = Q_f(q_n)$$

so that $Q_{g_n}(w_n) \to \infty$ as $n \to \infty$. Also, $k_\Omega(p_0, w_n) \to 0$ as $n \to \infty$, since

$$k_\Omega(p_0, w_n) = k_\Omega(\varphi_n(p_0), \varphi_n(w_n)) = k_\Omega(p_n, q_n) < \epsilon_n.$$

From this and Lemma 2.7, it follows that no subsequence of $\{g_n\}$ is normal on $B_k(p_0, \delta)$ for any $\delta > 0$. This in turn implies that f assumes every value of Σ with at most two exceptions infinitely often in the union

$$\bigcup_{n=1}^{\infty} B_k(p_n, \delta)$$

which is (b) of Theorem 7.1. Therefore, $\{p_n\}$ is a P-point sequence for f.

7.3. COROLLARY. *Let* $\{p_n\}$ *be a sequence in* Ω *such that* $Q_f(p_n) \to \infty$ *for some* $f \in \mathscr{H}(\Omega, \Sigma)$. *Then* $\{p_n\}$ *is a P-point sequence for* f.

The converse of Corollary 7.3 is not true in general as we see in Example 9.3.

7.4. COROLLARY. *Let* $\{p_n\}$ *be a P-point sequence for* $f \in \mathscr{H}(\Omega, \Sigma)$. *Then every sequence* $\{q_n\}$ *in* Ω *satisfying*

$$\lim_{n \to \infty} k_\Omega(p_n, q_n) = 0$$

is also a P-point sequence for f.

Proof. Since $\{p_n\}$ is a P-point sequence for f, there exists a sequence $\{r_n\}$ in Ω and a number $\delta > 0$ such that

$$k_\Omega(p_n, r_n) \to 0$$

but

$$\chi(f(p_n), f(r_n)) \geqq \delta \quad \text{for all } n.$$

If there exists an $\epsilon > 0$ such that

$$\chi(f(p_n), f(q_n)) \geqq \epsilon \quad \text{for all } n,$$

we are done. Therefore, we may assume that there is a subsequence so that

$$\chi(f(p_\nu), f(q_\nu)) \to 0 \quad \text{as } \nu \to \infty,$$

i.e.,

$$\chi(f(p_\nu), f(q_\nu)) \leqq \delta/2 \quad \text{for sufficiently large } \nu.$$

The triangle inequality now implies that

$$\chi(f(q_\nu), f(r_\nu)) \geqq \delta/2 \quad \text{for sufficiently large } \nu.$$

Since $\{q_\nu\}$ is a subsequence of $\{q_n\}$,

$$\delta/2 \leqq \limsup_{\nu \to \infty} \chi(f(q_\nu), f(r_\nu)) \leqq \limsup_{n \to \infty} \chi(f(q_n), f(r_n)),$$

and, hence, $\{q_n\}$ is a P-point sequence.

8. Asymptotic behavior of Bloch mappings. Let Ω be a bounded domain in $\mathbf{C}^m$ and let $\zeta \in \partial\Omega$. Denote by $T_\zeta(\partial\Omega)$ the $(2n - 1)$ dimensional real tangent space to $\partial\Omega$ at ζ. The unit outward normal ν_ζ at ζ is a unit vector which is orthogonal to $T_\zeta(\partial\Omega)$ such that $t\nu_\zeta \notin \Omega$ for all sufficiently small $t > 0$. By $\mathbf{C}\nu_\zeta$ we denote the complex line generated by ν_ζ. Then the complex tangent space at ζ is defined as the $(n - 1)$ dimensional complex subspace of $T_\zeta(\partial\Omega)$ and given by

$$\mathbf{C}T_\zeta(\partial\Omega) = \{z \in \mathbf{C}^m:(z, w) = 0 \forall w \in \mathbf{C}\nu_\zeta\}.$$

Clearly,

$$\mathbf{C}\nu_\zeta \perp \mathbf{C}T_\zeta \quad \text{and} \quad \mathbf{C}^m = \mathbf{C}\nu_\zeta \oplus_{\mathbf{C}} \mathbf{C}T_\zeta.$$

A set S in Ω is said to be *asymptotic* at $\zeta \in \partial\Omega$ if

$$\bar{S} \cap \partial\Omega = \{\zeta\}$$

and *non-tangentially asymptotic* at ζ if

$$S \subset \Gamma_\alpha(\zeta) \quad \text{for some } \alpha > 1,$$

where

$$(1) \qquad \Gamma_\alpha(\zeta) = \{z \in \Omega: |z - \zeta| < \alpha\delta_\zeta(z)\},$$

$$\delta_\zeta(z) = \min(\rho(z, \partial\Omega), \rho(z, T_\zeta(\partial\Omega))),$$

and ρ denotes the euclidean distance in $\mathbf{C}^m$. In particular, a curve $\gamma: [0, 1) \to \Omega$ is non-tangentially asymptotic at ζ if

$$\gamma(t) \in \Gamma_\alpha(\zeta) \quad \text{for some } \alpha > 1 \text{ and all } t \in [0, 1) \quad \text{and}$$

$$\lim_{t \to 1-} \gamma(t) = \zeta.$$

We shall say that a mapping $f:\Omega \to N$ has an *asymptotic limit* l at $\zeta \in \partial\Omega$ along the curve γ in Ω if γ is asymptotic at ζ and

$$\lim_{t \to 1-} d_N(f(\gamma(t)), l) = 0,$$

a *radial limit* l at ζ if

$$\lim_{\epsilon \to 0+} d_N(f(\zeta - \epsilon\nu_\zeta), l) = 0,$$

a *non-tangential limit* l at ζ if

$$\lim_{\Gamma_\alpha(\zeta) \ni z \to \zeta} d_N(f(z), l) = 0 \quad \text{for every } \alpha > 1,$$

and an *admissible limit* l at ζ if

$$\lim_{\mathscr{A}_\alpha(\zeta) \ni z \to \zeta} d_N(f(z), l) = 0$$

for every $\alpha > 0$, where

$$(2) \qquad \mathscr{A}_\alpha(\zeta) = \{z \in \Omega: |(z - \zeta, \nu_\zeta)| < (1 + \alpha)\delta_\zeta(z), \; |z - \zeta|^2 < \alpha\delta_\zeta(z)\}.$$

8.1. PROPOSITION. *Let Ω be a bounded domain and N any complex manifold. Let S_1 and S_2 be asymptotic sets at $\zeta \in \partial\Omega$ such that*

$$(3) \qquad \lim_{S_2 \ni z \to \zeta} \frac{\rho(z, S_1)}{\rho(z, \partial\Omega)} = 0.$$

If $f \in \mathscr{B}(\Omega, N)$ satisfies

$$\lim_{S_1 \ni z \to \zeta} d_N(f(z), l) = 0$$

then

$$\lim_{S_2 \ni z \to \zeta} d_N(f(z), l) = 0.$$

Proof. Let $\{z_n^{(2)}\}$ be any sequence along S_2 with $z_n^{(2)} \to \zeta$. Choose a sequence $\{z_n^{(1)}\}$ in S_1 so that

$$|z_n^{(2)} - z_n^{(1)}| \leqq 2\rho(z_n^{(2)}, S_1).$$

Join $z_n^{(1)}$ and $z_n^{(2)}$ by the complex line q_n. Then $q_n \cap \Omega$ contains the disc Δ_n of radius

$$r_n > \rho(z_n^{(2)}, \partial\Omega) \quad \text{about } z_n^{(2)}.$$

If $f \in \mathscr{B}(\Omega, N)$, there exists a number $Q > 0$ such that

$$(4) \qquad d_N(f(z_n^{(1)}), f(z_n^{(2)})) \leqq Qk_\Omega(z_n^{(1)}, z_n^{(2)}).$$

By the distance-decreasing property of Kobayashi metric,

$$\begin{aligned}(5) \qquad k_\Omega(z_n^{(1)}, z_n^{(2)}) &\leqq k_{\Delta_n}(z_n^{(1)}, z_n^{(2)}) \\ &= \tan h^{-1} \frac{|z_n^{(1)} - z_n^{(2)}|}{r_n} \\ &\leqq \tan h^{-1} \frac{2\rho(z_n^{(2)}, S_1)}{\rho(z_n^{(2)}, \partial\Omega)}.\end{aligned}$$

Inequalities (4) and (5) together with (3) yields that

$$\lim_{n \to \infty} f(z_n^{(1)}) = \lim_{n \to \infty} f(z_n^{(2)}) = l$$

in d_N-metric.

Proposition 8.1 and the next proposition show some improvements over Propositions 1 and 3 of [**5**] since $\mathscr{B}(\Omega, N)$ contains all bounded holomorphic mappings properly in many cases (see Proposition 3.3).

8.2. PROPOSITION. *Let Ω be a bounded domain in $\mathbf{C}^m$, ζ a boundary point of Ω at which the outward normal exists, and S an arbitrary asymptotic continuum at ζ such that*

$$(7) \qquad \lim_{S \ni z \to \zeta} \frac{\rho(z, \mathbf{C}\nu_\zeta)}{r(\nu(z))} = 0$$

where $r(\nu(z))$ *denotes the radius of the largest ball in* $\Omega \cap \mathbf{C}T_{\nu(z)}$, $\mathbf{C}T_{\nu(z)}$ *is the hyperplane through* $\nu(z)$, *the orthogonal projection of* z *to* $\mathbf{C}\nu_\zeta$, *and parallel to* $\mathbf{C}T_\zeta(\partial\Omega)$. *If* $f \in \mathscr{B}(\Omega, \Sigma)$ *and*

$$\lim_{S \ni z \to \zeta} \chi(f(z), l) = 0,$$

then

$$\lim_{\Gamma_\alpha(\zeta) \ni z \to \zeta} \chi(f(z), l) = 0 \quad \textit{for all } \alpha > 1.$$

Proof. We follow Cirka's technique [**5**] closely. By the definition of $r(\nu(z))$, $\Omega \cap \mathbf{C}T_{\nu(z)}$ contains the ball

$$B(\nu(z), r(\nu(z)))|_{\mathbf{C}T_{\nu(z)}}.$$

The distance-increasing property of the Kobayashi metric implies

$$(8) \qquad k_\Omega(z, \nu(z)) \leqq \tan h^{-1} \frac{|z - \nu(z)|}{r(\nu(z))}.$$

Since $f \in \mathscr{B}(\Omega, \Sigma)$,

$$\chi(f(z), f(\nu(z))) \leqq Qk_\Omega(z, \nu(z)) \quad \text{for some constant } Q > 0.$$

Let $\nu(S)$ denote the orthogonal projection of S onto $\mathbf{C}\nu_\zeta$. Thus, if $\eta \to \zeta$ along $\nu(S)$, then $f(\eta) \to l$. Clearly, the restriction $f|_{\Omega \cap \mathbf{C}\nu_\zeta}$ is also in $\mathscr{B}(\Omega \cap \mathbf{C}\nu_\zeta, \Sigma)$ and, hence, f is a normal meromorphic function in $\Omega \cap \mathbf{C}\nu_\zeta$. Therefore, it follows from a classical result of Lehto and Virtanen [**18**] that

$$\lim_{\tilde{\Gamma}_\alpha(\zeta) \ni \eta \to \zeta} \chi(f(\eta), l) = 0 \quad \text{for all } \alpha > 1,$$

where

$$(9) \qquad \tilde{\Gamma}_\alpha(\zeta) = \{\eta \in \Omega \cap \mathbf{C}\nu_\zeta : |\eta - \zeta| < \alpha\rho(\eta, T_\zeta)\}.$$

Let U be a sufficiently small neighborhood of ζ. The orthogonal projection of $\Gamma_\alpha(\zeta) \cap U$ onto $\mathbf{C}\nu_\zeta$ is then contained in some $\tilde{\Gamma}_\beta(\zeta)$ for $\beta \geqq \alpha$. Moreover, if $z \in \Gamma_\alpha(\zeta) \cap U$, then

$$(10) \qquad |z - \zeta| < \alpha\delta_\zeta(z) = \alpha|z - \tau(z)| + o(|\tau(z) - \zeta|),$$

where $\tau(z)$ is the orthogonal projection of z onto $T_\zeta(\partial\Omega)$. Since

$$|z - \zeta|^2 = |z - \tau(z)|^2 + |\tau(z) - \zeta|^2,$$

$$(11) \qquad |\tau(z) - \zeta| < \sqrt{\alpha^2 - 1}\, |z - \tau(z)| + o(|\tau(z) - \zeta|)$$

so that if U is sufficiently small, then

(12) $|\tau(z) - \zeta| < \alpha|z - \tau(z)|$.

On the section $\Gamma_\alpha(\zeta) \cap U \cap \mathbf{C}T_\eta$, $\eta \in \mathbf{C}\nu_\zeta$,

$$|z - \tau(z)| = |\nu(z) - \nu(\tau(z))| \leqq |\nu(z) - \zeta| \quad \text{and}$$

$$|z - \nu(z)| \leqq |\tau(z) - \zeta|$$

so that (12) yields:

(13) $|z - \eta| < \alpha|\eta - \zeta|$.

Namely, $\Gamma_\alpha(\zeta) \cap U \cap \mathbf{C}T_\eta$ lies in the ball $B(\eta, \alpha\delta)$ if $\delta = |\eta - \zeta|$ is sufficiently small. By the definition of $r(\eta)$, the section $\Omega \cap \mathbf{C}T_\eta(\partial\Omega)$ contains the ball $B(\eta, r(\eta))|_{\mathbf{C}T_\eta}$. Using the distance-decreasing property of k_Ω and the fact that $f|_{B(\eta, r(\eta))}$ is Bloch,

$$\text{(14)} \quad \chi(f(z), f(\nu(z))) \leqq Q \tan h^{-1}\frac{|z - \nu(z)|}{r(\nu(z))}$$

$$\leqq Q \tan h^{-1}\frac{\alpha\delta}{r(\nu(z))}$$

for some $Q > 0$ if $z \in \Gamma_\alpha(\zeta) \cap \mathbf{C}T_\eta$ and $|\eta - \zeta| = \delta$ is sufficiently small. If $z \in \Gamma_\alpha(\zeta) \cap U$, then $\nu(z) \in \widetilde{\Gamma}_\beta(\zeta)$ for some $\beta \geqq \alpha$. Therefore, it follows that

$$r(\nu(z)) \geqq r_\beta(\delta) = \inf\{r(\eta):\eta \in \widetilde{\Gamma}_\beta(\zeta), |\eta - \zeta| = \delta\}.$$

But by Lemma 1 of [**5**],

$$\lim_{\delta\to 0} \frac{r_\beta(\delta)}{\delta} = \infty$$

and, hence,

$$\lim_{\delta\to 0} \frac{\delta}{r(\nu(z))} = 0.$$

This, together with (14), implies that

$$\lim_{\Gamma_\alpha(\zeta)\ni z\to\zeta} \chi(f(z), f(\nu(z))) = 0.$$

The triangle inequality now yields:

$$\lim_{\Gamma_\alpha(\zeta)\ni z\to\zeta} f(z) = l \text{ in } \chi\text{-metric.}$$

Let Ω be a bounded domain in $\mathbf{C}^m$ with C^2-boundary. Then each boundary point $\zeta \in \partial\Omega$ has the outward unit normal ν_ζ and an $r_\zeta > 0$ such that $B(\zeta - r_\zeta\nu_\zeta, r_\zeta)$ is contained in Ω and tangent to $\partial\Omega$ at ζ from inside. The order of tangency in this case is no worse than along the set $\mathscr{A}_\alpha$ defined in (2). Therefore, we obtain the following version of Proposition 8.2 in this case.

8.3. PROPOSITION. *Let Ω be a bounded domain with C^2 boundary and S an arbitrary asymptotic continuum at $\zeta \in \partial\Omega$ such that*

$$(15)\qquad \lim_{S \ni z \to \zeta} \frac{\rho^2(z, \mathbf{C}\nu_\zeta)}{\rho(z, \mathbf{C}T_\zeta)} = 0.$$

If $f \in \mathscr{B}(\Omega, \Sigma)$ and

$$\lim_{S \ni z \to \zeta} \chi(f(z), l) = 0 \quad \textit{for some } l \in \Sigma,$$

then

$$\lim_{\Gamma_\alpha(\zeta) \ni z \to \zeta} \chi(f(z), l) = 0 \quad \textit{for all } \alpha > 1.$$

Proof. This can be carried out in the same line of argument as in Proposition 8.2, or that of [**4**, Lemma 3.1].

9. Examples. Let B be the open unit ball in $\mathbf{C}^m$ and Σ the Riemann sphere. It follows from Lemma 2.6 and inequalities (3.3) that if $f \in \mathscr{H}(B, \Sigma)$,

$$(1)\qquad \frac{(1 - |z|^2)}{1 + |f(z)|^2} \inf_{|\xi|=1} |df(z)\xi| \leqq Q_f(z) \leqq \frac{(1 - |z|^2)^{1/2}}{1 + |f(z)|^2} \sup_{|\xi|=1} |df(z)\xi|$$

for $z \in B$.

9.1. *Example.* Let $f \in \mathscr{H}(B, \Sigma)$ be defined by

$$(2)\qquad z \to \frac{1 + (z, b)}{1 - (z, b)}, \quad b \in \partial B.$$

A straightforward calculation shows that

$$(3)\qquad |df(z)\xi| = \frac{2|(\xi, b)|}{|1 - (z, b)|^2},$$

and

$$(4)\qquad Q_f(z) \leqq \frac{(1 - |z|^2)^{1/2}}{1 + |(z, b)|^2},$$

so that $f \in \mathscr{B}_0(B, \Sigma)$. By Theorem 5.4, every sequence of points in B are therefore regular for f.

9.2. *Example.* Let $g \in \mathscr{H}(B, \Sigma)$ be defined by

$$(5)\qquad z \to \exp(-f(z)),$$

where f, as defined in (2), maps B onto $\operatorname{Re} f > 0$.

Then g is a bounded function and bounded by 1 on B. By Proposition 3.3, $g \in \mathscr{B}(B, \Sigma)$ so that g can not have a P-point sequence in B by Theorem 5.1. However, g has many irregular sequences in B. To see this, we carry out a similar calculation as in Example 9.1 to obtain

$$(6) \qquad \frac{|dg(z)\xi|}{1 + |g(z)|^2} \geqq \frac{|(\xi, b)|}{|1 - (z, b)|^2} e^{-\operatorname{Re} f(z)} \quad (b \in \partial B)$$

and

$$(7) \qquad Q_g(z) \geqq \frac{(1 - |z|^2)}{|1 - (z, b)|^2} e^{-\operatorname{Re} f(z)}.$$

By setting $z = bt$, $t \in \mathbf{C}$, we have

$$(8) \qquad Q_g(bt) \geqq \frac{1 - |t|^2}{|1 - t|^2} \exp\left(\frac{|t|^2 - 1}{|t - 1|^2}\right).$$

Let $t = \cos \theta e^{i\theta}$. Then $|t - 1| = \sin \theta$, $|t| = \cos \theta$ and

$$Q_g(bt) \geqq e^{-1} > 0.$$

Therefore, along the curve $\gamma : t = \cos \theta e^{i\theta} b$ which tends to b as $\theta \to 0$, Q_g is bounded below by a positive number $1/e$. Thus, no sequence of points along γ is regular. Therefore, not every irregular sequence is a P-point sequence, although every P-point sequence is irregular. See the remark given after Proposition 4.3.

9.3. *Example*. Let $h \in \mathscr{H}(B, \Sigma)$ be defined by

$$(9) \qquad z \to \exp\left[\frac{i}{(z, b) - 1}\right] (b \in \partial B).$$

After some calculation, we find that Q_h satisfies the inequality:

$$(10) \quad Q_h(br) \geqq \frac{1 + r}{2(1 - r)}$$

along $z = br$ with $r \in (0, 1)$. By Corollary 7.3, any sequence in B along the line $z = br$ as $r \to 1$ is a P-point sequence for h. Therefore, by Theorem 5.1 h is not normal. Let

$$p_n = b\left(1 - \frac{1}{2n\pi}\right).$$

Then $\{p_n\}$ is a P-point sequence for h, but $h(p_n) = 1$ for all n. This shows that the converse of Corollary 7.3 does not hold. Let

$$q_n = b\left(1 - \frac{1}{2n\pi} + \frac{i}{n\sqrt{n}}\right) \quad \text{and}$$

$$r_n = b\left(1 - \frac{1}{2n\pi} - \frac{i}{n\sqrt{n}}\right).$$

It is easy to see, by using inequality (6.2), that $k_B(p_n, q_n)$ and $k_B(p_n, r_n)$ tend to 0 as $n \to \infty$. Since $\{p_n\}$ is a P-point sequence, by Corollary 7.4, $\{q_n\}$ and $\{r_n\}$ are both P-point sequences. But $h(q_n) \to \infty$, while $h(r_n) \to 0$ as $n \to \infty$. These facts also show that Proposition 4.4 is no longer true for non-normal mappings.

More generally, a similar computation yields that $Q_h(z) \to 0$ along the curve

$$\gamma: z = b \cos \theta e^{i\theta} \quad (0 \leqq \theta \leqq \pi/2)$$

as $\theta \to 0$, while $Q_h(z) \to \infty$ along the radial direction $z = rb$ $(r > 0)$ as $r \to 0$. This example seems to describe the typical behavior of a non-normal holomorphic mapping near a P-point sequence, as is already well-documented in Theorem 7.1.

9.4. *Example*. Let h be the non-normal function defined in Example 9.3. Let

$$p_n = b\left(1 - \frac{1}{2n\pi}\right)$$

as before. Then

$$(1 - |p_n|)\Lambda_h(p_n) \to \infty \quad \text{as } n \to \infty.$$

Let $r_n = 1/n^2$. Then

$$\lim_{n\to\infty} \frac{r_n}{\delta_B(p_n)} = 0.$$

Therefore, the hypotheses of Theorem 6.3 are satisfied with Condition (6.1) so that we have

$$(11) \quad h(p_n + r_n\zeta) = \exp\left\{\frac{-i}{1 - (p_n + r_n\zeta, b)}\right\}$$
$$= \exp\left\{-2n\pi i\left(1 - \frac{2\pi}{n}(\zeta, b)\right)^{-1}\right\}.$$

Since $\left|\frac{2\pi}{n}(\zeta, b)\right| < 1$ holds on any compact subset of $\mathbf{C}^m$ for n sufficiently large, the right hand side of (11) can be expanded into a series which converges uniformly on compact subsets of $\mathbf{C}^m$ to a nonconstant entire function $\exp\{-4\pi^2(\zeta, b)i\}$. This is the content of Theorem 6.3. On the other hand, let

$$p_n = b(1 - 1/n - i/n) \quad \text{and} \quad r_n = 1/n^2.$$

The same calculation shows that $r_n/\delta_B(p_n) \to 0$, but

$$(1 - |p_n|)\Lambda_h(p_n) \to 0 \quad \text{as } n \to \infty.$$

In this case,

$$(12) \quad \exp \frac{-i}{1 - (p_n + r_n\zeta, b)}$$

converges to a constant map: $g = 0 \in \Sigma$.

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