COMMUTATORS OF OPERATORS ON HILBERT SPACE

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1. Introduction. The purpose of this paper is to record some progress on the problem of determining which (bounded, linear) operators A on a separable Hilbert space $\mathbf{H}$ are commutators, in the sense that there exist bounded operators B and C on $\mathbf{H}$ satisfying $A = BC - CB$. It is thus natural to consider this paper as a continuation of the sequence (**2**; **3**; **5**). In §2 we show that many infinite diagonal matrices (with scalar entries) are commutators and that every weighted unilateral and bilateral shift is a commutator. In §3 we introduce some constructions involving matrices with operator entries to prove that every operator represented by a matrix $(A_{ij})_{i,\,j=0}^{\infty}$ with operator entries satisfying $\sum_{i,j} \|A_{ij}\| < \infty$ is a commutator. Using this result we then prove that every normal operator whose spectrum contains 0 as a limit point is a commutator. The major part of §4 is devoted to proving that every compact operator is a commutator. In the concluding section of the paper we mention several open questions concerning commutators.

2. Diagonal matrices and shifts. We begin with some preparatory definitions and notation. The algebra of all bounded operators on $\mathbf{H}$ is denoted by $\mathbf{L}(\mathbf{H})$. A sequence $\{\alpha_0, \alpha_1, \ldots\}$ of complex numbers is said to be *addable* if there is some permutation π of the set of non-negative integers such that the sequence $\{\alpha_{\pi(n)}\}$ has bounded partial sums. (It is easy to see that every addable sequence is bounded.) We now choose an orthonormal basis $\{x_0, x_1, \ldots\}$ for $\mathbf{H}$ which is to be held fixed until further notice. Any bounded sequence $\alpha = \{\alpha_0, \alpha_1, \ldots\}$ together with the basis $\{x_n\}$ determines three operators D_α, S_α, and T_α on $\mathbf{H}$ defined as follows:

$$
\begin{aligned}
&D_\alpha x_n = \alpha_n x_n, && n = 0, 1, 2, \ldots;\\
&S_\alpha x_n = \alpha_n x_{n+1}, && n = 0, 1, 2, \ldots;\\
&\begin{cases} T_\alpha x_0 = 0, & \\ T_\alpha x_{n+1} = \alpha_n x_n, & n = 0, 1, 2, \ldots . \end{cases}
\end{aligned}
$$

For convenience, we denote simply by S the particular operator S_α corresponding to the sequence $\{\alpha_n \equiv 1\}$.

LEMMA 2.1. *If $\alpha = \{\alpha_0, \alpha_1, \ldots\}$ is a sequence of complex numbers such that the sequence of partial sums*

$$s_n = \sum_{i=0}^{n} \alpha_i$$

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is bounded, then the operator D_α *is the commutator* $D_\alpha = T_\gamma S - ST_\gamma$, *where* $\gamma = \{s_0, s_1, \ldots\}$.

The proof of this lemma consists of noting that T_γ is a bounded operator and making the appropriate calculation. We omit the computation but observe for future use that not only is T_γ bounded, but also $||T_\gamma|| = \sup_n|s_n|$.

Before stating the next lemma, we need another definition, and for the purposes of this definition it is convenient to regard a sequence of complex numbers $\{\alpha_0, \alpha_1, \ldots\}$ as a complex-valued function α having as domain the set of non-negative integers. A subsequence $\{\alpha_{\beta(n)}\}$ is then the composite function $\alpha \circ \beta$ where β is a strictly increasing function mapping the set of non-negative integers into itself. By a *partition* **P** of $\{\alpha_n\}$ we mean a countable collection of subsequences $\{\alpha_{\beta_m(n)}\}$ of $\{\alpha_n\}$ such that the ranges of the β_m's form a partition of the set of all non-negative integers.

LEMMA 2.2. *If* $\{\alpha_0, \alpha_1, \ldots\}$ *is a sequence of complex numbers with bounded partial sums, then there exists a partition* **P** *of* $\{\alpha_n\}$ *and a positive number* M *such that every partial sum of every subsequence* $\{\alpha_{\beta_m(n)}\} \in$ **P** *is bounded in modulus by* M.

Proof. Consider the sequence

$$s_n = \sum_{i=0}^{n} \alpha_i$$

of partial sums of α_n. It follows from the hypothesis that all the numbers s_n are contained in some disk $\{z: |z| < r < \infty\}$ in the complex plane, and hence that the sequence $\{s_n\}$ has a point of accumulation z_0. Let $\{\epsilon_n\}$ be a sequence of positive numbers such that

$$\sum_{n=0}^{\infty} \epsilon_n < r.$$

We choose a sub-sequence $\{s_{n(k)}\}$ of $\{s_n\}$ by induction as follows. Let $s_{n(0)}$ be the first term of the sequence $\{s_n\}$ such that $|s_{n(0)} - z_0| < \epsilon_0$, and suppose that the first k terms of the subsequence $\{s_{n(k)}\}$ have been defined. We then define $s_{n(k)}$ to be the first term of the sequence $\{s_n\}$ such that $n(k) > n(k-1)$ and $|s_{n(k)} - z_0| < \epsilon_k$. We now use the sequence $\{s_{n(0)}, s_{n(1)}, \ldots\}$ to split $\{\alpha_n\}$ into countably many blocks—the 0th block consisting of the terms $\{\alpha_0, \ldots, \alpha_{n(0)}\}$, and, in general, the kth block consisting of the terms $\{\alpha_{n(k-1)+1}, \ldots, \alpha_{n(k)}\}$.

To obtain a partition with the desired properties we now simply piece these blocks together end to end so as to form infinite subsequences, taking care that each $\{\alpha_{\beta_m(n)}\} \in$ **P** contains infinitely many non-consecutive blocks. One possible way of accomplishing this is the "diagonal" procedure indicated in the following scheme:

$$\begin{aligned}
\alpha_{\beta_0} &= \{\alpha_0, \ldots, \alpha_{n(0)}; \alpha_{n(1)+1}, \ldots, \alpha_{n(2)}; \alpha_{n(4)+1}, \ldots, \alpha_{n(5)}; \ldots\}, \\
\alpha_{\beta_1} &= \{\alpha_{n(0)+1}, \ldots, \alpha_{n(1)}; \alpha_{n(3)+1}, \ldots, \alpha_{n(4)}; \ldots\}, \\
\alpha_{\beta_2} &= \{\alpha_{n(2)+1}, \ldots, \alpha_{n(3)}; \ldots\},
\end{aligned}$$

. . . .

To see that every subsequence $\{\alpha_{\beta_m(n)}\}$ in $\mathbf{P}$ has all its partial sums bounded in modulus by $M = 4r$, note that for any $t \geqslant 0$ we have

$$\begin{aligned}|\alpha_{n(t)+1} + \alpha_{n(t)+2} + \ldots + \alpha_{n(t+1)}| &= |s_{n(t+1)} - s_{n(t)}| = |(s_{n(t+1)} - z_0) + (z_0 - s_{n(t)})| \\ &< \epsilon_{n(t+1)} + \epsilon_{n(t)}.\end{aligned}$$

This fact, together with the observation that every partial sum of the sequence $\{\alpha_n\}$ is bounded in modulus by r, leads easily to the result.

THEOREM 1. *If $\alpha = \{\alpha_0, \alpha_1, \ldots\}$ is any bounded sequence of complex numbers containing an addable sub-sequence, then the operator D_α is a commutator.*

Proof. It suffices to prove the theorem in the case that the addable sub-sequence itself has bounded partial sums since, in any case, there is a permutation π of the set of non-negative integers such that the sequence $\pi(\alpha) = \{\alpha_{\pi(n)}\}$ contains a sub-sequence with bounded partial sums, and D_α is clearly unitarily equivalent to $D_{\pi(\alpha)}$. The pertinent fact to be established is that there is a countable collection of sequences $\alpha_{\beta_0}, \alpha_{\beta_1}, \ldots$ such that D_α is unitarily equivalent to the operator

$$\sum_{k=0}^{\infty} \oplus D_{\alpha\beta_k}$$

acting on the Hilbert space $\mathbf{K} = \mathbf{H} \oplus \mathbf{H} \oplus \ldots$, where each operator $D_{\alpha\beta_k}$ can be written as a commutator $D_{\alpha\beta_k} = T_{\gamma_k} S - S T_{\gamma_k}$ and where the operators T_{γ_k} are uniformly bounded in norm. Once this has been established, the proof is completed by noting that

$$\sum_{k=0}^{\infty} \oplus D_{\alpha\beta_k} = AB - BA,$$

where A and B are the operators

$$\sum_{k=0}^{\infty} \oplus T_{\gamma_k} \quad \text{and} \quad S \oplus S \oplus \ldots$$

on $\mathbf{K}$, respectively.

In order to verify the above-mentioned fact it suffices, in view of Lemma 2.1, to prove that there is a partition $\mathbf{R}$ of $\{\alpha_n\}$ and a positive number N such that every subsequence $\{\alpha_{\beta_m}\}$ in $\mathbf{R}$ has all its partial sums bounded by N. To see that there is such a partition, we first use Lemma 2.2 to obtain a partition $\mathbf{P}$ of the given addable subsequence of $\{\alpha_n\}$ with the property that every partial sum of every subsequence in $\mathbf{P}$ is bounded by a fixed positive number M. Now there are at most countably many terms of $\{\alpha_n\}$ that do not appear in the given addable subsequence, and therefore by inserting at most one such term at the beginning of each subsequence belonging to $\mathbf{P}$, we generate a partition $\mathbf{R}$ of the whole sequence $\{\alpha_n\}$. If P is a bound for the sequence $\{\alpha_n\}$, then the partition $\mathbf{R}$ has the desired properties with $N = M + P$. This completes the proof of the theorem.

We turn now to the consideration of the class of *weighted unilateral shifts* (with respect to the basis $\{x_n\}$). This class is by definition the collection of operators $\{S_\alpha\}$ where α runs over the collection of bounded sequences of complex numbers. The following lemma allows us to prove that every weighted unilateral shift is a commutator.

LEMMA 2.3. *If the matrix* $(\alpha_{ij})_{i,j=0}^{\infty}$ *of an operator* A *in* $\mathbf{L}(\mathbf{H})$ *with respect to the basis* $\{x_0, x_1, \ldots\}$ *is such that* $\alpha_{ij} = 0$ *whenever* $i + j$ *is an even integer, then* A *is a commutator.*

Proof. Let α be the sequence $\alpha = \{\alpha_0, \alpha_1, \ldots\}$ where $\alpha_n = (-1)^n$. An easy computation, which we omit, shows that $A = (\frac{1}{2}A)D_\alpha - D_\alpha(\frac{1}{2}A)$.

THEOREM 2. *Every weighted unilateral shift is a commutator.*

Proof. The matrix (α_{ij}) of any weighted unilateral shift clearly satisfies the hypothesis of the preceding lemma. (That the unweighted shifts, unilateral and bilateral, are commutators was first called to our attention by Donald Deckard.)

There is another interesting class of shifts on $\mathbf{H}$—namely, the *weighted bilateral shifts*. Just as in the case of unilateral shifts, the definition of a bilateral shift must be given in terms of a fixed basis for $\mathbf{H}$. Thus we now give up the basis $\{x_n\}$, and we consider instead an orthonormal basis $\{\ldots, y_{-1}, y_0, y_1, \ldots\}$ for $\mathbf{H}$. If $\alpha = \{\ldots, \alpha_{-1}, \alpha_0, \alpha_1, \ldots\}$ is any bounded, two-way infinite sequence of complex numbers, then α induces an operator B_α on $\mathbf{H}$ defined by

$$B_\alpha y_n = \alpha_n y_{n+1}, \qquad n = 0, \pm 1, \pm 2, \ldots,$$

and the collection of all such B_α is the class of weighted bilateral shifts (with respect to the basis $\{y_n\}$). It is quite easy to see that the analogue of Lemma 2.3 dealing with the basis $\{\ldots, y_{-1}, y_0, y_1, \ldots\}$ and two-way infinite matrices $(\alpha_{ij})_{i,j=-\infty}^{\infty}$ is valid; in fact, the proof of the analogous lemma is essentially the same as that of Lemma 2.3. Since the matrix (α_{ij}) of a bilateral shift also satisfies the condition $\alpha_{ij} = 0$ for $i + j$ an even integer, we obtain the following result.

THEOREM 3. *Every weighted bilateral shift is a commutator.*

3. Matrices with operator entries. In this section and the following one we employ various constructions involving matrices with operator entries in order to prove that certain classes of operators consist entirely of commutators. The underlying central idea of these constructions is as follows. Given an infinite-dimensional separable Hilbert space $\mathbf{H}$, one can construct many spatial isomorphisms ϕ of $\mathbf{H}$ onto a Hilbert space of the form

$$\mathbf{H}^{\sim} = \mathbf{K} \oplus \mathbf{K} \oplus \ldots$$

where $\mathbf{K}$ is also a Hilbert space. If A is an operator on $\mathbf{H}$, such an isomorphism

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ϕ carries A onto an operator on $\mathbf{H}^{\sim}$ which can be realized as an infinite matrix (A_{ij}) where the entries A_{ij} belong to $\mathbf{L}(\mathbf{K})$. The idea is to choose ϕ so that the operator having matrix (A_{ij}) can be shown to be a commutator on $\mathbf{H}^{\sim}$. Then, since A is unitarily equivalent to (A_{ij}) and the property of being a commutator is a unitary invariant, A is a commutator. This idea is made precise in the following lemma, whose proof we omit. For a thorough discussion of this circle of ideas see (**1**, chap. 1, §2).

LEMMA 3.1. *Suppose that* $\mathbf{H}$ *is an infinite-dimensional separable Hilbert space and that* $\{\mathbf{M}_0, \mathbf{M}_1, \ldots\}$ *is a sequence of mutually orthogonal, infinite-dimensional subspaces of* $\mathbf{H}$ *satisfying*

$$\sum_{n=0}^{\infty} \oplus \mathbf{M}_n = \mathbf{H}.$$

Suppose also that for each n, U_n *is a linear transformation from* $\mathbf{H}$ *to a fixed Hilbert space* $\mathbf{K}$ *that maps* $\mathbf{M}_n$ *isometrically onto* $\mathbf{K}$ *and annihilates* $\mathbf{H} \ominus \mathbf{M}_n$. *Then the* U_n*'s define an isomorphism of* $\mathbf{H}$ *onto the Hilbert space*

$$\mathbf{H}^{\sim} = \mathbf{K} \oplus \mathbf{K} \oplus \ldots,$$

and under this isomorphism every operator T *in* $\mathbf{L}(\mathbf{H})$ *is carried onto an operator* $\tilde{T}$ *in* $\mathbf{L}(\mathbf{H}^{\sim})$ *of the form* $\tilde{T} = (T_{ij})_{i,j=0}^{\infty}$, *where each* T_{ij} *belongs to* $\mathbf{L}(\mathbf{K})$ *and is given by the formula*

$$T_{ij} = U_i T U_j^*.$$

In the construction of commutators using infinite matrices the need arises of showing that the pertinent matrices represent *bounded* operators; the next lemma is useful in this connection.

LEMMA 3.2. *Suppose that* $(T_{ij})_{i,j=0}^{\infty}$ *is a matrix with operator entries,* $T_{ij} \in \mathbf{L}(\mathbf{K})$, *and suppose that* $(\alpha_{ij})_{i,j=0}^{\infty}$ *is a scalar matrix which, together with an orthonormal basis for* $\mathbf{H}$, *defines a bounded operator* A *on* $\mathbf{H}$. *If* $\|T_{ij}\| \leqslant \alpha_{ij}$ *for all* i *and* j, *then* (T_{ij}) *is the matrix of a bounded operator* T *on*

$$\mathbf{H}^{\sim} = \mathbf{K} \oplus \mathbf{K} \oplus \ldots,$$

and $\|T\| \leqslant \|A\|$.

Proof. An arbitrary vector $\tilde{x}$ in $\mathbf{H}^{\sim}$ of length one can be written as $\tilde{x} = (x_0, x_1, \ldots)$ where $x_i \in \mathbf{K}$ and

$$\sum_{i=0}^{\infty} \|x_i\|^2 = 1.$$

Hence

$$\begin{aligned}\|Tx\|^2 &= \left\| \sum_{i=0}^{\infty} T_{0i} x_i \right\|^2 + \left\| \sum_{i=0}^{\infty} T_{1i} x_i \right\|^2 + \ldots \\ &\leqslant \left[\sum_{i=0}^{\infty} \|T_{0i}\| \cdot \|x_i\| \right]^2 + \left[\sum_{i=0}^{\infty} \|T_{1i}\| \cdot \|x_i\| \right]^2 + \ldots\end{aligned}$$

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$$\leqslant \left[\sum_{i=0}^{\infty} \alpha_{0i} \|x_i\| \right]^2 + \left[\sum_{i=0}^{\infty} \alpha_{1i} \|x_i\| \right]^2 + \ldots$$
$$= \|Ay\|^2 \leqslant \|A\|^2,$$

where y denotes the unit vector $y = (\|x_0\|, \|x_1\|, \ldots)$ in $\mathbf{H}$. This shows that T is bounded and that $\|T\| \leqslant \|A\|$.

The usefulness of this result depends on having on hand a good supply of bounded scalar matrices, and in this connection we find it convenient to recall the notion of a *Toeplitz* matrix, i.e., a scalar matrix $(\alpha_{ij})_{i,j=0}^{\infty}$ satisfying $\alpha_{ij} = \alpha_{i+1,j+1}$ for all $i, j \geqslant 0$. A Toeplitz matrix $A = (\alpha_{ij})$ determines (and is determined by) the single two-way infinite sequence $\{\ldots, a_{-1}, a_0, a_1, \ldots\}$ obtained by setting

$$\alpha_{n0} = a_n, \qquad \alpha_{0n} = a_{-n},$$

for all $n \geqslant 0$. We suppose that the sequence $\{a_n\}$ belongs to (l^2). The sequence $\{a_n\}$ is then the sequence of Fourier coefficients of a unique function $\phi_A = \phi_A(z)$ on the unit circle $|z| = 1$ with respect to the orthonormal system $\{\ldots, e_{-1}, e_0, e_1, \ldots\}$ where $e_n(z) = z^n$. (Here and below the unit circle is understood to be equipped with normalized Lebesgue measure.)

The main result (for present purposes) concerning Toeplitz matrices is the well-known fact that

$$\|A\| = \operatorname*{ess.\,sup.}_{|z|=1} |\phi_A(z)|.$$

In particular, A is certainly bounded if

$$\sum_{n=-\infty}^{\infty} |a_n| < \infty .$$

Thus the following lemma is an immediate consequence of Lemma 3.2.

LEMMA 3.3. *Suppose that (T_{ij}) is a matrix with operator entries from $\mathbf{L}(\mathbf{K})$, and suppose that A is a Toeplitz matrix that entrywise dominates the matrix $(\|T_{ij}\|)$. If the sequence $\{\ldots, a_{-1}, a_0, a_1, \ldots\}$ associated with A satisfies*

$$\sum_{n=-\infty}^{\infty} |a_n| < \infty ,$$

then the matrix (T_{ij}) determines a bounded operator on $\mathbf{H}^{\sim} = \mathbf{K} \oplus \mathbf{K} \oplus \ldots$.

COROLLARY 3.4. *If (T_{ij}) is a matrix with operator entries from $\mathbf{L}(\mathbf{K})$ such that*

$$\sum_{i,j=0}^{\infty} \|T_{ij}\| < \infty ,$$

then (T_{ij}) determines a bounded operator on $\mathbf{H}^{\sim}$.

We are now ready to prove the following central result.

THEOREM 4. *Every operator* $T = (T_{ij})_{i,\,j=0}^{\infty}$ *on* $\mathbf{H}^{\sim} = \mathbf{K} \oplus \mathbf{K} \oplus \ldots$ *such that* $\sum_{i,j} ||T_{ij}|| < \infty$ *is a commutator.*

Proof. We remark first that it follows from Corollary 3.4 that T is automatically a bounded operator. Now a computation shows that formally T is the commutator $T = RW - WR$ where $R = (R_{ij})_{i,\,j=0}^{\infty}$ is defined by

$$R_{ij} = \delta_{i+1,j}1_{\mathbf{K}}$$

for all $i, j \geqslant 0$, and $W = (W_{ij})_{i,\,j=0}^{\infty}$ is given by

$$W_{0j} = 0, \qquad j = 0, 1, \ldots,$$

$$W_{ij} = \sum_{n=0}^{i-1} T_{i-n-1,\,j-n}, \qquad 1 \leqslant i \leqslant j,$$

$$W_{ij} = \sum_{n=0}^{j} T_{i-n-1,\,j-n}, \qquad i > j.$$

It is clear that the matrix R represents a bounded operator on $\mathbf{H}^{\sim}$, so that to complete the proof it suffices to show that W is also bounded. In order to do this we employ Lemma 3.3, and we begin by defining a sequence of positive numbers as follows:

$$a_n = \sum_{k=0}^{\infty} ||T_{k,k+n+1}||, \qquad n \geqslant 0,$$

$$a_{-n} = \sum_{k=0}^{\infty} ||T_{k+n-1,k}||, \qquad n \geqslant 1.$$

It follows from the hypothesis that

$$\sum_{n=-\infty}^{\infty} a_n < \infty,$$

and one verifies by inspection that the Toeplitz matrix associated with this sequence dominates $(||W_{ij}||)$ entrywise. An application of Lemma 3.3 now completes the argument.

Theorem 4 implies that many diagonal matrices with operator entries are commutators. Here is one example.

COROLLARY 3.5. *Suppose that* $T = (T_{ij})_{i,\,j=0}^{\infty}$ *is an operator on* $\mathbf{H}^{\sim}$ *such that* $T_{ij} = 0$ *for* $i \neq j$ *and such that*

$$\lim_{k\to\infty} ||T_{kk}|| = 0.$$

Then T *is a commutator.*

Proof. It is a direct consequence of Theorem 4 that any diagonal matrix (D_{ij}) whose entries satisfy

$$\sum_{i=0}^{\infty} ||D_{ii}|| < \infty$$

is a commutator. Thus to prove the corollary it suffices to obtain an isomorphism of $\mathbf{H}^{\sim} = \mathbf{K} \oplus \mathbf{K} \oplus \ldots$ onto another Hilbert space

$$\mathbf{N} = \mathbf{M} \oplus \mathbf{M} \oplus \ldots$$

so that the operator on $\mathbf{N}$ corresponding to T under the isomorphism is a diagonal matrix (D_{ij}) satisfying

$$\sum_{i=0}^{\infty} ||D_{ii}|| < \infty.$$

Now we may as well assume that the diagonal entries T_{kk} of T satisfy $||T_{kk}|| \geqslant ||T_{k+1,k+1}||$ for all non-negative integers k, because in any case T is unitarily equivalent to such an operator. Next choose a subsequence $\{T_{k_n k_n}\}$ of diagonal entries of (T_{ij}) satisfying $\sum_n ||T_{k_n, k_n}|| < \infty$. The desired isomorphism is obtained by identifying the direct sum of the first k_1 copies of $\mathbf{K}$ with the first copy of $\mathbf{M}$, the direct sum of the next $k_2 - k_1$ copies of $\mathbf{K}$ with the second copy of $\mathbf{M}$, and continuing in this fashion.

THEOREM 5. *Any normal operator T on $\mathbf{H}$ having zero as a limit point of its spectrum is a commutator.*

Proof. We recall that, according to common usage, a number α is a limit point of the spectrum of a normal operator T if either α is a point of accumulation of the spectrum in the topological sense or α is a proper value of T with infinite multiplicity. In either case it is an easy exercise in spectral theory to construct a sequence $\mathbf{M}_n$ of mutually orthogonal, infinite-dimensional subspaces of $\mathbf{H}$ such that

(1) $\sum_{n=0}^{\infty} \oplus \mathbf{M}_n = \mathbf{H}$,

(2) each $\mathbf{M}_n$ reduces T,

(3) $\lim_{n\to\infty} ||T|\mathbf{M}_n|| = 0$.

The argument is completed by an application of Lemma 3.1 and Corollary 3.5.

As a final observation in this section we note that the analogues of Theorems 2 and 3 concerning shifts with operator weights are valid. Every operator-weighted shift, unilateral or bilateral, is a commutator. Indeed, Lemma 2.3 might as well have been proved for matrices with operator entries.

4. Compact operators. In this section we first show that every compact (=completely continuous) operator C on $\mathbf{H}$ is a commutator. For this purpose it suffices, according to Theorem 4, to exhibit a sequence $\{\mathbf{M}_n\}$ of mutually orthogonal, infinite-dimensional subspaces in $\mathbf{H}$ such that $\sum \oplus \mathbf{M}_n = \mathbf{H}$ and such that the operator (C_{ij}) on $\mathbf{H}^{\sim} = \mathbf{K} \oplus \mathbf{K} \oplus \ldots$ corresponding to C under Lemma 3.1 satisfies $\sum_{i,j} ||C_{ij}|| < \infty$. The following two lemmas show that such a decomposition is always possible.

LEMMA 4.1. *Suppose that C is a compact operator on $\mathbf{H}$, $\mathbf{M}$ is any infinite-dimensional subspace of $\mathbf{H}$, and ϵ is any positive number. Then $\mathbf{M}$ can be split orthogonally as $\mathbf{M} = \mathbf{N} \oplus (\mathbf{M} \ominus \mathbf{N})$ where $\mathbf{N}$ and $\mathbf{M} \ominus \mathbf{N}$ are infinite-dimensional subspaces of $\mathbf{H}$ such that for x in $\mathbf{N}$, $||Cx|| < \epsilon||x||$ and $||C^* x|| < \epsilon||x||$.*

Proof. If we can find such an $\mathbf{N}$ so that $||Cx|| < \epsilon||x||$ for x in $\mathbf{N}$, then we can apply this result with C^* and $\mathbf{N}$ replacing C and $\mathbf{M}$ to complete the argument. Thus it is clear that it suffices to find a finite-dimensional subspace $\mathbf{S}$ of $\mathbf{M}$ such that for x in $\mathbf{M} \ominus \mathbf{S}$, $||Cx|| < \epsilon||x||$. Suppose that no such subspace exists. Then there is some unit vector x_1 in $\mathbf{M}$ satisfying $||Cx_1|| \geqslant \epsilon$. If $x_1, \ldots, x_n \in \mathbf{M}$ have been chosen to be orthonormal vectors such that $||Cx_i|| \geqslant \epsilon$ for $i = 1, \ldots, n$, then there must exist a unit vector x_{n+1} in $\mathbf{M}$ orthogonal to $x_1, x_2, \ldots, x_n$ such that $||Cx_{n+1}|| \geqslant \epsilon$. Thus we obtain by induction an orthonormal sequence $\{x_n\}$ of vectors such that $||Cx_n|| \geqslant \epsilon$ for every n. This contradicts the compactness of C, and the proof is complete.

LEMMA 4.2. *If $\{\epsilon_1, \epsilon_2, \ldots\}$ is any sequence of positive numbers, and C is any compact operator on $\mathbf{H}$, then there exists a sequence $\{\mathbf{M}_0, \mathbf{M}_1, \ldots\}$ of mutually orthogonal, infinite-dimensional subspaces of $\mathbf{H}$ such that*

(1) $\sum_{n=0}^{\infty} \oplus \mathbf{M}_n = \mathbf{H}$,

(2) $||Cx|| < \epsilon_n||x||$ *for* $x \in \mathbf{M}_n$, $n = 1, 2, \ldots$,

(3) $||C^*x|| < \epsilon_n||x||$ *for* $x \in \mathbf{M}_n$, $n = 1, 2, \ldots$.

Proof. We apply Lemma 4.1 to C with $\mathbf{M} = \mathbf{H}$ and $\epsilon = \epsilon_1$ to obtain infinite-dimensional subspaces $\mathbf{N}$ and $\mathbf{N}^\perp$ of $\mathbf{H}$ such that

$$||Cx|| < \epsilon_1||x|| \quad \text{and} \quad ||C^*x|| < \epsilon_1||x||$$

whenever $x \in \mathbf{N}$. Let $\mathbf{M}_1 = \mathbf{N}$ and suppose that the mutually orthogonal, infinite-dimensional subspaces $\mathbf{M}_1, \ldots, \mathbf{M}_n$ have already been defined in such a way that the appropriate inequalities are valid on each $\mathbf{M}_k$, $1 \leqslant k \leqslant n$, and such that $(\mathbf{M}_1 \oplus \ldots \oplus \mathbf{M}_n)^\perp$ is infinite-dimensional. Apply Lemma 4.1 again with $\mathbf{M} = (\mathbf{M}_1 \oplus \ldots \oplus \mathbf{M}_n)^\perp$ and $\epsilon = \epsilon_{n+1}$ to obtain an infinite-dimensional subspace $\mathbf{M}_{n+1} \subset (\mathbf{M}_1 \oplus \ldots \oplus \mathbf{M}_n)^\perp$ such that

$$(\mathbf{M}_1 \oplus \ldots \oplus \mathbf{M}_n)^\perp \ominus \mathbf{M}_{n+1}$$

is infinite-dimensional and such that $||Cx|| < \epsilon_{n+1}||x||$ and $||C^*x|| < \epsilon_{n+1}||x||$ for all $x \in \mathbf{M}_{n+1}$. Thus, by induction, we obtain the sequence $\mathbf{M}_1, \mathbf{M}_2, \ldots$. If

$$\mathbf{H} \ominus \left(\sum_{n=1}^{\infty} \oplus \mathbf{M}_n \right)$$

is infinite-dimensional, let

$$\mathbf{M}_0 = \mathbf{H} \ominus \left(\sum_{n=1}^{\infty} \oplus \mathbf{M}_n \right).$$

Otherwise, construct $\mathbf{M}_0$ by borrowing a one-dimensional subspace from each of the $\mathbf{M}_n$, $n \geqslant 1$, and redefine the $\mathbf{M}_n$ accordingly. It is clear that the resulting sequence $\{\mathbf{M}_0, \mathbf{M}_1, \ldots\}$ satisfies the requirements of the lemma.

THEOREM 6. *Every compact operator is a commutator.*

Proof. It is easily seen that every compact operator on a non-separable Hilbert space possesses a separable reducing subspace whose orthogonal complement is contained in its null space. But any such operator is known to be a commutator (**3**, Lemma 2). Accordingly, we assume from now on that C is a compact operator acting on a separable Hilbert space $\mathbf{H}$. Let $\{\epsilon_1, \epsilon_2, \ldots\}$ be a sequence of positive numbers such that

$$\sum_{n=1}^{\infty} (2n+1)\epsilon_n < \infty,$$

and apply Lemma 4.2 to obtain the sequence $\{\mathbf{M}_0, \mathbf{M}_1, \ldots\}$ of subspaces there described. Let $\mathbf{K}$ be any fixed separable Hilbert space and for each non-negative integer n, let U_n be a linear transformation from $\mathbf{H}$ to $\mathbf{K}$ that maps $\mathbf{M}_n$ isometrically onto $\mathbf{K}$ and annihilates $\mathbf{H} \ominus \mathbf{M}_n$. Then, according to Lemma 3.1, C is carried onto an operator $(C_{ij})_{i,j=0}^{\infty}$ on $\mathbf{H}^\sim = \mathbf{K} \oplus \mathbf{K} \oplus \ldots$ by the resulting isomorphism of $\mathbf{H}$ onto $\mathbf{H}^\sim$, and the proof will be complete, after an application of Theorem 4, provided we can show that (C_{ij}) satisfies $\sum_{i,j} ||C_{ij}|| < \infty$. To establish this inequality, we shall show that the matrix $(||C_{ij}||)_{i,j=0}^{\infty}$ is dominated entrywise by the matrix

$$\begin{bmatrix} ||C_{00}|| & \epsilon_1 & \epsilon_2 & \epsilon_3 & \ldots \\ \epsilon_1 & \epsilon_1 & \epsilon_2 & \ldots & \\ \epsilon_2 & \epsilon_2 & \epsilon_2 & \ldots & \\ \epsilon_3 & \vdots & \vdots & & \\ \cdot & & & & \\ \cdot & & & & \\ \cdot & & & & \end{bmatrix}.$$

Then

$$\sum ||C_{ij}|| < ||C_{00}|| + \sum_{n=1}^{\infty} (2n+1)\epsilon_n < \infty,$$

and the proof will be complete.

Accordingly, let i and j be fixed indices satisfying $i + j \geqslant 1$. Suppose first that $i \leqslant j$, and take x in $\mathbf{K}$ with $||x|| = 1$. Then

$$||C_{ij}x|| = ||U_i C U_j^* x|| \leqslant \epsilon_j ||U_j^* x|| \leqslant \epsilon_j$$

since $U_j^* x \in \mathbf{M}_j$ and $||U_j^* x|| = ||x|| = 1$. On the other hand, if $i > j$, we show in the same fashion that $||C_{ij}|| = ||C_{ij}^*|| \leqslant \epsilon_i$. Indeed, if $x \in \mathbf{K}$ with $||x|| = 1$, then $||C_{ij}^* x|| = ||U_j C^* U_i^* x|| \leqslant ||C^* U_i^* x|| \leqslant \epsilon_i ||U_i^* x|| = \epsilon_i$ since $U_i^* x \in \mathbf{M}_i$ and $||U_i^* x|| = 1$. This completes the proof of the theorem.

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The following result is a generalization of Theorem 6 as well as of **(5,** Theorem 2**)**.

THEOREM 7. *Let* **H** *be a separable Hilbert space, and let* $\mathbf{K} = \mathbf{H} \oplus \mathbf{H}$. *If this decomposition of* **K** *is used to write every operator* $T \in \mathbf{L}(\mathbf{K})$ *as a* 2×2 *operator matrix.*

$$T = \begin{pmatrix} A_T & C_T \\ B_T & D_T \end{pmatrix}$$

where the entries are operators on **H**, *then every operator* T *on* **K** *such that* C_T *and* D_T *are compact operators on* **H** *is a commutator.*

Sketch of the proof. Define the sequence $\{\epsilon_1, \epsilon_2, \ldots\}$ by setting $\epsilon_n = 1/2^n$ for each n. It is easy to see, using the circle of ideas connected with Lemmas 4.1 and 4.2, that we can choose a sequence $\{\mathbf{M}_0, \mathbf{M}_1, \ldots\}$ of mutually orthogonal, infinite-dimensional subspaces $\mathbf{M}_n \subset \mathbf{H}$ such that

(1) $$\sum_{n=0}^{\infty} \oplus \mathbf{M}_n = \mathbf{H},$$

(2) for each positive integer n and each x in $\mathbf{M}_n$, $||C_T x||$, $||C_T^* x||$, $||D_T x||$, and $||D_T^* x||$ are all dominated by $\epsilon_n ||x||$.

Let **G** be a fixed separable Hilbert space, and let U_0 be a linear mapping from **K** onto **G** that acts isometrically on $\mathbf{H} \oplus \mathbf{M}_0$ and annihilates $(\mathbf{H} \oplus \mathbf{M}_0)^\perp$. Also, for each positive integer n, let U_n be a linear mapping from **K** onto **G** that acts isometrically on $0 \oplus \mathbf{M}_n$ and annihilates $(0 \oplus \mathbf{M}_n)^\perp$. The sequence $\{U_0, U_1, \ldots\}$ then defines an isomorphism of $\mathbf{K} = \mathbf{H} \oplus \mathbf{H}$ onto

$$\mathbf{K}^{\sim} = \mathbf{G} \oplus \mathbf{G} \oplus \ldots$$

under which $\mathbf{H} \oplus \mathbf{M}_0$ is mapped onto the first copy of **G** and $(\mathbf{H} \oplus \mathbf{M}_0)^\perp$ is mapped by the U_n's, $n \geqslant 1$, onto the other $\aleph_0$ copies of **G**. If $(T_{ij})_{i,j=0}^{\infty}$ is the operator in $\mathbf{L}(\mathbf{K}^{\sim})$ corresponding to T under this isomorphism, it suffices to prove that (T_{ij}) is a commutator, and this argument runs as follows.

By virtue of the way the subspaces $\mathbf{M}_n$ were chosen, it is easy to verify that for all integers $i \geqslant 0$ and $j \geqslant 1$, $||T_{ij}|| \leqslant \min\{\epsilon_i, \epsilon_j\}$. Moreover it is not hard to see that, formally at least, (T_{ij}) is the commutator

$$(T_{ij}) = (W + Y)S - S(W + Y)$$

where $S = (S_{ij})$ is given by $S_{ij} = \delta_{i,j+1}.1_{\mathbf{G}}$ for $i, j \geqslant 0$, while W and Y denote the matrices

$$W = \begin{bmatrix} -T_{10} & T_{00} & 0 & 0 & \ldots \\ -T_{20} & 0 & T_{00} & 0 & \ldots \\ -T_{30} & 0 & 0 & T_{00} & \ldots \\ -T_{40} & 0 & 0 & 0 & \ldots \\ \cdot & \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \cdot & \end{bmatrix},$$

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$$Y = (Y_{ij}) = \begin{bmatrix} 0 & 0 & T_{01} & T_{02} & T_{03} & \cdots \\ 0 & 0 & T_{11} & (T_{01}+T_{12}) & (T_{02}+T_{13}) & \cdots \\ 0 & 0 & T_{21} & (T_{11}+T_{22}) & (T_{01}+T_{12}+T_{23}) & \cdots \\ 0 & 0 & T_{31} & (T_{21}+T_{32}) & (T_{11}+T_{22}+T_{33}) & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \end{bmatrix}.$$

Thus, to complete the proof, it suffices to show that W and Y represent bounded operators on $\mathbf{K}^{\sim}$.

The boundedness of W follows readily from that of (T_{ij}). The inequalities

$$||T_{ij}|| \leqslant \min\{1/2^i, 1/2^j\}; i \geqslant 0, j \geqslant 1,$$

imply that the matrix $(||Y_{ij}||)$ is dominated entrywise by the Toeplitz matrix

$$\begin{bmatrix} \frac{1}{2} & 1 & 1 & \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \cdots \\ \frac{1}{4} & \frac{1}{2} & 1 & 1 & \frac{1}{2} & \frac{1}{4} & \cdots \\ \frac{1}{8} & \frac{1}{4} & \frac{1}{2} & 1 & 1 & \frac{1}{2} & \cdots \\ \frac{1}{16} & \frac{1}{8} & \frac{1}{4} & \frac{1}{2} & 1 & 1 & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \end{bmatrix},$$

and therefore the boundedness of Y follows from Lemma 3.3.

Remark. The question whether Theorem 7 remains valid if the restriction that C_T be compact is dropped is a vexing open question. This problem is discussed more fully below (see Problem 6).

5. Some open questions. In this concluding section of the paper we mention some open questions concerning commutators that seem to be of interest. It appears probable that the solution of any of the problems to be enumerated below would contribute a new idea to the theory of commutators.

Let $\mathfrak{C}$ denote the collection of all commutators on a separable Hilbert space $\mathbf{H}$.

PROBLEM 1. *Does $\mathfrak{C}$ contain a subset $\mathbf{U}$ of $\mathbf{L}(\mathbf{H})$ that is open in the uniform operator topology?*

PROBLEM 2. *If $A \in \mathfrak{C}$ and E is a one-dimensional projection, is $A + E$ necessarily an element of $\mathfrak{C}$?*

PROBLEM 3. *Is every operator on $\mathbf{H} \oplus \mathbf{H}$ of the form*

$$\begin{bmatrix} A & B \\ C & -A \end{bmatrix}$$

a commutator?

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PROBLEM 4. *If* $A = A^* \in \mathfrak{C}$, *can* A *always be written in the form*

$$A = B^*B - BB^*?$$

Since no operator of the form $B^*B - BB^*$ can be positive and invertible, an affirmative answer to Problem 4 would imply that no positive invertible operator is a commutator.

PROBLEM 5. *Is* $\mathfrak{C}$ *closed in the uniform operator topology*?

In connection with Problem 5, we point out that a question that has been open for some time is whether $1_{\mathbf{H}}$ is a limit of commutators in the uniform topology, and an affirmative answer to Problem 5 would settle this question in the negative since it is known **(6; 7)** that $1_{\mathbf{H}}$ is not a commutator. The following proposition is concerned with this circle of ideas.

PROPOSITION 5.1. *If* $\{A_0, A_1, \ldots\}$ *and* $\{B_0, B_1, \ldots\}$ *are sequences of bounded operators on* $\mathbf{H}$ *such that* $||(A_n B_n - B_n A_n) - 1_{\mathbf{H}}|| \to 0$, *then at least one of the sequences* $\{||A_n||\}$, $\{||B_n||\}$ *must be unbounded.*

Proof. Suppose, to the contrary, that there is a positive number M such that $||A_n||$, $||B_n|| \leqslant M$ for all n. We consider the collection $\mathbf{B}$ of all sequences $(x_1, x_2, \ldots)$ where $x_i \in \mathbf{H}$ and the sequence $\{||x_n||\}$ is bounded. Then $\mathbf{B}$ is a Banach space under the obvious operations and the norm

$$||(x_1, x_2, \ldots)|| = \sup_n ||x_n||.$$

Let A be the bounded linear transformation on $\mathbf{B}$ defined by

$$A(x_1, x_2, \ldots) = (A_1 x_1, A_2 x_2, \ldots),$$

and let B be defined similarly.

Consider the submanifold $\mathbf{N}$ of $\mathbf{B}$ consisting of all sequences $(x_1, x_2, \ldots) \in \mathbf{B}$ such that $||x_n|| \to 0$. The manifold $\mathbf{N}$ is invariant under A and B and therefore A and B yield operators $A\hat{}$ and $B\hat{}$ on the quotient space $\mathbf{B}/\mathbf{N}$. But if $(x_1, x_2, \ldots)$ is any element of $\mathbf{B}$, then

$$\begin{aligned}&[(AB - BA) - 1_{\mathbf{B}}](x_1, x_2, \ldots) \\ &\qquad = ((A_1 B_1 - B_1 A_1)x_1 - x_1, (A_2 B_2 - B_2 A_2)x_2 - x_2, \ldots) \in \mathbf{N}\end{aligned}$$

and it follows that $A\hat{}B\hat{} - B\hat{}A\hat{} = 1_{\mathbf{B}}\hat{} = 1_{\mathbf{B}/\mathbf{N}}$, thus contradicting the known fact that the unit is never a commutator in any normed algebra.

PROBLEM 6. *Are any of the operators*

$$\begin{bmatrix} \alpha \cdot 1_{\mathbf{H}} & 0 \\ 0 & 1_{\mathbf{H}} \end{bmatrix}, \qquad \alpha > 0,$$

commutators on $\mathbf{H} \oplus \mathbf{H}$?

This question is perhaps the most significant open question in the theory of commutators, and we make some remarks concerning it.

We remind the reader that it is known (**4**) that no operator of the form $\lambda . 1_{\mathbf{H}} + C$ is a commutator, where $\lambda \neq 0$ and C denotes a compact operator. Furthermore, these seem to be the only known non-commutators. The results of this paper make it clear that commutators exist in abundance, and it would appear to us that further inquiry should be in the direction of trying to find some new non-commutators. The 2×2 operator matrices set forth above would seem, because of their simplicity and their nearness to the scalars, to be good candidates for non-commutators, and hence worthy of study. On the other hand, however, there is some evidence that these operators may, in fact, be commutators. The evidence is the following. It is quite easy to verify that any operator of the form

$$\begin{bmatrix} \alpha . 1_{\mathbf{H}} & 0 \\ 0 & 1_{\mathbf{H}} \end{bmatrix}, \qquad \alpha \neq 1,$$

is similar to an operator of the form

$$\begin{bmatrix} \lambda_1 . 1_{\mathbf{H}} & \lambda_2 . 1_{\mathbf{H}} \\ \lambda_2 . 1_{\mathbf{H}} & 0 \end{bmatrix},$$

and the latter operators look very similar to the operators of Theorem 7, which are commutators.

Added in proof (11 June 1965). Since this paper was written, it has been discovered (A. Brown and C. Pearcy, *Structure of commutators of operators*, to appear in Ann. Math.) that the only non-commutators on a separable space are the operators of the form $\lambda . 1 + C$ where $\lambda \neq 0$ and C is compact; see the discussion following Problem 6. It follows readily that the answers to Problems 1–6 are, respectively, yes, yes, yes, no, no, yes.

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