Nagoya Math. J., **253** (2024), 91–127
DOI 10.1017/nmj.2023.19

ON THE ESSENTIAL TORSION FINITENESS OF ABELIAN VARIETIES OVER TORSION FIELDS

JEFFREY D. ACHTER, LIAN DUAN AND XIYUAN WANG

Abstract. The classical Mordell–Weil theorem implies that an abelian variety A over a number field K has only finitely many K-rational torsion points. This finitude of torsion still holds even over the cyclotomic extension $K^{\mathrm{cyc}} = K\mathbb{Q}^{\mathrm{ab}}$ by a result of Ribet. In this article, we consider the finiteness of torsion points of an abelian variety A over the infinite algebraic extension K_B obtained by adjoining the coordinates of all torsion points of an abelian variety B. Assuming the Mumford–Tate conjecture, and up to a finite extension of the base field K, we give a necessary and sufficient condition for the finiteness of $A(K_B)_{tors}$ in terms of Mumford–Tate groups. We give a complete answer when both abelian varieties have dimension at most 3, or when both have complex multiplication.

§1. Introduction

Suppose A is an abelian variety defined over a number field K. The celebrated Mordell–Weil theorem states that for any number field L containing K, the subgroup $A(L)_{\mathrm{tors}}$ of torsion points of A defined over L is finite (e.g., [24, Appendix II]). At the opposite extreme, over the algebraic closure $\overline{K}$ of K, using the geometry of A, one easily sees that the geometric torsion group $A(\overline{K})_{\mathrm{tors}}$ is infinite. Then it is natural to ask whether the finiteness property of the torsion subgroup $A(L)_{\mathrm{tors}}$ is still preserved for various infinite algebraic extensions L/K. This kind of question can be traced back at least to [18], in which Mazur asked whether the group $A(K^{\mathrm{cyc},p})$, where $K^{\mathrm{cyc},p} = K(\cup_n \zeta_{p^n})$ is the field obtained by adjoining all p-power roots of unity to K, is still finitely generated. The torsion part $A(K^{\mathrm{cyc},p})_{\mathrm{tors}}$ of this group is then proved to be finite by Imai [11] and Serre [34] independently. Their results are then generalized by Ribet in his article [12, Appendix]. Let $K^{\mathrm{cyc}} := \cup_p K^{\mathrm{cyc},p} = K(\cup_n \zeta_n)$ be the infinite extension of K obtained by adjoining all roots of unity. Then Ribet showed that for every abelian variety A defined over the number field K, one has

$$|A(K^{\mathrm{cyc}})_{\mathrm{tors}}| < \infty. \tag{1.1}$$

Zarhin then further generalized this result [46] by showing that if A is a simple abelian variety over its ground field K, then over the maximal abelian extension K^{ab} of K, the torsion group $A(K^{ab})_{\mathrm{tors}}$ is finite if and only if A is not of CM type over K, that is, if and only if the K-endomorphism algebra $\mathrm{End}(A)_{\mathbb{Q}} := \mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$ is not a number field of degree $2\dim A$. As a cohomological generalization of Ribet's result, Rössler and Szamuely [33] proved that for any projective, smooth, and geometrically connected variety X over a number field K, the groups $H^i_{\text{ét}}(\overline{X}, \mathbb{Q}/\mathbb{Z}(j))^{\mathrm{Gal}(\overline{K}/K^{cyc})}$ are finite for all odd positive integers i and all integers j. In contrast, when K is a p-adic field, then the analog of Imai and Serre's result is generalized by Ozeki in [27]. In addition, an analog of Zarhin's result is proved for Drinfeld modules by Li [15]. Quite recently, Lombardo studied a problem which,

Received May 28, 2023. Revised July 20, 2023. Accepted July 30, 2023.
2020 Mathematics subject classification: Primary 14K15, 11G10, 11F80; Secondary 14K22.

while perhaps superficially different, turns out to be closely related [17]; we discuss his work at the end of this introduction.

In this article, we focus on the generalization of (1.1) in another direction. Notice that by the Kronecker–Weber theorem, the cyclotomic extension $K^{\mathrm{cyc}} = K(\mathbb{G}_{m,\mathrm{tors}})$ is exactly the extension of K obtained by adjoining all the geometric torsion points of the algebraic torus $\mathbb{G}_m$. Due to the fact that there is no nontrivial isogeny, or even nonconstant geometric morphism, between $\mathbb{G}_m$ and an abelian variety, one is naturally led to ask the following question.

QUESTION 1.1. *Suppose two abelian varieties A and B are defined over a number field K; assume that over $\overline{K}$ they share no common nontrivial isogeny factor. Let K_B denote the infinite extension of K obtained by adjoining the coordinates of all the geometric torsion points of B. Is the torsion group of A over K_B finite, that is, is*

$$|A(K_B)_{\mathrm{tors}}| < \infty?$$

We answer this question in the present article, up to a finite extension of the base field and under the Mumford–Tate conjecture. We state our results after introducing a few definitions.

DEFINITION 1.2. Given two abelian varieties A and B defined over a number field K, we say that A is *torsion finite* for B over K if $A(K_B)_{\mathrm{tors}}$ is finite. Otherwise, we say that A is *torsion infinite* for B over K.

Moreover, if there is a finite extension L/K such that $A(L_B)_{\mathrm{tors}}$ is infinite, we say A is *potentially torsion infinite* for B. If such L does not exist, we will say that A is *essentially torsion finite* for B.

Although not stated in this language, Serre gave a positive answer to Question 1.1 in [35, théoremè 6 and 7] when A and B are both elliptic curves which are not geometrically isogenous. In fact, Serre proved that for such A and B, the image of the adelic representation induced by $A \times B$ equals the product of the images induced by A and B, up to a finite index; the claim readily follows. Our strategy is inspired by Serre's work. However, one of the advantages of working with elliptic curves, as opposed to higher-dimensional abelian varieties, is the open image theorem [34, 35]. This theorem, together with its analog for CM elliptic curves [39], classifies the ℓ-adic representation images of elliptic curves in terms of $\mathbb{Q}$-algebraic groups. With the help of these algebraic groups, the answer to Question 1.1 is essentially (but nontrivially) a consequence of Goursat's lemma.

The open image theorem for higher-dimensional abelian varieties is not known in general—indeed, it cannot hold for an abelian variety which is not Hodge maximal. Nonetheless, the Mumford–Tate conjecture claims that, for an abelian variety A over a sufficiently large number field, its ℓ-adic Galois representation images are still classified by a $\mathbb{Q}$-algebraic group, the Mumford–Tate group $\mathrm{MT}(A)$, which is defined in terms of the Hodge structure $H_1(A(\mathbb{C}),\mathbb{Q})$. (For a quick review of the Mumford–Tate group and the related conjecture, see §3.2. For an abelian variety A defined over a subfield $K \subset \mathbb{C}$, we will often abuse notation and write $H_1(A,\mathbb{Q})$ for the homology group $H_1((A \times_{\mathrm{Spec}\,K} \mathrm{Spec}\,\mathbb{C})(\mathbb{C})^{\mathrm{an}},\mathbb{Q})$, endowed with its Hodge structure, and define $H^1(A,\mathbb{Q})$ in an analogous fashion.)

In this article, assuming the Mumford–Tate conjecture, using Galois theory, and generalizing Serre's idea to algebraic groups beyond GL_2, we are able to prove a criterion for the essential torsion finiteness of pairs of abelian varieties.

THEOREM 1.3 (See Theorem 4.10 for a more detailed version). *Suppose A and B are two absolutely simple abelian varieties defined over a number field K, and suppose that the Mumford–Tate conjecture holds for A and B.*

Then A is potentially torsion infinite for B if and only if

$$\dim \mathrm{MT}(A \times B) = \dim \mathrm{MT}(B). \tag{1.2}$$

When this holds, for each prime ℓ, there exists a finite extension L_ℓ/K such that

$$A[\ell^\infty](L_{\ell,B}) = A_{\ell^\infty} := A[\ell^\infty](\overline{K}).$$

Insofar as (the image of) the action of Galois on torsion points is constrained by the Mumford–Tate group, it is not surprising that a relation on Mumford–Tate groups can force a resonance among torsion fields. What is perhaps more interesting is that just the presence of infinite torsion—for example, if $A[\ell](K_B)$ is nontrivial for an infinite, but sparse, set of primes—is enough to constrain the relation between $\mathrm{MT}(A \times B)$ and $\mathrm{MT}(B)$. In particular, we will see that the existence of ℓ-torsion for infinitely many ℓ forces the presence of ℓ-torsion for ℓ in a set of positive density.

The Mumford–Tate group $\mathrm{MT}(A)$ is canonically an extension of $\mathbb{G}_m$ by the Hodge group, or special Mumford–Tate group, $\mathrm{sMT}(A)$. Ichikawa [10] and Lombardo [16] have investigated conditions under which

$$\mathrm{sMT}(A \times B) = \mathrm{sMT}(A) \times \mathrm{sMT}(B). \tag{1.3}$$

(For example, this holds if A and B satisfy a certain *odd relative dimension* condition and at least one is *not* of Type IV in the Albert classification.) When A and B satisfy (1.3), we have

$$\dim \mathrm{MT}(A \times B) = \dim(\mathrm{MT}(A)) + \dim(\mathrm{MT}(B)) - 1 > \max\{\dim \mathrm{MT}(A), \dim \mathrm{MT}(B)\}.$$

Theorem 1.3 then immediately implies that A and B are mutually essentially torsion finite. (See the last part of §4.4 for more details.)

Taken together with our main theorem, Ichikawa and Lombardo's results often imply a positive answer to our main question 1.1, except when both A and B are of Type IV. Thus, Question 1.1 is particularly interesting when both A and B are of Type IV, such as when both A and B have complex multiplication (CM) over $\overline{K}$.

In fact, there do exist examples where Question 1.1 has a negative answer. For instance, the Jacobian of a certain genus 4 curve [40, Exam. 6.1] decomposes into a product of a potentially CM elliptic curve and a simple potentially CM abelian threefold. However, one can check that the elliptic curve is torsion infinite for the threefold (see [17, Th. 1.2]). In addition, Lombardo [17] constructed infinitely many pairs of nonisogenous CM abelian varieties for which the answer to Question 1.1 is again negative. As a complement to Lombardo's work, we give a sufficient and necessary condition for answering our main question for CM pairs, as follows.

Let A be an isotypic abelian variety over a number field K with CM by a CM field E, and suppose that K contains E. (Recall that an abelian variety is said to be isotypic if it is isogenous to some power of a simple abelian variety.) We see in §5.1 that there is a surjection of algebraic tori

$$T^K := \mathrm{Res}_{K/\mathbb{Q}}\,\mathbb{G}_m \twoheadrightarrow \mathrm{MT}(A),$$

https://doi.org/10.1017/nmj.2023.19 Published online by Cambridge University Press

which induces an inclusion of character groups

$$X^*(\mathrm{MT}(A)) \hookrightarrow X^*(T^K).$$

In fact, $\mathrm{MT}(A)$ only depends on the CM type of A. With this reminder, we can state a version of our main theorem for abelian varieties with complex multiplication. For a torus T, let $X^*(T)_{\mathbb{Q}} = X^*(T) \otimes \mathbb{Q}$.

THEOREM 1.4 (See Theorem 5.4 for a more detailed version). *Let A_1 and A_2 be isotypic potentially CM abelian varieties over a sufficiently large number field K, with respective Mumford–Tate groups T_1 and T_2. Using the inclusions $X^*(T_i) \hookrightarrow X^*(T^K)$, either:*

(a) $X^*(T_1)_{\mathbb{Q}} \subset X^*(T_2)_{\mathbb{Q}}$. *Then A_1 is potentially torsion infinite for A_2.*
Moreover, if $X^*(T_1) \subset X^*(T_2)$ and if A_1 is simple with nondegenerate CM (5.1.3), then $A_1(K_{A_2})_{\mathrm{tors}} = A_1(\overline{K})_{\mathrm{tors}}$.

(b) $X^*(T_1)_{\mathbb{Q}} \not\subset X^*(T_2)_{\mathbb{Q}}$. *Then A_1 is essentially torsion finite for A_2.*

Theorem 1.4 is unconditional because the Mumford–Tate conjecture is known for CM abelian varieties (see Lemma 5.1).

We briefly compare this result to Zarhin's work [46]. Suppose B has complex multiplication over K, but A does not even have potential complex multiplication. Note that K_B is an abelian extension of K. Zarhin's result implies that $A(K^{\mathrm{ab}})_{\mathrm{tors}}$ is finite; *a fortiori*, A is essentially torsion finite for B. However, if A is also of CM type, then Theorem 1.4 gives finer information on whether A is essentially torsion finite for B.

A morphism $A \to B$ induces a map of homology groups $H_1(A,\mathbb{Q}) \to H_1(B,\mathbb{Q})$, and thus a morphism of Tannakian categories $\langle H_1(A,\mathbb{Q})\rangle \to \langle H_1(B,\mathbb{Q})\rangle$ and ultimately of Mumford–Tate groups $\mathrm{MT}(B) \to \mathrm{MT}(A)$. More generally, a correspondence between A^m and B^n induces a relation between $\mathrm{MT}(A)$ and $\mathrm{MT}(B)$; and the class of such a correspondence is a Hodge class on $A^m \times B^n$.

In §5.2, we will see that if the CM abelian variety A is torsion infinite for the CM abelian variety B, then there is a nonempty $\mathbb{Q}$-vector space of interesting Hodge classes on some product $A^m \times B^n$; perhaps not surprisingly, we call such a class a torsion infinite class. These Hodge classes are *extra*, in the sense that they are not in the span of classes pulled back from A^m and B^n. Conversely, we show that the presence of such a class implies that A is torsion infinite for B.

Of course, the Hodge conjecture predicts that torsion infinite classes are actually the classes of cycles on $A^m \times B^n$. It would be interesting to see, even in special cases, if one can geometrically realize torsion infinite classes.

In addition to the above applications, thanks to the work of Moonen and Zarhin on the Hodge groups of abelian varieties of low dimension [23], one can compare the Mumford–Tate groups of every possible pair of absolutely simple abelian varieties up to dimension 3. As a consequence, we give a positive answer to Question 1.1 for most pairs of such abelian varieties. Precisely, following the classification in their article, we prove the following theorem.

THEOREM 1.5 (Also Theorem 5.14). *Suppose A and B are absolutely simple abelian varieties over a common number field, and assume that they are nonisogenous over $\mathbb{C}$.*

Suppose that $\dim A \le \dim B \le 3$*. Then* A *and* B *are mutually essentially torsion finite except for the following cases:*

(a) A *is a CM elliptic curve, and* B *is a CM abelian threefold. Then* B *is essentially torsion finite for* A*, and* A *is potentially torsion infinite for* B *exactly when there is an embedding of* $\mathbb{Q}$*-algebras* $\mathrm{End}^0(A) \hookrightarrow \mathrm{End}^0(B)$*.*
(b) A *is a CM elliptic curve, and* B *is an abelian threefold of type IV but not CM. Then* B *is essentially torsion finite for* A*, and* A *is potentially torsion infinite for* B *exactly when there is an isomorphism of* $\mathbb{Q}$*-algebras* $\mathrm{End}^0(A) \cong \mathrm{End}^0(B)$*.*
(c) A *and* B *are both CM abelian threefolds.*

(In (c), the essential torsion finiteness depends on the CM types of A and B as in Theorem 5.4.)

Again, this result is unconditional since the Mumford–Tate conjecture is known to hold for simple abelian varieties of dimensions less than 4 [23].

This article is structured as follows: in §2, we collect some basic results on representations of algebraic groups. In particular, we introduce the notion of a collection of subgroups of bounded index (of the $\mathbb{F}_\ell$-points of a group scheme over $\mathbb{Z}[1/N]$); this allows us to infer information about a representation of an algebraic group from data about the behavior of abstract subgroups of its finite-field-valued points. In §3, we establish notation and review facts (and conjectures) concerning the Galois representations attached to abelian varieties. We finally turn to the torsion-finiteness question itself in §4.1, establishing our main result (Theorem 1.3) in §4.4. The article concludes with a detailed analysis of CM (§5.1) and low-dimensional (§5.3) pairs of abelian varieties, and of certain extra Hodge classes which are the hallmark of torsion-infinite pairs of CM abelian varieties (§5.2).

It turns out that while we were working out these results, Lombardo studied a similar problem with somewhat stronger restrictions [17]. Two abelian varieties A and B over a number field K are said to be *strongly iso-Kummerian* if for each positive integer d we have

$$K_{A,d} = K_{B,d}, \tag{1.4}$$

that is, if the d-torsion points of A and B generate the same extension of K. Using the theory of the (special) Mumford–Tate group and assuming the Mumford–Tate conjecture, Lombardo proves that condition (1.4) puts a strong restriction on the Hodge groups of A, B and $A \times B$. This constraint forces A to have the same isogeny factors as B when either $\dim A \le 3$ and $\dim B \le 3$ [17, Th. 1.2]; or every simple factor of A or B has dimension ≤ 2, or is of odd relative dimension and not of type IV [17, Th. 1.4]. As a complement, by studying certain simple CM types on cyclic CM fields, Lombardo also constructs infinitely many nonisogenous iso-Kummerian pairs [17, Th. 1.1]. In spite of the obvious similarities, our work is differs from Lombardo's in its emphasis and results.

1. Condition (1.4) is much stronger than our (potentially) torsion-infinite condition. In fact, (1.4) forces $K_A = K_B$, so $K_A K_B / K_B$ is a trivial extension. However, even if A is torsion-infinite for B, $K_A K_B / K_B$ can still be infinite (Example 5.7).
2. In assumption (1.4), by taking $d = \ell^n$ and letting $n \to \infty$, one can directly deduce $A[\ell^\infty](K_{B,\ell^\infty}) = A_{\ell^\infty}$. However, if one only assumes that A is torsion-infinite for B, it is possible that the subgroup $A[\ell^\infty](K_{B,\ell^\infty})$ is finite for every ℓ, but nontrivial for infinitely

many ℓ. One of our main contributions in this article is to rule out this possibility (under the Mumford–Tate conjecture, as usual).

3. Lombardo shows that if A and B are strongly iso-Kummerian, then the natural projections $\mathrm{MT}(A\times B)\to\mathrm{MT}(A)$ and $\mathrm{MT}(A\times B)\to\mathrm{MT}(B)$ are isogenies [17, Lem. 3.2]. We are able to deduce this conclusion from the weaker hypothesis that A and B are mutually potentially torsion infinite (Corollary 4.14).

§2. Preliminaries

2.1 Reminders on algebraic groups

We collect some standard, useful facts on algebraic groups.

LEMMA 2.1 (Goursat's lemma). *Let G_1, G_2, and G_{12} be either abstract groups or algebraic groups over a field. Suppose G_{12} is endowed with an inclusion $\iota\colon G_{12}\hookrightarrow G_1\times G_2$ such that $\pi_i\circ\iota$ is surjective for $i=1,2$:*

$$\begin{tikzcd} G_{12} \arrow[r, hook, "\iota"] & G_1\times G_2 \arrow[dl, two heads, "\pi_1"'] \arrow[dr, two heads, "\pi_2"] & \\ G_1 & & G_2 \end{tikzcd} \quad . \tag{2.1}$$

Let $M_{12}=\ker(\pi_2\circ\iota)$, and let $H_{12}\cong M_{12}$ be the image of M_{12} under the isomorphism $G_1\times\{e\}\cong G_1$; define M_{21} and H_{21} analogously.

Then, under the composite map

$$G_{12}\hookrightarrow G_1\times G_2 \longrightarrow \frac{G_1}{H_{12}}\times\frac{G_2}{H_{21}},$$

G_{12} is the inverse image in $G_1\times G_2$ of the graph of an isomorphism $\frac{G_1}{H_{12}}\to\frac{G_2}{H_{21}}$.

Proof. This is standard (see, e.g., [31, Lem. 5.2.1] for the case of abstract groups). The constructions of H_{ij} and M_{ij} also make sense in the category of algebraic groups, and the asserted properties may be verified pointwise, as in [31]. ∎

REMARK 2.2. Under the hypotheses of Lemma 2.1, suppose $H_{12}=G_1$. Then, clearly, $H_{21}=G_2$, and thus $G_{12}\cong G_1\times G_2$. At the opposite extreme, if H_{12} and H_{21} are trivial, then, up to a choice of isomorphism $G_1\cong G_2$, $\iota: G_{12}\hookrightarrow G_1\times G_2$ is the diagonal embedding.

LEMMA 2.3. *Assume G_1, G_2, and G_{12} are reductive groups over a field K of characteristic zero and satisfy a diagram (2.1). Then the following are equivalent:*

(a) $\dim G_{12}=\dim G_2$;
(b) $\operatorname{rank} G_{12}=\operatorname{rank} G_2$; *and*
(c) *the surjection $G_{12}\twoheadrightarrow G_2$ is an isogeny.*

Proof. The short exact sequence of algebraic groups

$$0\longrightarrow M_{12}\longrightarrow G_{12}\xrightarrow{\pi_2\circ\iota} G_2\longrightarrow 0$$

induces a corresponding exact sequence on Lie algebras. Since K has characteristic zero, the rank and dimension of a reductive group can be read off from its Lie algebra. Note that $\mathrm{Lie}(M_{12})=\mathrm{Lie}(M_{12}^\circ)$ and that M_{12}°, being a connected normal (Lemma 2.4) subgroup of a reductive group, is also reductive (e.g., [19, Cor. 21.53]). In particular, either (a) or (b) holds if and only if $\mathrm{Lie}(M_{12})=(0)$, that is, $\dim M_{12}=0$ and thus $\pi_2\circ\iota$ is an isogeny. ∎

https://doi.org/10.1017/nmj.2023.19 Published online by Cambridge University Press

LEMMA 2.4. *Let G be a connected algebraic group, and let $M \subset G$ be a normal algebraic subgroup. Then M° is normal in G.*

Proof. Since G and M° are connected, the image of M° under conjugation by G is connected and contains the identity element of G. Since this image is a subgroup of M, which is normal in G, it is contained in M°, and thus M° is stable under conjugation by G. ∎

Finally, when studying CM abelian varieties in §5, we will need to work with algebraic tori.

Let K be a perfect field. An algebraic torus T/K is an algebraic group such that $T_{\overline{K}} \cong \mathbb{G}_{m,\overline{K}}^{\oplus \dim T}$. Let $X^*(T)$ be the (absolute) character group $X^*(T) = \mathrm{Hom}(T_{\overline{K}}, \mathbb{G}_{m,\overline{K}})$, and let $X^*(T)_{\mathbb{Q}} = X^*(T) \otimes \mathbb{Q}$; then $T \mapsto X^*(T)$ gives a contravariant equivalence between the category of algebraic tori over K and the category of finite free $\mathbb{Z}$-modules with a continuous action by the absolute Galois group $\mathrm{Gal}(K) := \mathrm{Gal}(\overline{K}/K)$. This extends to a contravariant equivalence between the category of K-groups of multiplicative type and the category of finitely generated $\mathbb{Z}$-modules with continuous $\mathrm{Gal}(K)$ action. We have $\dim T = \mathrm{rank}_{\mathbb{Z}} X^*(T) = \dim_{\mathbb{Q}} X^*(T)_{\mathbb{Q}}$. If $\alpha \colon S \to T$ is a morphism of algebraic tori, then $\dim \ker(\alpha) = \dim_{\mathbb{Q}} X^*(T)_{\mathbb{Q}} / \alpha^* X^*(S)_{\mathbb{Q}}$, and α has connected kernel if and only if $X^*(T)/\alpha^* X^*(S)$ is torsion-free.

If $F/\mathbb{Q}$ is a finite extension, we let T^F denote $\mathrm{Res}_{F/\mathbb{Q}} \mathbb{G}_{m,F}$, the Weil restriction of the multiplicative group, and let $T^{F,1}$ denote the norm one torus $\mathrm{Res}^{(1)}_{F/\mathbb{Q}} \mathbb{G}_{m,F}$, which is the kernel of the norm map $\mathrm{N}_{F/\mathbb{Q}} \colon T^F \to \mathbb{G}_m$.

2.2 Representations of algebraic groups

2.2.1. Fixed spaces

Let G/K be an algebraic group over a field. Let V/K be a finite-dimensional representation of G, that is, a finite-dimensional vector space V equipped with a morphism $G \to \mathrm{GL}_V$ of algebraic groups. The schematic fixed space of V under G is

$$V^G = \{v \in V : g \cdot v_R = v_R \text{ (in } V_R) \text{ for all } K\text{-algebras } R \text{ and all } g \in G(R)\},$$

where v_R is the image of v under $V \hookrightarrow V \otimes_K R$ [19, §4i].

We define the naïve fixed space as

$$V^{G(K)} = \{v \in V : g \cdot v = v \text{ for all } g \in G(K)\}.$$

More generally, if $\Gamma \subset G(K)$ is an abstract subgroup, the subspace fixed by Γ is

$$V^\Gamma = \{v \in V : g \cdot v = v \text{ for all } g \in \Gamma\}.$$

LEMMA 2.5. *We have $V^G \subseteq V^{G(K)}$, with equality if K is infinite and G/G° is a split étale group.*

Proof. The first statement is trivial; for the second, use the fact that under the stated hypotheses, $G(K)$ is Zariski dense in G. ∎

LEMMA 2.6. *Let $\rho \colon G \to \mathrm{GL}_V$ be a morphism of algebraic groups over K, and let $M \subset G$ be a normal algebraic subgroup. Then V^M is stable under G, and thus is a sub-G-representation of G.*

Proof. It suffices to verify this after passage to the algebraic closure of K, so we may and do assume that $G(K)$ is dense in G, and that $M(K)$ is dense in M. It now suffices to show that, for each $g \in G(K)$, $gW \subset W$. Since W is fixed by the normal subgroup M, gW is fixed by $gMg^{-1} = M$, and so $gW \subset V^M = W$. ∎

LEMMA 2.7. *Let $V/\mathbb{F}_\ell$ be a finite-dimensional vector space, and let $\mathtt{G}$ be an abstract group equipped with a representation $\rho : \mathtt{G} \to \mathrm{GL}_V(\mathbb{F}_\ell)$. Let $\mathtt{H} \trianglelefteq \mathtt{G}$ be a normal subgroup such that $V^{\rho(\mathtt{H})} \supsetneq (0)$ and $[\mathtt{G} : \mathtt{H}]$ is a power of ℓ. Then $V^{\rho(\mathtt{G})} \supsetneq (0)$.*

Proof. Choose some $w \in V^{\rho(\mathtt{H}_\ell)} \setminus \{0\}$, and let $W = \mathbb{F}_\ell[\mathtt{G}]w$ be the subspace spanned by its $\mathtt{G}$-orbit. The representation ρ_W of $\mathtt{G}$ on $W \subseteq V$ factors as $\mathtt{G} \to \mathtt{G}/\mathtt{H} \to \mathrm{GL}_W(\mathbb{F}_\ell)$. Since $\mathtt{G}/\mathtt{H}$ is an ℓ-group, by the Sylow theorem, $\rho_W(\mathtt{G})$ is contained in the $\mathbb{F}_\ell$-points of a maximal unipotent subgroup U of GL_W. (Differently put, after a suitable choice of basis, the image of $\mathtt{G}$ in GL_W is contained in the $\mathbb{F}_\ell$-points of the group U of unipotent upper-triangular matrices.) Since U has a nontrivial fixed vector, so does $\mathtt{G}$. ∎

2.2.2. Bounded subgroups

Let $H/\mathbb{Z}[1/N]$ be a smooth affine algebraic group scheme with geometrically connected fibers. Suppose that for each $\ell \nmid N$, an abstract group $\mathtt{H}_\ell \subset H(\mathbb{F}_\ell)$ is specified.

DEFINITION 2.8. The collection $\{\mathtt{H}_\ell\}_\ell$ is bounded (in H, independently of ℓ) if there exists some finite B such that, for each ℓ,

$$[H(\mathbb{F}_\ell) : \mathtt{H}_\ell] < B. \tag{2.2}$$

Equivalently, there is some positive $C = 1/B$ such that $\#\mathtt{H}_\ell > C\#H(\mathbb{F}_\ell)$.

LEMMA 2.9. *Let $\alpha \colon H \to G$ be a surjective morphism of smooth algebraic groups over $\mathbb{Z}[1/N]$, with G connected. Then $\{\alpha(H(\mathbb{F}_\ell))\}_\ell$ is bounded in G.*

Proof. Let $P = \ker(\alpha)$. Its formation commutes with base change, and it is the extension of a finite group scheme D by a connected group P°. Taking $\mathbb{F}_\ell$-points, we have

$$1 \longrightarrow P(\mathbb{F}_\ell) \longrightarrow H(\mathbb{F}_\ell) \xrightarrow{\alpha(\mathbb{F}_\ell)} G(\mathbb{F}_\ell) \longrightarrow H^1(\mathbb{F}_\ell, P).$$

It suffices to show that $H^1(\mathbb{F}_\ell, P)$ is bounded independently of ℓ. By Lang's theorem, $H^1(\mathbb{F}_\ell, P^\circ)$ is trivial, and so it suffices to show that $H^1(\mathbb{F}_\ell, D)$ is bounded. Now, use the fact that D is finite and $\#H^1(\mathbb{F}_\ell, D) = \#D(\mathbb{F}_\ell)$ (see [29, p. 290]). ∎

LEMMA 2.10. *Let*

$$0 \longrightarrow P \longrightarrow H \xrightarrow{\alpha} G \longrightarrow 0$$

be an exact sequence of algebraic groups over $\mathbb{Z}[1/N]$. Suppose that $\{\mathtt{H}_\ell\}_\ell$ has bounded index in H. Then:

(a) *$\{\alpha(\mathtt{H}_\ell)\}_\ell$ has bounded index in G, and*
(b) *$\{(\ker \alpha|_{\mathtt{H}_\ell}) \cap P^\circ(\mathbb{F}_\ell)\}_\ell$ has bounded index in P°.*

Proof. Suppose $\#\mathtt{H}_\ell > C_{\mathtt{H}} \cdot \#H(\mathbb{F}_\ell)$ and (using Lemma 2.9) $\#\alpha(H(\mathbb{F}_\ell)) > C_\alpha \cdot \#G(\mathbb{F}_\ell)$ for $\ell \gg 0$. We then have the easy estimates

https://doi.org/10.1017/nmj.2023.19 Published online by Cambridge University Press

$$\#\alpha(\mathtt{H}_\ell) = \frac{\#\mathtt{H}_\ell}{\#\ker\alpha|_{\mathtt{H}_\ell}} \geq \frac{\#\mathtt{H}_\ell}{\#P(\mathbb{F}_\ell)} > C_{\mathtt{H}}\frac{\#H(\mathbb{F}_\ell)}{\#P(\mathbb{F}_\ell)} = C_{\mathtt{H}}\cdot\#\alpha(H(\mathbb{F}_\ell)) > C_{\mathtt{H}}C_\alpha\cdot\#G(\mathbb{F}_\ell).$$

This proves (a).

Let $\mathtt{P}_\ell = \ker\alpha|_{\mathtt{H}_\ell}$. Then

$$\#\mathtt{P}_\ell = \frac{\#\mathtt{H}_\ell}{\#\alpha(\mathtt{H}_\ell)} \geq \frac{\#\mathtt{H}_\ell}{\#\alpha(H(\mathbb{F}_\ell))} > C_{\mathtt{H}}\frac{\#H(\mathbb{F}_\ell)}{\#\alpha(H(\mathbb{F}_\ell))} = C_{\mathtt{H}}\#P(\mathbb{F}_\ell).$$

Let P' be any irreducible component of $P_{\mathbb{F}_\ell}$. Then $\mathtt{P}_\ell \cap P'(\mathbb{F}_\ell)$, if nonempty, is a torsor under $\mathtt{P}_\ell \cap P^\circ(\mathbb{F}_\ell)$, and so

$$\#(\mathtt{P}_\ell \cap P^\circ(\mathbb{F}_\ell)) \geq \frac{1}{[P:P^\circ]}\#\mathtt{P}_\ell. \qquad \square$$

2.2.3. Representations of connected groups

Let $G/\mathbb{Z}[1/N]$ be a smooth affine algebraic group with connected fibers. Let V be a free $\mathbb{Z}[1/N]$-module of rank n, and let $\rho\colon G \to \mathrm{GL}_V$ be a representation. For a field k equipped with a ring map $\mathbb{Z}[1/N] \to k$, let $r_k(G,\rho)$ be the multiplicity of the trivial representation of $G(k)$:

$$r_k(G,\rho) = \dim_k (V\otimes k)^{G(k)}. \tag{2.3}$$

Let $V_\ell = V\otimes\mathbb{F}_\ell$, and let $r_\ell(G,\rho) = r_{\mathbb{F}_\ell}(G,\rho)$. Note that, when $\ell \gg 0$, by specialization, we always have $r_{\mathbb{Q}}(G,\rho) \leq r_\ell(G,\rho)$.

If $g \in G(k)$, let $m(g,\rho) = m_k(g,\rho)$ be the multiplicity of 1 as a root of the characteristic polynomial of $\rho(g)$. Let $G_{\rho,\geq m}$ be the locus of those g for which $m(g,\rho) \geq m$. (Schematically, $G_{\rho,\geq 1}$ may be constructed by pulling back the composite morphism

$$G \xrightarrow{\rho} \mathrm{GL}_V \xrightarrow{\text{charpoly}} \mathbb{G}_a^n \xrightarrow{\text{eval}_1} \mathbb{G}_a$$

by the zero section Spec $\mathbb{Z}[1/N] \to \mathbb{G}_a$, where the map eval_1 means evaluating the characteristic polynomial at 1; for other values of m, $G_{\rho,\geq m}$ may be constructed by considering higher derivatives of the characteristic polynomial.)

LEMMA 2.11. *Suppose that there is an infinite collection of primes $\mathbb{L}$ such that, if $\ell \in \mathbb{L}$, then $r_\ell(G,\rho) = r$. Then we have:*

(a) *for each $g \in G(\mathbb{Q})$, $m(g,\rho) \geq r$;*
(b) *$r_{\mathbb{Q}}(G,\rho) = r$; and*
(c) *$\dim V_{\mathbb{Q}}^{G_{\mathbb{Q}}} = r$.*

Proof. We assume $r > 0$, since (by specialization) the statement is trivial if $r = 0$.

For (a), it suffices to apply, to the characteristic polynomial of $\rho(g)$, the following elementary observation. Let $f(T) \in \mathbb{Q}[T]$ be any polynomial; since *clearing denominators* does not alter the roots of f, we may and do assume $f(T) \in \mathbb{Z}[T]$. Suppose $\lambda \in \mathbb{Z}$. If ℓ is sufficiently large, relative to the coefficients of f and to λ, then $f(\lambda) = 0$ if and only if $f(\lambda) \equiv 0 \bmod \ell$; and, by taking the first $r-1$ derivatives of f, a similar result holds for roots of higher multiplicity.

We now prove (b). For each $\ell \in \mathbb{L}$, let $Y_\ell \subset V_\ell$ be the subspace fixed by $G_{\mathbb{F}_\ell}$.

Let m_0 be the integer such that $G_{\mathbb{Q}} = G_{\mathbb{Q}_{\rho,\geq m_0}} \supsetneq G_{\mathbb{Q}_{\rho,\geq m_0+1}}$; by (a), we have $m_0 \geq r \geq 1$. Let ${G_{\mathbb{Q}}}^{ss}$ be the open and dense semisimple locus (e.g., [9, Th. 22.2]), and let $G^* = {G_{\mathbb{Q}}}^{ss} \setminus$

https://doi.org/10.1017/nmj.2023.19 Published online by Cambridge University Press

$G_{\mathbb{Q}_{\rho,\geq m_0+1}}$. Since $G_{\mathbb{Q}}$ is connected, G^* is open and dense in $G_{\mathbb{Q}}$. Like any connected affine algebraic group, $G_{\mathbb{Q}}$ is unirational. Consequently, $G^*(\mathbb{Q})$ is Zariski dense in $G_{\mathbb{Q}}$.

For $g \in G^*(\mathbb{Q})$, let $W_g \subset V_{\mathbb{Q}}$ be the m_0-dimensional subspace fixed by g. After choosing an integral model of W_g, for all but finitely many ℓ, the reductions $g_\ell \in G(\mathbb{F}_\ell)$ and $W_{g,\ell} \subset V_\ell$ are well defined; and for $\ell \in \mathbb{L}$, we have $W_{g,\ell} \supseteq Y_\ell$.

Let $W = \cap_{g \in G^*(\mathbb{Q})} W_g$; since $V_{\mathbb{Q}}$ is finite-dimensional, there is a finite list of elements $g_1, \ldots, g_n \in G^*(\mathbb{Q})$ such that $W = \cap_i W_{g_i}$. If ℓ is sufficiently large as to avoid the finitely many primes of bad reduction for the W_{g_i}, then $W \otimes \mathbb{F}_\ell$ contains Y_ℓ. This shows that $\dim_{\mathbb{Q}} W \geq \dim_{\mathbb{F}_\ell} Y_\ell = r$. By the density of $G^*(\mathbb{Q})$, W is fixed by all of $G_{\mathbb{Q}}$, and so $r_{\mathbb{Q}}(G,\rho) \geq r$; again, by specialization, we find that equality holds. This proves (b). Since $W = V_{\mathbb{Q}}^{G_{\mathbb{Q}}}$, we may conclude (c), as well. ∎

Now, let $\{\mathtt{G}_\ell\}$ be a collection of bounded subgroups of G, and let

$$r_\ell(\mathtt{G}_\ell, \rho) = \dim V_\ell^{\mathtt{G}_\ell}.$$

In the statement below, *sufficiently large* depends only on $\dim V$ and the constant in (2.2); however, in our applications, we do not have control over this constant.

LEMMA 2.12. *Let $\{\mathtt{G}_\ell\}$ be a collection of bounded subgroups of G. If $r_\ell(\mathtt{G}_\ell, \rho) = r$ for some sufficiently large ℓ, then $r_\ell(G,\rho) = r$.*

Proof. For any ℓ, let $W_\ell \subset V \otimes \mathbb{F}_\ell$ be a subspace of dimension r, and let $\mathrm{Fix}_{G,W_\ell} \subset G_{\mathbb{F}_\ell}$ be the subgroup scheme which fixes W_ℓ. By Bézout's theorem, since Fix_{G,W_ℓ} is the intersection of G and $\mathrm{Fix}_{\mathrm{GL}_{V,\mathbb{F}_\ell},W_\ell}$ in $\mathrm{GL}_{V,\mathbb{F}_\ell}$, there is a constant B such that $\#\pi_0(\mathrm{Fix}_{G,W_\ell}) \leq B$, independent of the choice of W_ℓ and of ℓ.

For any connected group H of dimension d over a finite field $\mathbb{F}$, we have $(\#\mathbb{F}-1)^d \leq \#H(\mathbb{F}) \leq (\#\mathbb{F}+1)^d$ (see [25, Lem. 3.5] or [14, Prop. 3.1]). Fix a prime ℓ, and suppose that $r_\ell(\mathtt{G}_\ell,\rho) = r$; let $W_\ell \subset V \otimes \mathbb{F}_\ell$ be the subspace fixed by $\mathtt{G}_\ell$. We then have

$$\#\mathrm{Fix}^\circ_{G,W_\ell}(\mathbb{F}_\ell) \geq \frac{1}{B}\#\mathrm{Fix}_{G,W_\ell}(\mathbb{F}_\ell) \geq \frac{1}{B}\#\mathtt{G}_\ell \geq \frac{C}{B}\#G(\mathbb{F}_\ell) \geq \frac{C}{B}(\ell-1)^d.$$

If $\ell \gg_{d,C/B} 0$, this forces $\dim \mathrm{Fix}_{G,W_\ell} = \dim G_{\mathbb{F}_\ell}$, so that $\mathrm{Fix}_{G,W_\ell} = G_{\mathbb{F}_\ell}$. ∎

LEMMA 2.13. *Let $\{\mathtt{G}_\ell\}$ be a collection of bounded subgroups of G. Suppose that there is an infinite collection of primes $\mathbb{L}$ such that, if $\ell \in \mathbb{L}$, then $r_\ell(\mathtt{G}_\ell,\rho) \geq r$. Then:*

(a) *$r_{\mathbb{Q}}(G,\rho) \geq r$ and*
(b) *$r_\ell(G,\rho) \geq r$ for all but finitely many ℓ.*

Proof. By Lemma 2.12, we find that for $\ell \gg 0$, we have $r_\ell(G_\ell,\rho) \geq r$. Lemma 2.11 then shows that $r_{\mathbb{Q}}(G,\rho) \geq r$. This proves (a); then (b) follows by specialization. ∎

LEMMA 2.14. *Let $\mathtt{G}_{\ell_0^\infty}$ be a Zariski dense subgroup of $G_{\mathbb{Q}_{\ell_0}}$. Suppose $r(\mathtt{G}_{\ell_0^\infty},\rho) \geq r$. Then:*

(a) *$r_{\mathbb{Q}}(G,\rho) \geq r$ and*
(b) *$r_\ell(G,\rho) \geq r$ for all but finitely many ℓ.*

Proof. Under the hypothesis, $r(G_{\mathbb{Q}_{\ell_0}},\rho) \geq r$; for (a), it then suffices to note that, since $G_{\mathbb{Q}}$ is connected, the formation of the fixed points of the action of G is stable under the base change $\mathbb{Q} \hookrightarrow \mathbb{Q}_{\ell_0}$ (§2.2.1). Part (b) follows by specialization. ∎

https://doi.org/10.1017/nmj.2023.19 Published online by Cambridge University Press

2.2.4. Interlude on étale group schemes

If $(S,\bar{s})$ is a geometrically pointed connected scheme, then a (not necessarily connected) finite étale group scheme $G \to S$ is tantamount to an action, by group automorphisms, of $\pi_1(S,\bar{s})$ on the abstract finite group $G_{\bar{s}}$. We will say that G is split if this action is trivial.

LEMMA 2.15. *Let $G/\mathbb{Z}[1/N]$ be an étale group scheme, and let $M \subset G$ be a normal sub-group scheme. Suppose that there exists an $\ell_0 \nmid N$ such that $M(\mathbb{F}_{\ell_0}) = G(\mathbb{F}_{\ell_0})$. Then, for ℓ in a set of positive density, $M(\mathbb{F}_\ell) = G(\mathbb{F}_\ell)$; and if G is split, then $M(\mathbb{F}_\ell) = G(\mathbb{F}_\ell)$ for every $\ell \nmid N$.*

Proof. Let $\mathrm{Spec}(R) \to \mathrm{Spec}(\mathbb{Z}[1/N])$ be a Galois étale cover which trivializes G and M; let $K = \mathrm{Frac}(R)$. Let $\ell \nmid N$ be a prime, and let λ be a prime of R lying over ℓ. The Artin symbol $(\lambda, K/\mathbb{Q})$ determines $G(\mathbb{F}_\ell)$ and $M(\mathbb{F}_\ell)$ as abstract groups, and the equality $M(\mathbb{F}_\ell) = G(\mathbb{F}_\ell)$ depends only on the conjugacy class $(\ell, K/\mathbb{Q})$.

Under the hypotheses, for any ℓ in the set of positive density for which $(\ell, K/\mathbb{Q}) = (\ell_0, K/\mathbb{Q})$, we have $M(\mathbb{F}_\ell) = G(\mathbb{F}_\ell)$.

The claim when G is split is trivially true, since then M is split, too, and $M(\mathbb{F}_\ell) = M(\mathbb{Z}[1/N])$ and $G(\mathbb{F}_\ell) = G(\mathbb{Z}[1/N])$. ∎

2.2.5. Representations of group schemes

We now turn to working with a smooth group scheme $G/\mathbb{Z}[1/N]$. We will often assume that G has reductive connected component of identity, that is, that for each $s \in \mathrm{Spec}\,\mathbb{Z}[1/N]$, $(G_s)^\circ$ is reductive. With this hypothesis, G° is a reductive group scheme over $\mathbb{Z}[1/N]$, and G/G° is étale [7, Prop. 3.1.3]. (In fact, we will use the same nomenclature, and deduce the same conclusions, for group schemes over an arbitrary base.) (Without this assumption, it is known that G/G° is an étale algebraic space [1, Lem. 2.1], but one may need to enlarge N to ensure that G/G° is representable by a scheme [2, Prop. 5.1.1].)

LEMMA 2.16. *Let $G/\mathbb{Z}[1/N]$ be a smooth affine algebraic group scheme with reductive connected component of identity. Let V be a free $\mathbb{Z}[1/N]$-module of finite rank, and let $\rho\colon G \to \mathrm{GL}_V$ be a homomorphism of algebraic groups. Let $\{\mathtt{G}_\ell\}$ be a collection of bounded subgroups of G. Suppose that there is an infinite collection of primes $\mathbb{L}$ such that, if $\ell \in \mathbb{L}$, then $r_\ell(\mathtt{G}_\ell, \rho) \ge r$.*

(a) *Then $r_\mathbb{Q}(G^\circ, \rho) \ge r$.*
(b) *Suppose that for some $\ell_0 \in \mathbb{L}$, $\mathtt{G}_{\ell_0}$ meets every geometrically irreducible component of G_{ℓ_0}. Then, for ℓ in a set of positive density, $r_\ell(G, \rho) \ge r$.*
(c) *In the setting of (b), suppose that G/G° is split. Then $r_\ell(G, \rho) \ge r$ for all $\ell \nmid N$.*

Proof. By Lemma 2.13, applied to the bounded subgroups $\mathtt{G}_\ell \cap G^\circ(\mathbb{F}_\ell)$ of G°, we find that $r(G^\circ_\mathbb{Q}, \rho) \ge r$; this proves (a).

Now suppose that there is some $\ell_0 \in \mathbb{L}$ for which $\mathtt{G}_{\ell_0}$ meets every geometrically irreducible component of G_{ℓ_0}. Let $\overline{G} = G/G^\circ$. The image of $\mathtt{G}_{\ell_0}$ in $\overline{G}(\mathbb{F}_{\ell_0})$ is all of $\overline{G}(\mathbb{F}_{\ell_0})$. Replace V with the eigenspace where G° acts with eigenvalue one; then V has rank at least r. The representation $\rho\colon G \to \mathrm{GL}_V$ factors through $\bar{\rho}\colon \overline{G} \to \mathrm{GL}_W$. There is some subrepresentation $\bar{\rho}_W : \overline{G} \to \mathrm{GL}_W \subset \mathrm{GL}_V$ of rank at least r such that $\mathtt{G}_{\ell_0}$ acts trivially on $W \otimes \mathbb{F}_{\ell_0}$.

Let M be the group scheme $M = \ker(\bar{\rho}_W) \subset \overline{G}$. We have $\mathtt{G}_{\ell_0} \subseteq M(\mathbb{F}_{\ell_0})$. Parts (b) and (c) now follow from Lemma 2.15. ∎

https://doi.org/10.1017/nmj.2023.19 Published online by Cambridge University Press

LEMMA 2.17. *Let $G/\mathbb{Z}_\ell$ be a smooth affine group scheme with reductive connected component of identity, and suppose that $\ell \nmid [G : G^\circ]$. Let V be a free $\mathbb{Z}_\ell$-module of finite rank, and let $\rho\colon G \to \mathrm{GL}_V$ be a homomorphism of algebraic groups. Suppose that $\dim V_{\mathbb{Q}_\ell}^{G^\circ_{\mathbb{Q}_\ell}} \geq r$ and $r_\ell(G,\rho) \geq r$. Then $r(G_{\mathbb{Q}_\ell},\rho) \geq r$.*

Proof. Replacing V with the subspace fixed by $G^\circ(\mathbb{Q}_\ell)$, we assume that the representation $G \to \mathrm{GL}_V$ factors through $\overline{G} := (G/G^\circ)$. Since $\overline{G}$ is étale, specialization of sections gives a bijection $\overline{G}(\mathbb{Z}_\ell) \to \overline{G}(\mathbb{Z}/\ell)$. Moreover, by Lang's theorem, $\overline{G}(\mathbb{Z}/\ell) = G(\mathbb{Z}/\ell)/G^\circ(\mathbb{Z}/\ell)$ and $\overline{G}(\mathbb{Z}_\ell) = G(\mathbb{Z}_\ell)/G^\circ(\mathbb{Z}_\ell)$.

Since $\ell \nmid [G : G^\circ]$, we may write V uniquely as a direct sum of irreducible $\overline{G}$ representations over $\mathbb{Z}_\ell$. By hypothesis, there is a free $\mathbb{Z}_\ell$-module $W \subset V$, stable under G and of rank at least r, such that $G(\mathbb{F}_\ell)$ acts trivially on $W \otimes \mathbb{F}_\ell$. We have a commutative diagram:

$$\begin{tikzcd}
G(\mathbb{Z}_\ell) \arrow[r] \arrow[d] & G(\mathbb{Z}_\ell)/G^\circ(\mathbb{Z}_\ell) \arrow[r] \arrow[d, "\sim"] & \mathrm{GL}_W(\mathbb{Z}_\ell) \arrow[d] \\
G(\mathbb{Z}/\ell) \arrow[r] & G(\mathbb{Z}/\ell)/G^\circ(\mathbb{Z}/\ell) \arrow[r] & \mathrm{GL}_W(\mathbb{Z}/\ell) \\
g \arrow[r, mapsto] & \bar{g} \arrow[r, mapsto] & \mathrm{id}_{W\otimes \mathbb{Z}/\ell}.
\end{tikzcd}$$

Suppose $g \in G(\mathbb{Z}_\ell)$, and let α be any eigenvalue of $\rho_W(g)$. On one hand, α is an mth root of unity for some $m | [G : G^\circ]$. On the other hand, by the commutativity of the diagram, $\alpha \equiv 1 \bmod \ell$. Consequently, $\alpha = 1$. Since $\rho_W(g) \in \mathrm{GL}_W(\mathbb{Z}_\ell)$ has finite order, it is semisimple, and we conclude that $\rho_W(g) = \mathrm{id}_W$. ∎

LEMMA 2.18. *Let $G/\mathbb{Z}[1/N]$ be a smooth affine group scheme with reductive connected component of identity. Let V be a free $\mathbb{Z}[1/N]$-module of finite rank, and let $\rho\colon G \to \mathrm{GL}_V$ be a homomorphism of algebraic groups. Let $\mathtt{G}_{\ell_0^\infty}$ be a Zariski dense subgroup of $G_{\mathbb{Q}_{\ell_0}}$. If $r(\mathtt{G}_{\ell_0^\infty},\rho) \geq r$, then $r_{\mathbb{Q}_\ell}(G,\rho) \geq r$ for ℓ in a set of positive density. If G/G° is split, then this holds for all ℓ.*

Proof. By Lemma 2.14, $r_\ell(G^\circ,\rho) \geq r$ for all ℓ. Using the same technique as in the proof of Lemma 2.16 to move from G° to G, we find that $r_{\mathbb{Q}}(G,\rho) \geq r$; for ℓ in a set of positive density, $r_\ell(G,\rho) \geq r$; and that this holds for all sufficiently large ℓ if G/G° is split. In particular, by Lemma 2.11(c), $\dim V_{\mathbb{Q}}^{G^\circ_{\mathbb{Q}}} \geq r$.

Now, let ℓ be any prime for which $r_\ell(G,\rho) \geq r$ and $\ell \nmid [G : G^\circ]$; by Lemma 2.17, $r_{\mathbb{Q}_\ell}(G,\rho) \geq r$. ∎

§3. Torsion points on abelian varieties

3.1 Torsion points and Galois representations

For the purpose of establishing notation, let A/K be an abelian variety over a perfect field. For a natural number N, we let $K_{A,N}$ be the field of definition of the N-torsion of A. We further let $K_{A,\ell^\infty} = \bigcup_n K_{A,\ell^n}$, and let $K_A = \bigcup_N K_{A,N}$ be the field obtained by adjoining the coordinates of all torsion points of A. Finally, we let $A_N = A[N](\overline{K})$ be the geometric N-torsion, and $A_{\ell^\infty} = \cup_n A_{\ell^n}$.

For a fixed prime ℓ, we have the usual representations

$$\rho_{A/K,\ell} : \mathrm{Gal}(K) \longrightarrow \mathrm{GL}(A_\ell) \tag{3.1}$$

$$\rho_{A/K,\ell^\infty} : \mathrm{Gal}(K) \longrightarrow \mathrm{GL}(T_\ell A)$$

with respective images $\Gamma_{A/K,\ell}$ and $\Gamma_{A/K,\ell^\infty}$.

3.1.1. Independence

Serre has shown that, while the ℓ-adic representations attached to an abelian variety are compatible, they are also independent.

For an abelian variety A/K and a prime ℓ, briefly let $K'_{A,\ell} := \bigcup_{\ell\nmid N} K(A_N)$. Say that A/K has *independent torsion fields* if, for each prime ℓ, the Galois extensions K_{A,ℓ^∞} and $K'_{A,\ell}$ are linearly disjoint over K. (Note that the compositum $K_{A,\ell^\infty}K'_{A,\ell}$ is simply K_A, the field generated by the torsion points of A.)

Lemma 3.1. *Let A/K be an abelian variety over a number field.*

(a) *There exists a finite extension K^{ind}/K such that A/K^{ind} has independent torsion fields.*
(b) *If L/K^{ind} is any algebraic extension, then A/L has independent torsion fields.*

Proof. See [38, théorème 1 and §3] or [3, §1]. ∎

Lemma 3.2. *Let A and B be abelian varieties over a number field K, and suppose that $A\times B$ has independent torsion fields. Then*

$$A[\ell^\infty](K_B) = A[\ell^\infty](K_{B,\ell^\infty}).$$

Proof. It suffices to show that $A[\ell^\infty](K_B) \subset A[\ell^\infty](K_{B,\ell^\infty})$. Assuming that $P \in A[\ell^\infty](K_B)$, we denote by $K(P)$ the extension of K by adjoining the coordinates of P. Then $K(P) \subset K_{A\times B,\ell^\infty}$. We also have $K(P) \subset K_B = K_{B,\ell^\infty}\cdot K'_{B,\ell}$. Notice that $A\times B$ has independent torsion fields, so $K \subset K_{A\times B,\ell^\infty} \cap K'_{B,\ell} \subset K_{A\times B,\ell^\infty} \cap K'_{A\times B,\ell} = K$, which tells us that every inclusion here is actually an equality. Hence, one has that

$$K(P) \subset K_{A\times B,\ell^\infty} \cap K_B = K_{A\times B,\ell^\infty} \cap K_{B,\ell^\infty} = K_{B,\ell^\infty}.$$

This means that $P \in A[\ell^\infty](K_{B,\ell^\infty})$. ∎

3.1.2. Connectedness

We let $\mathcal{G}_{A/K,\ell}$ be the Zariski closure of $\Gamma_{A/K,\ell^\infty}$ in $\mathrm{GL}_{H_1(A_{\overline{K}},\mathbb{Q}_\ell)} = \mathrm{GL}_{T_\ell A\otimes\mathbb{Q}_\ell}$, with connected component $\mathcal{G}^\circ_{A/K,\ell}$. In general, $\mathcal{G}_{A/K,\ell}$ does not have to be connected, but when K is a number field, $\mathcal{G}_{A/K,\ell}$ will be connected after a finite extension of K which is independent of ℓ.

Lemma 3.3. *Suppose $K/\mathbb{Q}$ is a finite extension. Then:*

(a) *The finite quotient group $\mathcal{G}_{A/K,\ell}/\mathcal{G}^\circ_{A/K,\ell}$ is independent of ℓ.*
(b) *There exists a finite extension K^{conn} of K such that, if L is any finite extension of K^{conn} and ℓ is any prime number, the corresponding $\mathcal{G}_{A/L,\ell}$ is connected.*

Proof. See [37] or [13, Prop. 6.14]. ∎

https://doi.org/10.1017/nmj.2023.19 Published online by Cambridge University Press

(In contrast, Example 5.7 will show that if K is algebraic but infinite, then such a finite connectedness extension need not exist.)

3.2 Mumford–Tate conjecture

This section is devoted to recalling the Mumford–Tate conjecture. In particular, we will review a result of Cadoret and Moonen [4, §1] and of Hindry and Ratazzi [8] which states that as ℓ varies, the ℓ-adic image of the Galois group is a bounded index subgroup of the $\mathbb{Z}_\ell$-points of the Mumford–Tate group.

Let K be a number field, embedded in $\mathbb{C}$. To ease notation slightly, we write $\mathrm{MT}(A)$ for the Mumford–Tate group of an abelian variety A over K, that is, $\mathrm{MT}(A) := \mathrm{MT}(H_1(A_{\mathbb{C}}, \mathbb{Q}))$ (cf. §4.1). This is a connected $\mathbb{Q}$-algebraic group. Let G_A be the Zariski closure of $\mathrm{MT}(A)$ in $\mathrm{GL}_{H_1(A_{\mathbb{C}},\mathbb{Z})}$; it is a group scheme over $\mathbb{Z}$. Then G_A is smooth over $\mathbb{Z}[1/N_A]$ for some positive integer N_A, and $G_{A,\mathbb{Q}} = \mathrm{MT}(A)$.

If $K = K^{\mathrm{conn}}$, then it is known that there is a natural inclusion $\Gamma_{A/K,\ell^\infty} \subset \mathrm{MT}(A)(\mathbb{Q}_\ell)$, and thus an inclusion $G_{A/K,\ell^\infty} \hookrightarrow \mathrm{MT}(A)_{\mathbb{Q}_\ell}$ of algebraic groups over $\mathbb{Q}_\ell$. The Mumford–Tate conjecture asserts that this inclusion is actually an isomorphism. More precisely, for every prime $\ell \nmid N$, both $G_{A,\ell} := G_A \times_{\mathbb{Z}[\frac{1}{N}]} \mathrm{Spec}\mathbb{Z}_\ell$ and $\mathcal{G}^\circ_{A/K,\ell}$ are subgroup schemes of $\mathrm{GL}_{H^1(A,\mathbb{Z}_\ell)}$. The following conjecture claims the comparison result of the two group schemes.

CONJECTURE 3.4 [4, Mumford–Tate conjecture]. *With the above notations,* $G_{A,\ell} = \mathcal{G}^\circ_{A/K,\ell}$.

REMARK 3.5. Conjecture 3.4 is equivalent to the usual statement

$$\mathcal{G}^\circ_{A/K,\mathbb{Q}_\ell} = \mathrm{MT}(A) \times_{\mathbb{Q}} \mathbb{Q}_\ell$$

for every prime ℓ [4].

In this article, the Mumford–Tate conjecture is a standing assumption we require in order to make any significant progress. The conjecture is known to be true for large classes of abelian varieties. For example, it is known that an absolutely simple abelian variety A of dimension g satisfies the Mumford–Tate conjecture in any of the following settings:

1. g is prime [32, 42, 43];
2. $g \le 3$ [23];
3. $\mathrm{End}_{\overline{K}}(A) = \mathbb{Z}$, and g satisfies certain numerical conditions (e.g., g is odd) [28];
4. A has complex multiplication [30, 45].

Our list is far from complete. See also [44] and the discussion in [22, §2.4.] for additional references and known results. Moreover, if the Mumford–Tate conjecture is true for abelian varieties A and B, then it is also true for their product $A \times B$ [6].

In the presence of the Mumford–Tate conjecture, we have good control over $\Gamma_{A/K,\ell^\infty}$.

THEOREM 3.6 [4, Th. A], [8, théorème 10.1]. *Let A be an abelian variety defined over K, assume $K = K^{\mathrm{conn}}$, and assume that the Mumford–Tate conjecture is true for A. Then the index $[G_A(\mathbb{Z}_\ell) : \Gamma_{A/K,\ell^\infty}]$ is bounded when ℓ varies.*

In particular, $\{\Gamma_{A/K,\ell}\}$ is a collection of bounded subgroups of G_A.

https://doi.org/10.1017/nmj.2023.19 Published online by Cambridge University Press

§4. Torsion-finite pairs of abelian varieties

4.1 Mumford–Tate groups for a pair of abelian varieties

Let A and B be abelian varieties over a subfield K of $\mathbb{C}$. Let G_A, G_B, and $G_{A\times B}$ denote the ($\mathbb{Z}$-models of) the Mumford–Tate groups of, respectively, A, B, and $A\times B$, and let sG_A, sG_B, and $sG_{A\times B}$ denote their respective Hodge groups. Recall that the Mumford–Tate group G_C of a complex abelian variety C is the $\mathbb{Q}$-algebraic hull of the morphism $\mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m \to \mathrm{Aut}(H_1(C,\mathbb{Q})\otimes_{\mathbb{Q}}\mathbb{R})$ defining the Hodge structure on $H_1(C,\mathbb{Q})$. Equivalently, it is the Tannakian fundamental group (really, the group which represents automorphisms of the fiber functor which sends a Hodge structure V to its underlying vector space $|V|$) of $\langle H_1(C,\mathbb{Q})\rangle$, the tensor category generated by the Hodge structure $H_1(C,\mathbb{Q})$. Since $H^1(C,\mathbb{Q})$ is dual to $H_1(C,\mathbb{Q})$, the tensor categories $\langle H_1(C,\mathbb{Q})\rangle$ and $\langle H^1(C,\mathbb{Q})\rangle$ coincide, and we may use either description to compute $\mathrm{MT}(C)$. We have $G_C/sG_C \cong \mathbb{G}_m$.

Since $H_1(A\times B,\mathbb{Q}) \cong H_1(A,\mathbb{Q})\oplus H_1(B,\mathbb{Q})$ is an object of $\langle H_1(A,\mathbb{Q}),H_1(B,\mathbb{Q})\rangle$, there is a canonical inclusion $\iota\colon G_{A\times B}\hookrightarrow G_A\times G_B$. Moreover, $H_1(A,\mathbb{Q})$ and $H_1(B,\mathbb{Q})$ are both objects of $\langle H_1(A\times B,\mathbb{Q})\rangle$. The corresponding inclusions $\langle H_1(A,\mathbb{Q})\rangle \hookrightarrow \langle H_1(A\times B,\mathbb{Q})\rangle$ and $\langle H_1(B,\mathbb{Q})\rangle \hookrightarrow \langle H_1(A\times B,\mathbb{Q})\rangle$ yield surjections $G_{A\times B}\twoheadrightarrow G_A$ and $G_{A\times B}\twoheadrightarrow G_B$. Thus, the three algebraic groups G_A, G_B, and $G_{A\times B}$ satisfy the hypotheses of Goursat's lemma (Lemma 2.1), and fit in a diagram as follows.

$$\begin{tikzcd} G_{A\times B} \arrow[r, hook, "\iota"] & G_A\times G_B \arrow[dl, two heads, "\pi_A"] \arrow[dr, two heads, "\pi_B"] & \\ G_A & & G_B \end{tikzcd} \tag{4.1}$$

Let $M_{A,B} = \ker(\pi_B\circ\iota)$; under the isomorphism $G_A\times\{e\}\cong G_A$, it is isomorphic to a normal algebraic subgroup $H_{A,B}$ of G_A. Define $M_{B,A}$ and $H_{B,A}$ in an analogous fashion. Because $H_1(A,\mathbb{Q})$ and $H_1(B,\mathbb{Q})$ have the same nonzero weight, $H_{A,B}\subset sG_A$ and $H_{B,A}\subset sG_B$. Consequently, the Hodge groups also satisfy the hypotheses of Goursat's lemma, that is, fit in a diagram:

$$\begin{tikzcd} sG_{A\times B} \arrow[r, hook, "\iota"] & sG_A\times sG_B \arrow[dl, two heads, "\pi_A"] \arrow[dr, two heads, "\pi_B"] & \\ sG_A & & sG_B \end{tikzcd}$$

In particular, $sG_{A\times B}$ is the inverse image in $sG_A\times sG_B$ of the graph of an isomorphism

$$\tfrac{sG_A}{H_{A,B}} \xrightarrow{\sim} \tfrac{sG_B}{H_{B,A}}\,. \tag{4.2}$$

For future use, we record the following observation.

Lemma 4.1. *Let A and B be complex abelian varieties. The following are equivalent.*

(a) *A and B are isogenous.*
(b) *$H_1(A,\mathbb{Q})$ and $H_1(B,\mathbb{Q})$ are isomorphic representations of* $\mathrm{sMT}(A\times B)$.

https://doi.org/10.1017/nmj.2023.19 Published online by Cambridge University Press

(c) *The canonical surjections* $\mathrm{sMT}(A\times B)\twoheadrightarrow \mathrm{sMT}(A)$ *and* $\mathrm{sMT}(A\times B)\twoheadrightarrow \mathrm{sMT}(B)$ *are isomorphisms, and* $H_1(A,\mathbb{Q})$ *and* $H_1(B,\mathbb{Q})$ *are isomorphic representations of this common group.*

Proof. The category $\langle H_1(A\times B,\mathbb{Q})\rangle$ is equivalent to the category of representations of $\mathrm{MT}(A\times B)$. So $H_1(A,\mathbb{Q})$ and $H_1(B,\mathbb{Q})$ are isomorphic in the category of Hodge structures, or equivalently in the full subcategory generated by $H_1(A\times B,\mathbb{Q})$, if and only if they are isomorphic representations of $\mathrm{MT}(A\times B)$. For weight reasons, it suffices to verify this for the Hodge group $\mathrm{sMT}(A\times B)$. Riemann's theorem—that the isogeny class of an abelian variety is determined by its Hodge structure—proves the equivalence of (a) and (b).

If A and B are isogenous, it is well known that $\mathrm{MT}(A\times B)\cong \mathrm{MT}(A)\cong \mathrm{MT}(B)$ (e.g., [21, Rem. 1.8]). Conversely, under the hypothesis of (c), weight considerations show that the corresponding hypothesis holds for Mumford–Tate groups, too. Now, use the fact that $\mathrm{MT}(A)$ is canonically isomorphic to the image of $\mathrm{MT}(A\times B)$ in $\mathrm{GL}_{H_1(A,\mathbb{Q})}$ and the analogous statement for B in order to deduce (b). ∎

Now, suppose that A and B have complex multiplication (see §5.1 for a review of this concept). Then $A\times B$ does, too, and the Mumford–Tate groups G_A, G_B, and $G_{A\times B}$ are all tori. Taking character groups in (4.1) yields a diagram of $\mathbb{Z}$-modules

$$\begin{tikzcd} X^*(G_{A\times B}) & & X^*(G_A)\times X^*(G_B) \arrow[ll, two heads] & & \\ & X^*(G_A) \arrow[ur, hook] & & & X^*(G_B) \arrow[ull, hook'] \end{tikzcd}. \tag{4.3}$$

In particular, we may use this diagram to compute $H_{A,B}$, a group of multiplicative type; it is the group whose character group is

$$X^*(H_{A,B}) = \frac{X^*(G_{A\times B})}{X^*(G_B)}. \tag{4.4}$$

If we identify $X^*(G_A)$ and $X^*(G_B)$ with their images under, respectively, the inclusions $(\pi_A\circ\iota)^*$ and $(\pi_B\circ\iota)^*$, we may rewrite this as

$$X^*(H_{A,B}) = \frac{X^*(G_A)+X^*(G_B)}{X^*(G_B)} \cong \frac{X^*(G_A)}{X^*(G_A)\cap X^*(G_B)}. \tag{4.5}$$

4.2 Galois representations for a pair of abelian varieties

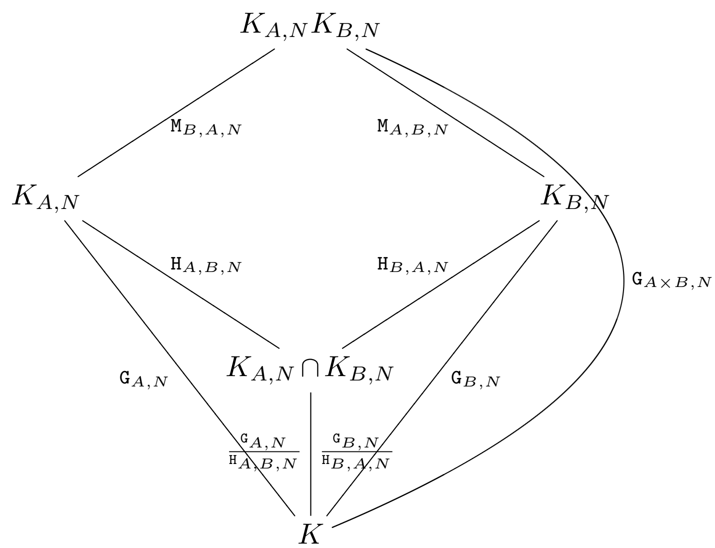
Now, further suppose that K is finitely generated over $\mathbb{Q}$, and assume that A and B satisfy the Mumford–Tate conjecture.

Since our main results concern potentially infinite torsion, we will assume that $A\times B$ has connected, independent Galois representations.

For a positive integer N, we identify the Galois group $\Gamma_{A/K,N}$ with a subgroup $\mathtt{G}_{A,N}$ of $G_A(\mathbb{Z}/N)$, and make similar identifications of the image of $\mathrm{Gal}(K)$ acting on the N-torsion

https://doi.org/10.1017/nmj.2023.19 Published online by Cambridge University Press

of B and of $A\times B$. The N-torsion fields of A and B are then arranged in the following tower, where each extension is labeled with its corresponding Galois group.



Let $\mathtt{H}_{A,B,\ell^\infty}=\varprojlim_n \mathtt{H}_{A,B,\ell^n}\subset H_{A,B}(\mathbb{Z}_\ell)$.

We have $\operatorname{Gal}(K_{B,\ell}K_{A,\ell}/K_{B,\ell})=\mathtt{M}_{A,B,\ell}\cong\mathtt{H}_{A,B,\ell}\subseteq H_{A,B}(\mathbb{F}_\ell)$, and $A[\ell](K_{B,\ell})$ is the set of elements of A_ℓ fixed by $\mathtt{H}_{A,B,\ell}$.

LEMMA 4.2. *Let A and B be abelian varieties over a number field K. Suppose that $A\times B$ has independent Galois representations. Then, for each prime ℓ, $A[\ell](K_B)$ is nontrivial if and only if $A[\ell](K_{B,\ell})$ is nontrivial.*

Proof. By independence, $A[\ell](K_B)=A[\ell](K_{B,\ell^\infty})$ (Lemma 3.2). Suppose that, for some $n>1$, $\#A[\ell](K_{B,\ell^n})>1$; equivalently, $\operatorname{Gal}(K_{B,\ell^n})$ has a nontrivial fixed point in A_ℓ. By Lemma 2.7, in order to show that $\operatorname{Gal}(K_{B,\ell})$ has a nontrivial fixed point in A_ℓ, it suffices to show that $[\operatorname{Gal}(K_{B,\ell^n}):\operatorname{Gal}(K_{B,\ell})]$ is a power of ℓ. This last claim follows from the inclusions

$$\frac{\operatorname{Gal}(K_{B,\ell})}{\operatorname{Gal}(K_{B,\ell^n})}\cong\operatorname{Gal}(K_{B,\ell^n}/K_{B,\ell})\hookrightarrow\{g\in\operatorname{Aut}(B_{\ell^n}):g\equiv\operatorname{id}\bmod\ell\}\subset 1+\ell\operatorname{End}(B_{\ell^n})\ .$$

∎

LEMMA 4.3. *Let A and B be abelian varieties over a number field K. Suppose that A and B satisfy the Mumford–Tate conjecture, and that $A\times B$ has connected Galois representations. Then:*

(a) $\left\{\mathtt{H}_{A,B,\ell}\cap H^\circ_{A,B}(\mathbb{F}_\ell)\right\}$ *is a collection of bounded subgroups of $H^\circ_{A,B}$; and*
(b) $\mathtt{H}_{A,B,\ell^\infty}\cap H^\circ_{A,B}(\mathbb{Z}_\ell)$ *is Zariski dense in $H^\circ_{A,B,\mathbb{Q}_\ell}$.*

Proof. Since $A\times B$ satisfies the Mumford–Tate conjecture, $\{\mathtt{G}_{A\times B,\ell}\}$ is bounded in $G_{A\times B}$ (Theorem 3.6). Now, apply Lemma 2.10(b) to the exact sequence

$$0\longrightarrow M_{A,B}\longrightarrow G_{A\times B}\longrightarrow G_B\longrightarrow 0$$

to deduce (a).

For a fixed ℓ, there exists an $n=n_\ell$ such that $\mathtt{G}_{A,\ell^\infty}$ contains $\ker(G_A(\mathbb{Z}_\ell)\to G_A(\mathbb{Z}_\ell/\ell^n))$, and so $\mathtt{H}_{A,B,\ell^\infty}\cap H^\circ_{A,B}(\mathbb{Z}_\ell)$ contains $\ker(H^\circ_{A,B}(\mathbb{Z}_\ell)\to H^\circ_{A,B}(\mathbb{Z}_\ell/\ell^n))$. Then counting points

https://doi.org/10.1017/nmj.2023.19 Published online by Cambridge University Press

(with values in $\mathbb{Z}_\ell/\ell^{m+n}$ for $m \gg 0$) shows that the Zariski closure of $\mathtt{H}_{A,B,\ell^\infty} \cap H^\circ_{A,B}(\mathbb{Z}_\ell)$, a closed subgroup of the irreducible variety $H^\circ_{A,B,\mathbb{Q}_\ell}$, must be all of $H^\circ_{A,B,\mathbb{Q}_\ell}$. ∎

4.3 Preliminaries on torsion finiteness

If two abelian varieties are isomorphic, or more generally isogenous, then it is easy to see that each is torsion infinite for the other:

LEMMA 4.4. *Let A and B be abelian varieties over a number field K.*

(a) *If A and A' are isogenous over K, and if B and B' are isogenous over K, then A is torsion finite for B over K if and only if A' is torsion finite for B' over K.*
(b) *If L/K is a finite extension, and if A_L is torsion finite for B_L over L, then A is torsion finite for B over K.*
(c) *If m and n are two positive integers, then A is torsion finite for B over K if and only if A^m is torsion finite for B^n over K.*

Proof. Let $g\colon B \to B'$ be an isogeny of exponent N; there is an isogeny $g'\colon B' \to B$ such that $g' \circ g = [N]_B$. Then, for any (not necessarily finite) field extension F/K, one has

$$\frac{1}{N} \cdot \#B'(F)_{\mathrm{tors}} \leq \#B(F)_{\mathrm{tors}} \leq N \cdot \#B'(F)_{\mathrm{tors}}.$$

In particular, $B(F)_{\mathrm{tors}}$ and $B'(F)_{\mathrm{tors}}$ are either both finite or infinite. Moreover, one can also deduce that $K_B = K_{B'}$. We deduce (a) after applying the same argument to A and A'.

Part (b) is obvious since L_B contains K_B.

Part (c) follows from the observation that $K_{C,N} = K_{C^r,N}$ for any abelian variety C/K and any natural numbers r and N. ∎

LEMMA 4.5. *Suppose A and B are abelian varieties over a field K. There exist a finite extension L and an isogeny of L-abelian varieties $\oplus_{i=1}^r A_i^{m_i} \to A_L$ with each A_i absolutely simple; and A is essentially torsion finite for B if and only if each A_i is essentially torsion finite for B_L.*

Proof. The existence of such an L and a factorization of A_L is standard; since we are only concerned with essential torsion finiteness, we may and do assume $L = K$. Then A is essentially torsion finite for B if and only if $\oplus_i A_i^{m_i}$ is (Lemma 4.4(a)), which obviously holds if and only if each summand $A_i^{m_i}$ is essentially torsion finite for B. By Lemma 4.4(c), this holds if and only if each A_i is essentially torsion finite for B. ∎

4.4 Potentially torsion-infinite pairs

If $H_{A,B}$ is connected and if A acquires infinite torsion over K_B, then A acquires ℓ-power torsion for all ℓ:

LEMMA 4.6. *Let A and B be abelian varieties over a number field K. Suppose that A and B satisfy the Mumford–Tate conjecture, and that $A \times B$ has connected independent Galois representations.*

Suppose that $H_{A,B}$ is connected. Then the following are equivalent:

(a) *$A(K_B)_{\mathrm{tors}}$ is infinite.*
(b) *For all ℓ, $A[\ell](K_B)$ is nontrivial.*
(c) *For all ℓ, $A[\ell^\infty](K_B)$ is infinite.*

https://doi.org/10.1017/nmj.2023.19 Published online by Cambridge University Press

Proof. It suffices to show that (a) implies each of (b) and (c). Note that, since $A\times B$ has independent representations, $A[\ell^\infty](K_B) = A[\ell^\infty](K_{B,\ell^\infty})$.

By Lemma 4.3, $\{\mathtt{H}_{A,B,\ell}\}$ is a collection of bounded subgroups of the connected group $H_{A,B}$, and each $\mathtt{H}_{A,B,\ell^\infty}$ is Zariski dense in $H_{A,B,\mathbb{Q}_\ell}$.

Suppose that (a) holds; then $A[\ell](K_{B,\ell^\infty})$ is nontrivial for infinitely many ℓ, or $A[\ell_0^\infty](K_{B,\ell_0^\infty})$ is infinite for some ℓ_0. In the former case, $A[\ell](K_{B,\ell})$ is nontrivial for infinitely many ℓ (Lemma 4.2), and thus $r_\ell(H_{A,B},\rho_A)$ is positive for infinitely many ℓ; in the latter, $r_{\mathbb{Q}_{\ell_0}}(H_{A,B},\rho_A)$ is positive. Thus, by Lemma 2.13 or 2.14, $\rho_{\mathbb{Q}}(H_{A,B},\rho_A)$ is positive; therefore, so is $\rho_\ell(H_{A,B},\rho_A)$ and $\rho_{\mathbb{Q}_\ell}(H_{A,B},\rho_A)$ for each ℓ. Therefore, both (b) and (c) hold. ∎

In the absence of a connectedness hypothesis on $H_{A,B}$, our results are less balanced. Moreover, the Mumford–Tate conjecture does not immediately imply that $\mathtt{H}_{A,B,\ell}$ meets every geometrically irreducible component of $H_{A,B,\ell}$. In situations where this is known, however, we can deduce the following statement.

LEMMA 4.7. *Let A and B be abelian varieties over a number field K. Suppose that A and B satisfy the Mumford–Tate conjecture and that $A\times B$ has connected, independent Galois representations.*

Suppose that $A(K_B)_{\mathrm{tors}}$ is infinite, and that there exists some ℓ_0 such that $A[\ell_0](K_{B,\ell_0})$ is nontrivial. Additionally, assume that $\mathtt{H}_{A,B,\ell_0}$ meets every geometrically irreducible component of H_{A,B,ℓ_0}, and $H_{A,B,\mathbb{Z}_{\ell_0}}$ is smooth. Then, for ℓ in a set of positive density, $A[\ell](K_B)$ is nontrivial and $A[\ell^\infty](K_B)$ is infinite.

Proof. If $A[\ell](K_B)$ is nontrivial for infinitely many ℓ, this follows from Lemma 2.16, applied to the collection of bounded subgroups $\{\mathtt{H}_{A,B,\ell}\}$ of $H_{A,B}$. If instead there exists some ℓ_1 such that $A[\ell_1^\infty](K_B)$ is infinite, then as in the proof of Lemma 2.18, we find that $r_{\mathbb{Q}}(H^\circ_{A,B},\rho_A)$ is positive; and that for ℓ in a set of positive density (namely, the set of ℓ relatively prime to $[H_{A,B}:H^\circ_{A,B}]$ and with the same Artin symbol as ℓ_0 in some finite splitting field for $H_{A,B}/H^\circ_{A,B}$), $r_\ell(H_{A,B},\rho_A)$ is also positive. ∎

The hypothesis on ℓ_0 in Lemma 4.7 seems difficult to work with abstractly, although in explicit examples one can compute $H_{A,B}/H^\circ_{A,B}$ (e.g., Example 5.7) and thereby make progress. However, we can still make a uniform statement purely in terms of torsion, at the cost of surrendering some control of the precise field over which A acquires infinite ℓ-torsion.

LEMMA 4.8. *Let A and B be abelian varieties over a number field K. Suppose that A and B satisfy the Mumford–Tate conjecture and that $A\times B$ has connected independent Galois representations.*

Suppose that $A(K_B)_{\mathrm{tors}}$ is infinite. Let $N_{A,B}=[H_{A,B}:H^\circ_{A,B}]$.

(a) *We have $r_{\mathbb{Q}}(H^\circ_{A,B},\rho_A)>0$.*
(b) *For each ℓ, there exists a finite extension $\widetilde{K_{B,\ell^\infty}}$ of K_{B,ℓ^∞} such that $[\widetilde{K_{B,\ell^\infty}}:K_{B,\ell^\infty}]|N_{A,B}$ and $A[\ell^\infty](\widetilde{K_{B,\ell^\infty}})$ is infinite.*
(c) *For each ℓ, there exists some n_ℓ such that $A[\ell^\infty](K_{B,\ell^\infty}K_{A,\ell^{n_\ell}})$ is infinite; and if $\ell\nmid N_{A,B}$, then we may take $n_\ell=1$. (In fact, $K_{B,\ell^\infty}\subseteq\widetilde{K_{B,\ell^\infty}}\subseteq K_{B,\ell^\infty}K_{A,\ell^{n_\ell}}$.)*

Proof. Since $A(K_B)_{\mathrm{tors}}$ is infinite, there exist infinitely many ℓ such that $A[\ell](K_B)$ is nontrivial, or there is some ℓ_0 such that $A[\ell_0^\infty](K_B)$ is infinite. By Lemma 2.16 or 2.14 as appropriate, $r:=r_{\mathbb{Q}}(H^\circ_{A,B},\rho_A)>0$.

https://doi.org/10.1017/nmj.2023.19 Published online by Cambridge University Press

Now, fix a prime ℓ. Let $\widetilde{K_{B,\ell^\infty}}$ be the smallest extension of K_{B,ℓ^∞} for which $\rho_{A,\ell^\infty}(\mathrm{Gal}(\widetilde{K_{B,\ell^\infty}})) \subseteq H^\circ_{A,B}(\mathbb{Z}_\ell)$. Then $[\widetilde{K_{B,\ell^\infty}} : K_{B,\ell^\infty}] | N_{A,B}$, and $\mathrm{rank}_{\mathbb{Z}_\ell} A[\ell^\infty](\widetilde{K_{B,\ell^\infty}}) \geq r > 0$.

Moreover, we have inclusions $K_{B,\ell^\infty} \subseteq \widetilde{K_{B,\ell^\infty}} \subseteq (K_{B,\ell^\infty})_{A,\ell^\infty}$. Since the first extension is finite, there exists some n such that $\widetilde{K_{B,\ell^\infty}} \subseteq (K_{B,\ell^\infty})_{A,\ell^n}$. Now, if $n \geq 2$, then $\mathrm{Gal}((K_{B,\ell^\infty})_{A,\ell^n}/(K_{B,\ell^\infty})_{A,\ell})$ is a group whose order is a power of ℓ. Consequently, if $\ell \nmid N_{A,B}$, then $\widetilde{K_{B,\ell^\infty}} \subseteq (K_{B,\ell^\infty})_{A,\ell} = K_{A,\ell}K_{B,\ell^\infty}$. ∎

Remark 4.9. In the context of Lemma 4.8(b), one might hope that there is a finite extension L_B of K_B such that $A[\ell](L_B)$ is nontrivial for each ℓ. (Even more optimistically, one might hope that such an L_B is the compositum of K_B and a finite extension of K.) However, there is no independence-of-ℓ connectedness result for Galois representations of infinite algebraic extensions of $\mathbb{Q}$ to which one might appeal. In fact, we will see below (5.7) that in general, no such uniform-in-ℓ finite extension exists. In this sense, Lemma 4.8(b) is optimal without additional hypotheses.

Theorem 4.10. *Let A and B be abelian varieties over a number field K. Suppose that A and B satisfy the Mumford–Tate conjecture and that A is absolutely simple. Then the following are equivalent:*

(a) *A is potentially torsion infinite for B;*
(b) $\dim H_{A,B} = 0$*;*
(c) $\dim G_{A\times B} = \dim G_B$*;*
(d) $\mathrm{rank}\, G_{A\times B} = \mathrm{rank}\, G_B$*; and*
(e) *there exists a finite extension K' of K such that for all sufficiently large ℓ, $K'_{A,\ell^\infty} \subset K_B K'_{A,\ell}$, and thus $A[\ell^\infty](K_B K'_{A,\ell}) = A_{\ell^\infty}$.*

Proof. Suppose that A is potentially torsion infinite for B. Then, possibly after replacing K with a finite extension, we find that $r := r_{\mathbb{Q}}(H^\circ_{A,B}, \rho_A) > 0$ (Lemma 4.8(a)). Note that $H_{A,B}$, and thus $H^\circ_{A,B}$, are normal subgroups of $\mathrm{MT}(A)$ (Lemma 2.4).

Because A is absolutely simple and $\mathrm{MT}(A)$ is reductive, $H_1(A,\mathbb{Q})$ is an irreducible representation of $\mathrm{MT}(A)$. Now, $H_1(A,\mathbb{Q})^{H^\circ_{A,B}}$ is a sub-$\mathrm{MT}(A)$-representation (Lemma 2.6) of $H_1(A,\mathbb{Q})$. Since $\dim H_1(A,\mathbb{Q})^{H^\circ_{A,B}} = r > 0$ and $H_1(A,\mathbb{Q})$ is irreducible, it follows that $H_1(A,\mathbb{Q})^{H^\circ_{A,B}} = H_1(A,\mathbb{Q})$. This implies that $H^\circ_{A,B}$ is trivial, and thus $\dim H_{A,B} = 0$.

The converse, that (b) implies (a), is easy. Indeed, suppose $\dim H_{A,B} = 0$, and fix a prime ℓ. Then $\mathrm{Gal}(K_B)$ acts on $T_\ell A$ through a subgroup of $H_{A,B}(\mathbb{Z}_\ell)$, which is by hypothesis a finite group; after passage to a finite extension, the Galois group acts trivially on all ℓ-power torsion points.

The equivalence of (b)–(d) is a standard observation about reductive groups (Lemma 2.3).

Now suppose (b) holds. After replacing K with a suitable finite extension, we may and do assume that $A\times B$ has connected independent Galois representations. Then Lemma 4.8(c) shows that there exists some n_ℓ such that $\mathrm{Gal}(K_B K_{A,\ell^{n_\ell}})$ acts trivially on $T_\ell A$ and that if $\ell \nmid [H_{A,B} : H^\circ_{A,B}]$, then we may take $n_\ell = 1$.

The proof is completed with the trivial observation that (e) implies (a). ∎

Remark 4.11. In Theorem 4.10, by Lemma 4.5, it suffices to assume that A is absolutely isotypic.

https://doi.org/10.1017/nmj.2023.19 Published online by Cambridge University Press

Corollary 4.12. *Let B/K be an abelian variety over a number field for which the Mumford–Tate conjecture holds, and let E/K be an elliptic curve with complex multiplication. Then either E is potentially torsion infinite for B or $sG_{E\times B} = sG_E \times sG_B$.*

Proof. The Mumford–Tate conjecture holds for E and thus for $E \times B$, and E is visibly absolutely simple; it therefore suffices to show that if E is essentially torsion finite for B, then the special Mumford–Tate group of the product is the product of the special Mumford–Tate groups. By Theorem 4.10, $\dim H_{E,B} > 0$. Since $H_{E,B}$ is a positive-dimensional subgroup of the one-dimensional torus sG_E, it follows that $H_{E,B} = sG_E$, and thus $sG_{E\times B} = sG_E \times sG_B$ (Remark 2.2). □

Corollary 4.13. *Let A and B be abelian varieties over a number field, of respective dimensions d_A and d_B. Suppose that A and B satisfy the Mumford–Tate conjecture, that A is absolutely simple, and that*

$$\log_2 d_A \geq 3d_B - 1. \tag{4.6}$$

Then A is essentially torsion finite for B.

Proof. Let r_A and r_B denote the respective ranks of the Mumford–Tate groups of A and B. On one hand, we have the trivial bound $r_B \leq d+1$. On the other hand, a weak form of [26, Th. 1.2] implies that $r_A \geq \frac{1}{3}(\log_2 d_A + 2)$. Therefore, hypothesis (4.6) implies that $r_A > r_B$. Since $\operatorname{rank}(G_A) - \operatorname{rank}(G_B) = \operatorname{rank}(H_{A,B}) - \operatorname{rank}(H_{B,A})$, we find that the rank of $H_{A,B}$ is positive, and thus $\dim H_{A,B} > 0$. Now, apply Theorem 4.10. □

Corollary 4.14. *Suppose that A and B are two absolutely simple abelian varieties over K and that the Mumford–Tate conjecture holds for $A \times B$. Then the following are equivalent.*

(a) *A and B are mutually potentially torsion infinite.*
(b) *The natural surjections $G_{A\times B} \to G_A$ and $G_{A\times B} \to G_B$ are isogenies.*
(c) *The natural surjections $sG_{A\times B} \to sG_A$ and $sG_{A\times B} \to sG_B$ are isogenies.*
(d) $\dim(G_{A\times B}) = \dim G_A = \dim G_B$.
(e) $\dim(sG_{A\times B}) = \dim sG_A = \dim sG_B$.
(f) $\operatorname{rank}(G_{A\times B}) = \operatorname{rank} G_A = \operatorname{rank} G_B$.
(g) $\operatorname{rank}(sG_{A\times B}) = \operatorname{rank} sG_A = \operatorname{rank} sG_B$.

At the opposite extreme, we have the following corollary.

Corollary 4.15. *Suppose that A and B are two absolutely simple abelian varieties over K and that the Mumford–Tate conjecture holds for $A \times B$. If $sG_{A\times B} = sG_A \times sG_B$, then A and B are mutually essentially torsion finite.*

As we noted in the introduction, Ichikawa [10] and Lombardo [16] have given sufficient criteria for a pair of abelian varieties (A,B) to satisfy $sG_{A\times B} = sG_A \times sG_B$; a typical example is when A and B are nonisogenous abelian varieties of odd dimension with absolute endomorphism ring $\mathbb{Z}$.

§5. Applications

In this section, we apply Theorem 4.10 to the following two classes of abelian varieties:

(a) CM abelian varieties;
(b) abelian varieties with dimension smaller than 4.

https://doi.org/10.1017/nmj.2023.19 Published online by Cambridge University Press

Since the Mumford–Tate groups, and in fact the Mumford–Tate conjecture, are known for these classes of varieties, we are able to obtain some unconditional results.

More precisely, in §5.1, after a brief review of CM abelian varieties, we give a criterion (Theorem 5.4) to decide the essential torsion finiteness of a pair of CM abelian varieties in terms of their CM types. This criterion, compared with Theorem 4.10, has the advantage of being effective. In particular, one can use it to create examples of non-isogenous but potentially torsion-infinite pairs. This is demonstrated in Examples 5.12 and 5.7. Moreover, since the Mumford–Tate group of a CM abelian variety is a torus—indeed, this characterizes CM abelian varieties—our theorem indicates a way to describe the extra Hodge classes on the product of CM abelian varieties via relations among the characters of their Mumford–Tate groups. We also explore the relation between extra Hodge classes and torsion infiniteness in the CM case; this is the main content of §5.2.

In a different direction, thanks to the classification of Mumford–Tate groups of low-dimension abelian varieties (see [23, §2] for instance), one is able to analyze the equivalent conditions in Theorem 4.10 for each pair of realizable Mumford–Tate groups. The result of this study is Theorem 5.14. In particular, this theorem allows the comparison of Type IV varieties, which is not covered by the result of [16]. Details are given in §5.3.

5.1 CM abelian varieties

5.1.1. CM types and Mumford–Tate groups

We start by briefly reviewing some background on CM abelian varieties. See [20] and [39] for more details.

Let A be a g-dimensional abelian variety over a number field K, and let E be a CM-algebra of dimension $2g$. We say A has complex multiplication by E if there exists an embedding of algebras $i: E \hookrightarrow \mathrm{End}^0(A) := \mathrm{End}(A) \otimes \mathbb{Q}$. In this case, we call A a CM abelian variety and say that A has CM by E. (If there is some finite extension L/K such that A_L has CM by E, we will say that A has potential CM by E.)

The embedding i induces an E-action on the Lie algebra $\mathrm{Lie}(A_{\mathbb{C}})$. The character of this E-representation is given by $\sum_{\varphi \in \Phi} \varphi$ for some subset $\Phi \subset \mathrm{Hom}_{\mathbb{Q}}(E, \mathbb{C})$. Let c be the complex conjugation on $\mathbb{C}$. Then

$$\Phi \sqcup \Phi^c = \mathrm{Hom}_{\mathbb{Q}}(E, \mathbb{C}), \tag{5.1}$$

where $\Phi^c = \{c \circ \varphi | \varphi \in \Phi\}$. The pair (E, Φ) is called the *CM type* of A (or (A, i)). Conversely, for any pair (E, Φ) where E is a CM algebra and the subset $\Phi \subset \mathrm{Hom}_{\mathbb{Q}}(E, \mathbb{C})$ satisfies (5.1), there exists a CM abelian variety A_0 defined over a number field equipped with an action $i: E \hookrightarrow \mathrm{End}^0(A_0)$ with CM type (E, Φ). The correspondence between CM types and isogeny classes of abelian varieties with action by a CM algebra is well understood (see, e.g., [20, Prop. 3.12]).

Fix an embedding $K \hookrightarrow \mathbb{C}$, and let $\overline{K}$ be the algebraic closure of K in $\mathbb{C}$. Then any embedding $E \hookrightarrow \mathbb{C}$ factors through $\overline{K}$, and we have a bijection $\mathrm{Hom}_{\mathbb{Q}}(E, \overline{K}) \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{Q}}(E, \mathbb{C})$. Let $\widetilde{E}$ be a Galois extension of $\mathbb{Q}$ in $\overline{K}$ which splits E. Then the Galois group $\mathrm{Gal}(\widetilde{E}/\mathbb{Q})$ acts on $\mathrm{Hom}_{\mathbb{Q}}(E, \overline{K})$ by left composition. Let $H = H_{(E,\Phi)}$ be the group $\{\sigma \in \mathrm{Gal}(\widetilde{E}/\mathbb{Q}) | \Phi^{\sigma} = \Phi\}$. The fixed field $E^* := \widetilde{E}^H$ of H is called the reflex field of the CM type (E, Φ).

We now introduce the reflex norm associated with the CM abelian variety A. Recall (§2.1) that for any finite extension F of $\mathbb{Q}$, we let $T^F := \mathrm{Res}_{F/\mathbb{Q}} \mathbb{G}_m$ be the Weil restriction

https://doi.org/10.1017/nmj.2023.19 Published online by Cambridge University Press

of the multiplicative group. We define $\widetilde{\Phi} := \{\widetilde{\varphi} \in \mathrm{Hom}_{\mathbb{Q}}(\widetilde{E}, \overline{K}) \mid \widetilde{\varphi}|_E \in \Phi\}$. Since $\widetilde{E}/\mathbb{Q}$ is Galois, $\mathrm{Hom}_{\mathbb{Q}}(\widetilde{E}, \overline{K})$ is a torsor under $\mathrm{Gal}(\widetilde{E}/\mathbb{Q})$, and the choice of embedding $\widetilde{E} \hookrightarrow \overline{K}$ gives a bijection between these two sets. We use this to define, for $\phi \in \mathrm{Hom}_{\mathbb{Q}}(\widetilde{E}, \overline{K})$, the (group-theoretic) inverse $\phi^{-1} \in \mathrm{Hom}_{\mathbb{Q}}(\widetilde{E}, \overline{K})$.

Let $\widetilde{\Phi}^{-1}$ be the set $\{\widetilde{\varphi}^{-1} \mid \widetilde{\varphi} \in \widetilde{\Phi}\}$. The map $N_{\widetilde{\Phi}^{-1}} : \widetilde{E}^* \to \widetilde{E}^*$ given by $N_{\widetilde{\Phi}^{-1}}(a) = \prod_{\sigma \in \widetilde{\Phi}^{-1}} \sigma(a)$ defines a map of algebraic tori $N_{\widetilde{\Phi}^{-1}} : T^{\widetilde{E}} \to T^{\widetilde{E}}$. This map factors through T^{E^*} and has image contained in T^E. More precisely, we have a commutative diagram of $\mathbb{Q}$-tori

$$\begin{array}{ccc} T^{\widetilde{E}} & \xrightarrow{N_{\widetilde{\Phi}^{-1}}} & T^{\widetilde{E}} \\ {\scriptstyle N_{\widetilde{E}/E^*}}\big\downarrow & & \big\uparrow \\ T^{E^*} & \xrightarrow{N_{\Phi}} & T^{E} \end{array},$$

where $N_{\widetilde{E}/E^*}$ is the usual norm map and N_Φ is called the reflex norm of (E, Φ). For any finite extension L/E^*, we define $N_{L,\Phi} = N_\Phi \circ N_{L/E^*}$. We will call $N_{L,\Phi}$ the L-reflex norm of (E, Φ). Note that N_{L/E^*} is a surjective map. The image of the map $N_{L,\Phi} : T^L \to T^E$ is independent of L, and we denote it by T_Φ. See, for example, [45, Lem. 4.2] for an explicit calculation of $X^*(T_\Phi)$.

Consider the special case where E is a Galois extension of $\mathbb{Q}$. Then E contains the reflex field E^*, and T_Φ, as the image of T^E under $N_{E,\Phi} : T^E \to T^{E^*} \to T^E$, is a *quotient* of T^E. Inside

$$X^*(T^E) \cong \mathbb{Z}\langle \sigma | \sigma \in \mathrm{Gal}(E/\mathbb{Q})\rangle, \tag{5.2}$$

we find that

$$X^*(T_\Phi) \cong \mathbb{Z}\langle \sum_{\varphi \in \Phi} (\sigma \circ \varphi^{-1}) \mid \sigma \in \mathrm{Gal}(E/\mathbb{Q})\rangle \subset X^*(T^E). \tag{5.3}$$

LEMMA 5.1. *If A is a CM abelian variety over a number field, then the Mumford–Tate conjecture holds for A. Moreover, if (E, Φ) is a CM-type for A, then $\mathrm{MT}(A) \cong T_\Phi$.*

Proof. This is due to Pohlmann [30, Th. 5] (see also [45, Lem. 4.2] for a modern proof). (In fact, while [45] focuses on simple abelian varieties, the proof given there works verbatim in the non-simple case, too.) ∎

REMARK 5.2. Note in particular that (the character group of) the torus $\mathrm{MT}(A)$ can be explicitly described using a CM type (E, Φ) of A. It is usually more convenient to assume that E is Galois over $\mathbb{Q}$; and in studying the essential torsion finiteness problem for CM abelian varieties, we can always do this. Indeed, if $E/\mathbb{Q}$ is not Galois, choose a CM Galois extension $E'/\mathbb{Q}$ such that $E \subseteq E'$, and let $n = [E' : E]$. Then A^n has a CM type (E', Φ') where $\Phi' = \{\sigma \in \mathrm{Gal}(E'/\mathbb{Q}) \mid \sigma|_E \in \Phi\}$. Moreover, $\mathrm{MT}(A) \cong \mathrm{MT}(A^n)$; and if A is defined over a field K, then $K_A = K_{A^n}$ (Lemma 4.5). Thus, for our purposes, we may restrict our attention to *Galois* CM fields.

5.1.2. Galois representations

For use in later examples, we recall the calculation of the Galois representation of a CM abelian variety following [39, §7] and [45, §3]. Let A/K be an abelian variety with complex multiplication by E. Let ℓ be a rational prime which does not divide the index

https://doi.org/10.1017/nmj.2023.19 Published online by Cambridge University Press

$[\mathcal{O}_E : \mathrm{End}(A)]$. (This condition only rules out finitely many primes. Alternatively, for our applications, we may replace K by a finite extension and adjust A in its isogeny class, in which case we may assume that $\mathrm{End}(A) = \mathcal{O}_E$.) Then $\mathcal{O}_{E_\ell} := \mathcal{O}_E \otimes \mathbb{Z}_\ell$ is a direct sum of discrete valuation rings, and the Tate module $T_\ell A$ is free of rank one over $\mathcal{O}_{E_\ell}$. The Galois group of K acts E-linearly on $T_\ell A$, and so the Galois representation $\rho_{A/K,\ell^\infty}$ of $\mathrm{Gal}(K)$ factors through $\mathrm{Gal}(K)^{\mathrm{ab}}$. Composing the ℓ-adic representation with the Artin reciprocity map (in the idelic formulation of class field theory), one obtains a continuous group homomorphism which we still denote by $\rho_{A/K,\ell^\infty}$:

$$\begin{tikzcd} \mathbb{A}_K^\times \arrow[r, "\widetilde{\rho}_{A/K,\ell^\infty}"] \arrow[d, "\mathrm{art}"'] & \mathcal{O}_{E_\ell}^\times = T^E(\mathbb{Z}_\ell) \subset \mathrm{GL}(T_\ell A). \\ \mathrm{Gal}(K)^{\mathrm{ab}} \arrow[ur, "\rho_{A/K,\ell^\infty}"'] & \end{tikzcd}$$

After possibly replacing K with a finite extension, we now assume that K contains E^*, the reflex field of the CM type of A, so that the reflex norm $N_{K,\Phi}$ is defined (§5.1). Then, by [39, Ths. 10 and 11], we can concretely describe the representation $\widetilde{\rho}_{A/K,\ell^\infty}$ by

$$\mathbb{A}_K^\times \xrightarrow{\widetilde{\rho}_{A/K,\ell^\infty}} T^E(\mathbb{Z}_\ell) = \mathcal{O}_{E_\ell}^\times, \tag{5.4}$$

$$a = (a_v)_v \longmapsto \varepsilon(a) N_{K,\Phi,\ell}(a_\ell^{-1}).$$

Here, $a_\ell = (a_v)_{v|\ell}$ denotes the component of a in $K_\ell^\times = \prod_{v|\ell} K_v^\times$; and the map

$$N_{K,\Phi,\ell} = N_{K,\Phi} \times \mathbb{Q}_\ell : K_\ell^\times = (K \otimes \mathbb{Q}_\ell)^\times \longrightarrow E_\ell^\times = (E \otimes \mathbb{Q}_\ell\ )^\times$$

is induced by the reflex norm map from T^K to T^E; and

$$\varepsilon : \mathbb{A}_K^\times \longrightarrow E^\times$$

is the unique homomorphism satisfying the following conditions:

(a) The restriction of ε to $K^\times$ is the reflex norm map $N_{K,\Phi} : K^\times \to E^\times$.
(b) The homomorphism ε is continuous, in the sense that its kernel is open in $\mathbb{A}_K^\times$.
(c) There is a finite set S of places of K, including the infinite ones and those where A has bad reduction, such that

$$\varepsilon(a) = \prod_{v \notin S} \pi_v^{\nu(a_v)} \quad \text{for all } a = (a_v) \text{ with } a_v = 1 \text{ when } v \in S,$$

where each π_v is the Frobenius element attached to v [39, p. 511].

5.1.3. Nondegenerate abelian varieties

Let A/K be an abelian variety with CM type (E, Φ). Recall that A is called nondegenerate if $\dim \mathrm{MT}(A)$ is maximal, that is, $\dim \mathrm{MT}(A) = \dim A + 1$. Recall that $\mathcal{G}_{A/K,\mathbb{Q}_\ell}$ is the Zariski closure of the ℓ-adic representation image (cf. §3.1.2).

LEMMA 5.3. *Let A/K be an abelian variety with nondegenerate CM type (E, Φ). Then, for each ℓ, $\mathcal{G}_{A/K,\mathbb{Q}_\ell} = \mathrm{MT}(A) \times_{\mathbb{Q}} \mathbb{Q}_\ell$, that is, $K = K^{conn,A}$.*

https://doi.org/10.1017/nmj.2023.19 Published online by Cambridge University Press

Proof. By Lemma 3.3, it suffices to prove the statement for a single ℓ, and so we assume that $\ell \nmid [\mathcal{O}_E : \mathrm{End}(A)]$. As we have seen above, $T_\ell A$ admits commuting actions by $\mathcal{O}_{E_\ell}$ and $\mathrm{Gal}(K)$, and thus the ℓ-adic representation $\mathrm{Gal}(K) \to \mathrm{Aut}(T_\ell A)$ factors through $\mathcal{O}_{E_\ell}^\times$. A choice of K-rational polarization on A induces a symplectic form ψ on $T_\ell A$, which is also preserved by $\mathrm{Gal}(K)$ up to a scaling. Thus, the image of $\mathrm{Gal}(K)$ in $\mathrm{Aut}(T_\ell A)$ is contained in $\mathcal{O}_{E_\ell}^\times \cap \mathrm{GSp}(T_\ell A, \psi)$. Since these are the $\mathbb{Z}_\ell$-points of a maximal torus in $\mathrm{GSp}_{2\dim A}$—indeed, both $\mathrm{rank}\,\mathrm{GSp}_{2\dim A}$ and $\dim \mathrm{MT}(A)$ are $1 + \dim A$—the result follows. ∎

5.1.4. Torsion finiteness

With this preparation, we can now use Theorem 4.10 to characterize the essential torsion finiteness of CM abelian varieties in terms of CM types.

Recall that an abelian variety A over a field K is called *isotypic* if it is isogenous to a power of a simple abelian variety over the same field K, that is, up to isogeny, A has a unique simple factor (see [5, Defn. 1.2.5.2]). Any CM abelian variety is isogenous to a product of isotypic CM abelian varieties (see [5, Prop. 1.3.2.1]), and an isotypic CM abelian variety is geometrically isotypic (see [5, Cor. 1.3.7.2]).

To state our next result, it is more convenient to name our abelian varieties A_1 and A_2. In this case, we will change notation slightly and write, for example, G_1 and H_{12} for G_{A_1} and H_{A_1,A_2}.

THEOREM 5.4. *Let A_1 and A_2 be two isotypic abelian varieties over a number field K, with A_i of potential CM type (E_i, Φ_i). Let $T_i = T_{\Phi_i} = \mathrm{MT}(A_i)$ be the Mumford–Tate group of A_i, and let $T_{12} = \mathrm{MT}(A_1 \times A_2)$. Use the surjection $T_{12} \twoheadrightarrow T_i$ to identify $X^*(T_i)$ with a submodule of $X^*(T_{12})$. Then either:*

(a) $X^*(T_1) \otimes \mathbb{Q} \subseteq X^*(T_2) \otimes \mathbb{Q}$. *Then A_1 is potentially torsion infinite for A_2. For each ℓ, there exists a finite extension $\widetilde{K}_\ell/K$ such that*

$$A_1[\ell^\infty]((\widetilde{K}_\ell)_{A_2,\ell^\infty}) = A_{1,\ell^\infty}.$$

Moreover, if $X^*(T_1) \subseteq X^*(T_2)$, and if A_1 is simple and nondegenerate, then

$$A_1(K_{A_2})_{\mathrm{tors}} = A_1(\overline{K})_{\mathrm{tors}}.$$

(b) $X^*(T_1) \otimes \mathbb{Q} \not\subseteq X^*(T_2) \otimes \mathbb{Q}$. *Then A_1 is essentially torsion finite for A_2.*

Proof. By Theorem 4.10 and Remark 4.11, A_1 is potentially torsion infinite for A_2 if and only if $\dim H_{12} = 0$. Since H_{12} is of multiplicative type, this happens if and only if $\dim_{\mathbb{Q}} X^*(H_{12}) \otimes \mathbb{Q} = 0$. After tensoring both sides of (4.5) with $\mathbb{Q}$, we find that this happens if and only if $X^*(T_1) \otimes \mathbb{Q} \subseteq X^*(T_2) \otimes \mathbb{Q}$.

If A_1 is potentially torsion infinite for A_2, then the description of $\widetilde{K}_\ell$, etc. is in Lemma 4.8.

Finally, suppose that we have an inclusion of integral lattices $X^*(T_1) \subseteq X^*(T_2)$ and that A_1 is simple and nondegenerate. The calculation (4.5) shows that H_{12} is trivial. Briefly suppose that A_1/K has CM actually defined over K, and thus (Lemma 5.3) has connected Galois representations. Then, for each natural number N, we have a containment $\Gamma_{A_1,N} \subset T_1(\mathbb{Z}/N)$, and thus $\mathrm{Gal}(K_{A_2})$ acts trivially on $A_{1,N}$.

Now, suppose that A_1/K merely has potential complex multiplication. The surjection $T_2 \to T_1$ means that the splitting field of T_2 contains the splitting field of T_1; equivalently, we have an inclusion of reflex fields $E_1^* \subset E_2^*$. Suppose $N \geq 3$ is an integer. Then all geometric

https://doi.org/10.1017/nmj.2023.19 Published online by Cambridge University Press

endomorphisms of A_2 are defined over $K_{A_2,N}$ [41]. Therefore, $K_{A_2,N}$ contains E_2^*, and thus E_1^* (see [20, Prop. 7.11]). Because E_1^* is simple, all geometric endomorphisms of A_1 are defined over $K_{A_2,N}$. Therefore, the image of the action of $\mathrm{Gal}(K_{A_2,N})$ on $A_{1,N}$, and *a fortiori* that of $\mathrm{Gal}(K_{A_2})$, is contained in $T_1(\mathbb{Z}/N)$ (Lemma 5.3), and we conclude as before. □

REMARK 5.5. In the context of Theorem 5.4, let $E/\mathbb{Q}$ be a Galois CM field containing E_1 and E_2, and assume that A_1 and A_2 share no common geometric isogeny factor. As in Remark 5.2, after replacing A_1 and A_2 by suitable powers, we may assume A_1 and A_2 have CM by the same field E, with respective CM types $\Phi_{E,1}$ and $\Phi_{E,2}$; then $A_1 \times A_2$ has a CM type $(E \times E, \Phi_{12})$, where $\Phi_{12} = \Phi_{E,1} \sqcup \Phi_{E,2}$. Then $T_{\Phi_i} = T_{E,\Phi_i}$, and the compatibility of the various (reflex) norm maps is expressed in the commutativity of the following diagram of tori:

$$\begin{tikzcd}
 & & T_{\Phi_1} \\
T^E \arrow[r, two heads, "N_{E,\Phi_{12}}"] \arrow[urr, two heads, bend left, "N_{E,\Phi_1}"] \arrow[drr, two heads, bend right, "N_{E,\Phi_2}"'] & T_{\Phi_{12}} \arrow[ur, two heads, "\pi_1"'] \arrow[dr, two heads, "\pi_2"] & \\
 & & T_{\Phi_2}
\end{tikzcd}.$$

In particular, in Theorem 5.4, we may compare $X^*(T_{\Phi_1})$ and $X^*(T_{\Phi_2})$ inside $X^*(T^E)$ (or $X^*(T^E) \otimes \mathbb{Q}$).

5.1.5. Examples

In concrete cases, Theorem 5.4 gives a way to explicitly analyze essential torsion finiteness for pairs of CM abelian varieties.

EXAMPLE 5.6. Let $E = \mathbb{Q}(\zeta_{13})$. Then $\mathrm{Gal}(E/\mathbb{Q}) \cong \langle \sigma | \sigma^{12} = 1 \rangle$. There are exactly six isomorphism classes of CM types for E, with representatives

$$\begin{aligned}
\Phi_1 &= \{1, \sigma, \sigma^2, \sigma^3, \sigma^4, \sigma^5\}, & \Phi_4 &= \{1, \sigma^7, \sigma^8, \sigma^3, \sigma^4, \sigma^5\}, \\
\Phi_2 &= \{1, \sigma^7, \sigma^2, \sigma^3, \sigma^4, \sigma^5\}, & \Phi_5 &= \{1, \sigma^7, \sigma^8, \sigma^3, \sigma^{10}, \sigma^5\}, \\
\Phi_3 &= \{1, \sigma, \sigma^8, \sigma^3, \sigma^4, \sigma^5\}, & \Phi_6 &= \{1, \sigma^4, \sigma^8, \sigma, \sigma^5, \sigma^9\}.
\end{aligned}$$

Let A_i be an abelian sixfold with CM type (E, Φ_i). Then any abelian variety with CM by E is geometrically isogenous to one of the A_i; and for $1 \le i \le 5$, A_i is geometrically simple. By an explicit computation, one can check that in $X^*(T^E) \otimes \mathbb{Q}$, we have

$$\begin{aligned}
&X^*(T_{\Phi_i}) \otimes \mathbb{Q} = X^*(T_{\Phi_j}) \otimes \mathbb{Q}, \quad 1 \le i, j \le 5, \text{ and} \\
&X^*(T_{\Phi_6}) \otimes \mathbb{Q} \subsetneq X^*(T_{\Phi_i}) \otimes \mathbb{Q}, \quad 1 \le i \le 5.
\end{aligned}$$

Then, for any $i \le 5$, A_i is essentially torsion finite for A_6; and for any $j \le 6$, A_j is potentially torsion infinite for A_i.

We also can compute $X^*(H_{ij})$ explicitly. For example, consider A_1 and A_2. By (5.3), $X^*(T_{\Phi_1})$ is generated by the Galois orbit of $1 + \sigma^{-1} + \sigma^{-2} + \sigma^{-3} + \sigma^{-4} + \sigma^{-5}$ and $X^*(T_{\Phi_2})$ is generated by the Galois orbit of $1 + \sigma^{-7} + \sigma^{-2} + \sigma^{-3} + \sigma^{-4} + \sigma^{-5}$. Note that the Galois orbit of $1+\sigma^{-1}+\sigma^{-2}+\sigma^{-3}+\sigma^{-4}+\sigma^{-5}$ (resp. $1+\sigma^{-7}+\sigma^{-2}+\sigma^{-3}+\sigma^{-4}+\sigma^{-5}$) equals the Galois orbit of $1+\sigma^{1}+\sigma^{2}+\sigma^{3}+\sigma^{4}+\sigma^{5}$ (resp. $1+\sigma^{7}+\sigma^{2}+\sigma^{3}+\sigma^{4}+\sigma^{5}$). We compute

$$\begin{aligned}1+\sigma^7+\sigma^2+\sigma^3+\sigma^4+\sigma^5 &= (1+\sigma+\sigma^2+\sigma^3+\sigma^4+\sigma^5)-(\sigma+\sigma^2+\sigma^3+\sigma^4+\sigma^5+\sigma^6)\\ &\quad +(\sigma^2+\sigma^3+\sigma^4+\sigma^5+\sigma^6+\sigma^7)\\ &= (1+\sigma+\sigma^2+\sigma^3+\sigma^4+\sigma^5)-\sigma(1+\sigma+\sigma^2+\sigma^3+\sigma^4+\sigma^5)\\ &\quad +\sigma^2(1+\sigma+\sigma^2+\sigma^3+\sigma^4+\sigma^5).\end{aligned} \tag{5.5}$$

Then $X^*(T_{\Phi_2}) \subseteq X^*(T_{\Phi_1})$. So, by (4.5),

$$X^*(H_{21}) \cong \frac{X^*(T_{\Phi_2})}{X^*(T_{\Phi_2}) \cap X^*(T_{\Phi_1})} \cong \frac{X^*(T_{\Phi_2})}{X^*(T_{\Phi_2})} \cong 0.$$

The types Φ_1 and Φ_2 are primitive. Then $X^*(T_{\Phi_1})$ is a rank 7 free $\mathbb{Z}$-module with a basis $\{\sigma^i+\sigma^{i+1}+\sigma^{i+2}+\sigma^{i+3}+\sigma^{i+4}+\sigma^{i+5} \mid 0 \le i \le 6\}$ and $X^*(T_{\Phi_2})$ is a rank 7 free $\mathbb{Z}$-module with a basis $\{\sigma^i+\sigma^{i+7}+\sigma^{i+2}+\sigma^{i+3}+\sigma^{i+4}+\sigma^{i+5} \mid 0 \le i \le 6\}$. A more detailed linear algebra calculation shows that

$$X^*(H_{12}) \cong \frac{X^*(T_{\Phi_1})}{X^*(T_{\Phi_2}) \cap X^*(T_{\Phi_1})} \cong \frac{X^*(T_{\Phi_1})}{X^*(T_{\Phi_2})} \cong (\mathbb{Z}/2)^2.$$

In Lemma 4.8, one might hope that $\widetilde{K}_\ell$ could be chosen independently of ℓ. To end this section, we will explain that this is impossible in general by considering the following example.

Example 5.7. Let E be $\mathbb{Q}(\zeta_{11})$. Then $\mathrm{Gal}(E/\mathbb{Q}) \cong \langle \sigma | \sigma^{10} = 1\rangle$. There are exactly four isomorphism classes of CM type for E, with representatives

$$\Phi_1 = \{1,\sigma^2,\sigma^4,\sigma^6,\sigma^8\}, \qquad \Phi_3 = \{1,\sigma^3,\sigma^6,\sigma^9,\sigma^2\},$$
$$\Phi_2 = \{1,\sigma^6,\sigma^2,\sigma^3,\sigma^4\}, \qquad \Phi_4 = \{1,\sigma,\sigma^2,\sigma^3,\sigma^4\}.$$

Let K be a number field containing E (and, in particular, the reflex fields of each Φ_i), and let A_i/K be an abelian fivefold with CM type (E,Φ_i). Further, assume that each A_i has independent representations over K and that each A_i has everywhere good reduction. Let S be a sufficiently large finite set of primes of K so that the description of the Galois representations in §5.1.2 holds. Then any abelian variety with CM by E is geometrically isogenous to one of the A_i. For $2 \le i \le 4$, A_i is geometrically simple, while A_1 is geometrically isogenous to the cube of an elliptic curve with complex multiplication by $\mathbb{Q}(\sqrt{-11})$. By an explicit computation, one can check that in $X^*(T^E) \otimes \mathbb{Q}$, we have

$$X^*(T_{\Phi_i}) \otimes \mathbb{Q} = X^*(T_{\Phi_j}) \otimes \mathbb{Q}, \quad 2 \le i,j \le 4, \text{ and}$$
$$X^*(T_{\Phi_1}) \otimes \mathbb{Q} \subsetneq X^*(T_{\Phi_i}) \otimes \mathbb{Q}, \quad 2 \le i \le 4.$$

For any $i \ge 2$, A_i is essentially torsion finite for A_1; and for any $j \le 4$, A_j is potentially torsion infinite for A_i.

Now, we focus on A_1 and A_2.

Identifying $X^*(T^E)$ with the group ring $\mathbb{Z}[\langle\sigma\rangle]$, we may present $X^*(T_{\Phi_1})$ as $X^*(T_{\Phi_1}) \cong \mathbb{Z} \oplus \mathbb{Z}$, with basis elements $\sum_{0 \le j \le 5} \sigma^{2j}$ and $\sum_{0 \le j \le 5} \sigma^{2j+1}$. The action of the generator σ

https://doi.org/10.1017/nmj.2023.19 Published online by Cambridge University Press

of $\mathrm{Gal}(E/\mathbb{Q})$ is to exchange these two basis vectors. In particular, the torus is split by the fixed field $\mathbb{Q}(\zeta_{11})^{\sigma^5} = \mathbb{Q}(\sqrt{-11})$.

An explicit computation shows that

$$X^*(T_{\Phi_1}) \cap X^*(T_{\Phi_2}) = 3X^*(T_{\Phi_1}).$$

In particular, $X^*(H_{12}) = X^*(T_{\Phi_1})/3X^*(T_{\Phi_1})$. Thus, $H_{12\bar{\mathbb{Q}}} \cong (\mathbb{Z}/3)_{\bar{\mathbb{Q}}} \oplus (\mathbb{Z}/3)_{\bar{\mathbb{Q}}}$, and the action of $\mathrm{Gal}(\mathbb{Q}(\sqrt{-11})/\mathbb{Q})$ exchanges the two components.

On one hand, H_{12} is zero-dimensional. Consequently, by Theorem 4.10 (and the following remark), A_1 is potentially torsion-infinite for A_2.

On the other hand, H_{12} is not split over $\mathbb{Q}$, although it does admit $\mathbb{Q}$-points. Indeed, $H_{12}(\mathbb{Q}) = \{(0,0),(1,1),(2,2)\} \subsetneq (\mathbb{Z}/3)\oplus(\mathbb{Z}/3)$. Consequently, Lemma 4.8 only implies that, for each ℓ, there exists some finite extension $\widetilde{K}_{2,\ell^\infty}$ of K_{2,ℓ^∞} such that $A_1[\ell^\infty](\widetilde{K}_{2,\ell^\infty})$ is infinite. We will now use the explicit calculation of the action of Galois to show that for any finite extension L/K, $A_1[\ell^\infty](L_{2,\ell^\infty})$ is finite for all but finitely many primes ℓ. This shows that Lemma 4.8 is essentially optimal.

Let $\rho_{1,\ell}$ and $\rho_{2,\ell}$ denote $\rho_{A_1/K,\ell^\infty}$ and $\rho_{A_2/K,\ell^\infty}$, respectively; and let $\widetilde{\rho}_{i,\ell}$ be the pullback of $\rho_{i,\ell}$ to $\mathbb{A}_K^\times$.

Now, suppose $\ell \notin S$, and embed $K_\ell^\times = \prod_{v|\ell} K_v^\times$ naturally into the ℓ-adic component of $\mathbb{A}_K^\times$. Then the restriction of the ℓ-adic representations (5.4) to $\mathcal{O}_{K_\ell}^\times$ reads as

$$\widetilde{\rho}_{i,\ell}(a) = N_{K,\Phi_i}(a_\ell^{-1}) \quad \text{if } a \in \mathcal{O}_{K_\ell}^\times, \text{ and } i = 1,2. \tag{5.6}$$

Now, further assume that ℓ is a prime integer that totally splits in K, and use the fact that the reflex norm $N_{K,\Phi_i} = N_{E,\Phi_i} \circ N_{K/E}$, where $N_{K/E}$ is the usual norm map of fields. Then K_ℓ is unramified over E_ℓ, and thus $N_{K/E}(\mathcal{O}_{K_\ell}^\times) = \mathcal{O}_{E_\ell}^\times$ [36, V, §2, Cor. of Prop. 3]. Hence, (5.6) factors through (which will still be denoted by $\widetilde{\rho}_{i,\ell}$ to ease notation)

$$\widetilde{\rho}_{i,\ell}(a) = N_{E,\Phi_i}(a_\ell^{-1}) \quad \text{if } a \in \mathcal{O}_{E_\ell}^\times, \text{ and } i = 1,2.$$

Since ℓ splits in K, it is also totally split in the subextension E. Recall that $\mathrm{Gal}(E/\mathbb{Q}) = \langle\sigma\rangle \simeq \mathbb{Z}/(10)$, and hence

$$\mathcal{O}_{E_\ell}^\times = (\mathbb{Z}_\ell \otimes E)^\times \simeq \prod_{\tau \in \mathrm{Gal}(E/\mathbb{Q})} \mathbb{Z}_{\ell,\tau}^\times = \prod_{i=0}^{9} \mathbb{Z}_{\ell,\sigma^i}^\times,$$

With respect to this isomorphism, every element a_ℓ of $\mathcal{O}_{E_\ell}^\times$ can be expressed by a vector of the form $a_\ell = (a_1,a_2,a_3,\ldots,a_{10})$, and $\sigma \in \mathrm{Gal}(E/\mathbb{Q})$ acts on a_ℓ by cyclically permuting its coordinates. By definition,

$$N_{E,\Phi_i}(a_\ell) = \prod_{\tau \in \Phi_i} \tau^{-1}(a_\ell).$$

Fix $x,y \in \mathbb{Z}_\ell^\times$, and consider the element $a_\ell \in \mathcal{O}_{E_\ell}^\times$ with coordinates

$$a_\ell = (x^{-2}y^{-1}, x^{-1}y^{-1}, xy, x^2y^2, x^2y, y, 1, 1, 1, x).$$

Direct computation then shows that

$$\begin{aligned} N_{E,\Phi_1}(a_\ell) &= (xy, x^2y^2, xy, x^2y^2, xy, x^2y^2, xy, x^2y^2, xy, x^2y^2), \\ N_{E,\Phi_2}(a_\ell) &= (x^3y^3, x^3y^3, x^3y^3, x^3y^3, 1, 1, 1, 1, 1, x^3y^3). \end{aligned}$$

https://doi.org/10.1017/nmj.2023.19 Published online by Cambridge University Press

Now, suppose that xy is a primitive third root of unity; this is possible, of course, exactly if $\ell \equiv 1 \bmod 3$. On one hand, $N_{E,\Phi_1}(a_\ell) = 1$; on the other hand, because $xy \not\equiv 1 \bmod \ell$ and $(xy)^2 \not\equiv 1 \bmod \ell$, $N_{E,\Phi_2}(a_\ell)$ acts without fixed points on $\mathcal{O}_E \otimes \mathbb{Z}/\ell$. In particular, $a_\ell \in \ker \widetilde{\rho}_{2,\ell} \setminus \ker \widetilde{\rho}_{1,\ell}$. Let $g \in \mathrm{Gal}(K)$ be the image of a_ℓ under the reciprocity map. By (5.6) and the formula of Serre-Tate, $g \in \ker \rho_{2,\ell} \setminus \ker \rho_{1,\ell}$. Moreover, since $\rho_{1,\ell}(g) = \widetilde{\rho}_{1,\ell}(a_\ell) \in E^\times$, we know that it does not have eigenvalue 1, even when working with $\mathbb{Z}/\ell$-coefficients. In particular, $A_1[\ell](K_{A_2,\ell^\infty})$ is trivial.

Finally, notice that we can produce such an a_ℓ for each ℓ with $\ell \equiv 1 \bmod 3$ which is totally split in K. Since A_1 has independent extensions, there is not a finite extension L/K on which each $\mathrm{art}(a_\ell)$ acts trivially. In particular, there is no finite extension L/K such that, for each ℓ, $A_1[\ell^\infty](L_{A_2,\ell^\infty}) = A_1[\ell^\infty](\overline{L})$.

Example 5.8. In [17, Th. 5.1], Lombardo gives a construction of an infinite family of iso-Kummerian CM pairs of abelian varieties. We briefly interpret his work in the framework developed here.

Given a CM field E which is the compositum of a cyclic totally real field of dimension g and a quadratic imaginary field, and the auxiliary choice of two integers r and h, Lombardo defines two different CM types Φ_1 and Φ_2, and chooses corresponding abelian varieties A_1 and A_2. After passage to a suitably large common field of definition K, one shows that the kernels of the ℓ-adic representations $\widetilde{\rho}_{A_k,K,\ell^\infty}$ coincide.

This calculation shows that the characters of T^E which vanish on the image of T^K under N_{K,Φ_1} are the same as those characters which vanish on the image of T^K under N_{K,Φ_2}. Consequently, $\mathrm{MT}(A_1)$ and $\mathrm{MT}(A_2)$ are the same subtorus of T_E, and thus A_1 and A_2 are mutually torsion-infinite.

5.2 Extra Hodge classes and torsion infiniteness

Following the notation in the previous section, let A_1 and A_2 be two isotypic CM abelian varieties over K. In this section, we will see that if A_1 is potentially torsion-infinite for A_2, then this is explained by a certain sort of Hodge class in some degree $2w$ on some product $A_1^m \times A_2^n$. The particular values of w, m, and n are not unique; and even once these are specified, the class itself, or even its $\mathbb{Q}$-span is not canonical. Consequently, we will call any such Hodge class a *torsion-infinite Hodge class from A_1 to A_2*, even though it actually lives on some unspecified product $A_1^m \times A_2^n$.

Suppose that A_i has a CM type (E, Φ_i) where E is a CM Galois extension of $\mathbb{Q}$. We also assume that the base field K is sufficiently large (e.g., it contains E).

We first describe the Hodge classes on $A_1^m \times A_2^n$ (see [30] for more details). Let $V_i = H^1(A_i, \mathbb{Q})$. Recall that $\langle V_i \rangle$, the tensor category generated by V_i, is equivalent to the category $\mathrm{Rep}_{\mathbb{Q}}(\mathrm{MT}(A_i))$ of representations of $\mathrm{MT}(A_i)$. By Lemma 5.1 and our assumption for E, the reflex norm defines a quotient map $N_{E,\Phi_i} : T^E \twoheadrightarrow \mathrm{MT}(A_i)$, which induces a fully faithful map on the categories of representations $\mathrm{Rep}_{\mathbb{Q}}(\mathrm{MT}(A_i)) \to \mathrm{Rep}_{\mathbb{Q}}(T^E)$. This allows us to describe the Hodge classes on $A_1^m \times A_2^n$ using the representation theory of the algebraic torus T^E for any positive integers m and n.

We denote the representation $T^E \twoheadrightarrow \mathrm{MT}(A_i) \hookrightarrow \mathrm{GL}_{V_i}$ by ρ_i. Note that $X^*(T^E) \cong \bigoplus_{\sigma \in \mathrm{Gal}(E/\mathbb{Q})} \mathbb{Z}\langle\sigma\rangle$, and the Galois group $\mathrm{Gal}(E/\mathbb{Q})$ acts on it by left multiplication. Since $E/\mathbb{Q}$ is Galois, we can identify Φ_i with a subset of $\mathrm{Gal}(E/\mathbb{Q})$.

For any representation $\rho : T^E \to \mathrm{GL}_V$, we let Ξ_V (or Ξ_ρ) be the collection of weights of this representation. The set Ξ_V is a finite submultiset of $X^*(T^E)$; the support $\mathrm{supp}(\Xi_V)$

https://doi.org/10.1017/nmj.2023.19 Published online by Cambridge University Press

of Ξ_V—that is, those elements of $X^*(T^E)$ with nonzero multiplicity—is finite, and all multiplicities are finite. For future use, we note that if $\Xi_V = \{\alpha_1,\dots,\alpha_d\}$ is a set of distinct characters, then $\mathrm{supp}(\Xi_{V^{\oplus m}})$ is the same set, and each weight now occurs with multiplicity m. Moreover, the support of $\Xi_{\wedge^r(V^{\oplus m})}$ is then

$$\mathrm{supp}(\Xi_{\wedge^r(V^{\oplus m})}) = \left\{\sum e_i\alpha_i : \sum e_i = r \text{ and } 0 \le e_i \le m \text{ for each } i\right\}. \tag{5.7}$$

By the definition of the reflex norm,

$$\Xi_{V_i} = \left\{\sum_{\sigma\in\Phi_i} (\sigma\circ\varphi^{-1}) \mid \varphi \in \mathrm{Gal}(E/\mathbb{Q})\right\}.$$

Since $\mathrm{Gal}(E/\mathbb{Q})$ acts transitively on this set, we have

$$X^*(T_{\Phi_i})\otimes\mathbb{Q} \cong \sum_{\alpha\in\Xi_{V_i}} \mathbb{Q}\langle\alpha\rangle \subset X^*(T^E)\otimes\mathbb{Q}$$

as Galois modules. We also denote the one-dimensional representation $\mathrm{Nm}: T^E \to \mathbb{G}_m$ by $\mathbb{Q}(1)$. The weight of the representation $\mathbb{Q}(1)$ is $\chi := \sum_{\sigma\in\mathrm{Gal}(E/\mathbb{Q})}\sigma \in X^*(T^E)$. If $n \ge 0$, let $\mathbb{Q}(n) = \mathbb{Q}(1)^{\otimes n}$, and if $n < 0$, set $\mathbb{Q}(n) = \mathbb{Q}(1)^{\vee,\otimes -n}$. Finally, let $\mathbb{Q}(0)$ denote the trivial representation of T^E. If V is any representation of T^E, then we let $V(n) = V\otimes\mathbb{Q}(n)$. If $V = H^1(A,\mathbb{Q})$ for some abelian variety A, then $V^\vee \cong V(1)$.

If V is any Hodge structure, the group of Hodge classes in V is $\mathrm{Hom}_{\mathbb{Q}-\mathrm{HS}}(\mathbb{1}, V)$, where $\mathbb{1}$ is the trivial Hodge structure. For an integer $w \ge 0$, we have

$$\begin{aligned}
\mathrm{Hom}_{\mathbb{Q}-\mathrm{HS}}(\mathbb{1}, H^{2w}(A_1^m\times A_2^n,\mathbb{Q})(w)) &\cong \mathrm{Hom}_{T^E}(\mathbb{Q}(0), H^{2w}(A_1^m\times A_2^n,\mathbb{Q})(w)) \\
&\cong H^{2w}(A_1^m\times A_2^n,\mathbb{Q})(w)^{T^E} \\
&\cong \bigoplus_{r+s=2w} (H^r(A_1^m,\mathbb{Q})\otimes H^s(A_2^n,\mathbb{Q}))(w)^{T^E} \\
&\cong \bigoplus_{r+s=2w} (\wedge^r V_1^{\oplus m}\otimes(\wedge^s V_2^{\oplus n}(w))^{T^E} \\
&\cong \bigoplus_{r+s=2w} ((\wedge^r V_1^{\vee,\oplus m})(-r)\otimes(\wedge^s V_2^{\oplus n})(w))^{T^E} \\
&\cong \bigoplus_{r+s=2w} (\wedge^r V_1^{\vee,\oplus m}\otimes(\wedge^s V_2^{\oplus n})(w-r))^{T^E}.
\end{aligned}$$

So the $\mathbb{Q}$-span of a Hodge class can be identified with an element $\alpha\in X^*(T^E)$ such that

$$-\alpha \in \Xi_{\wedge^r V_1^{\vee,\oplus m}} \quad\text{and}\quad \alpha\in\Xi_{(\wedge^s V_2^{\oplus n})(w-r)}.$$

Using the polarization on V_1, we rewrite these conditions as

$$\alpha \in \Xi_{\wedge^r V_1^{\oplus m}} \quad\text{and}\quad \alpha\in\Xi_{(\wedge^s V_2^{\oplus n})(w-r)}.$$

Moreover, the existence of a Hodge class in degree $2w$ on some product $A_1^m\times A_2^n$ is equivalent to the existence of $\alpha\in X^*(T^E)$ and r with $0\le r\le 2w$ such that α is a $\mathbb{Z}_{\ge 0}$-linear combination of the weights in Ξ_{V_1} and $\alpha + (w-r)\chi$ is a $\mathbb{Z}_{\ge 0}$-linear combination of the weights in Ξ_{V_2}. Let $s = 2w - r$. The choices $(r,s) = (0,2w)$ and $(r,s) = (2w,0)$ correspond to Hodge classes which come from A_1^m and A_2^n by pullback, while classes with r and s positive are conjecturally the classes of nontrivial correspondences between A_1^m and A_2^n.

https://doi.org/10.1017/nmj.2023.19 Published online by Cambridge University Press

For such $\alpha \in \Xi_{\wedge^r V_1^{\oplus m}}$, if moreover $\alpha = r\alpha_0$ for some $\alpha_0 \in \Xi_{V_1}$ and some positive integer r, that is,

$$\alpha \in r\Xi_{V_1} \subset r\Xi_{V_1^{\oplus m}} \subset \Xi_{\wedge^r V_1^{\oplus m}} \quad \text{and} \quad \alpha \in \Xi_{(\wedge^s V_2^{\oplus n})(w-r)}.$$

we call the related Hodge classes *torsion-infinite Hodge classes from A_1 to A_2*, regardless of the choice of m and n (and of r and s). As a consequence of our definition, these classes are in $H^{r,0}(A_1^m)^\vee \otimes H^s(A_2^n,\mathbb{C})(w-r)$ (or $H^{0,r}(A_1^m)^\vee \otimes H^s(A_2^n,\mathbb{C})(w-r)$). In particular, these classes are *not* in the $\mathbb{Q}$-span of those classes which are pulled back from A_1^m or A_2^n, and thus the torsion-infinite Hodge classes are extra Hodge classes.

PROPOSITION 5.9. *Let A_1 and A_2 be two isotypic CM abelian varieties over a number field K. Suppose that $\mathrm{Hom}_{\overline{K}}(A_1, A_2) = (0)$, that is, that $A_{1,\overline{K}}$ and $A_{2,\overline{K}}$ have no common nontrivial isogeny factor.*

Then A_1 is potentially torsion-infinite for A_2 if and only if there is a torsion-infinite Hodge class from A_1 to A_2.

Proof. Assume that A_1 is potentially torsion-infinite for A_2 over K. By Theorem 5.4,

$$X^*(T_{\Phi_1}) \otimes \mathbb{Q} \subset X^*(T_{\Phi_2}) \otimes \mathbb{Q}.$$

Suppose that $\alpha \in \Xi_{V_1}$. Then $\alpha = \sum_{\beta\in\Xi_{V_2}} c_\beta \beta$ for certain rational numbers c_β. Note that, if $\beta \in \Xi_{V_2}$, then its complex conjugate $\beta \circ c = \chi - \beta$ is in Ξ_{V_2}, too. Using this fact, we can rewrite α as

$$\alpha = \left(\sum_{\beta\in\Xi_{V_2}} c_\beta^+ \beta\right) - c_\chi^+ \chi,$$

where c_β^+ and c_χ^+ are nonnegative rational numbers. Choose a positive integer m such that mc_β^+ and mc_χ^+ are integers. Then

$$m\alpha = \left(\sum_{\beta\in\Xi_{V_2}} mc_\beta^+ \beta\right) - mc_\chi^+ \chi. \tag{5.8}$$

Let $n = \sum_{\beta\in\Xi_{\rho_2}} mc_\beta^+$.

Consider the embedding $\mathbb{G}_m \to T^E$ induced by $\mathbb{Q} \hookrightarrow E$, and let τ be the standard (positive) generator of $X^*(\mathbb{G}_m)$. Then $\gamma|_{\mathbb{G}_m} = g\tau$ for any $\gamma \in \Xi_{V_1} \cup \Xi_{V_2}$, while $\chi|_{\mathbb{G}_m} = 2g\tau$. Thus, restricting (5.8) to $\mathbb{G}_m$ and computing coefficients of τ yields

$$mg = ng - 2mc_\chi^+.$$

In particular, $m+n = n-m+2m = 2mc_\chi^+ + 2m = 2(mc_\chi^+ + m)$ is even. Then $H^m(A_1^m,\mathbb{Q}) \otimes H^n(A_2^n,\mathbb{Q})(mc_\chi^+ + m)$ contains a torsion-infinite Hodge class from A_1 to A_2.

Conversely, if there is a torsion-infinite Hodge class from A_1 to A_2, then there exists a weight $\alpha_0 \in \Xi_{V_1}$ which is a $\mathbb{Q}$-linear combination of the weights in Ξ_{V_2}. Since $\mathrm{Gal}(E/\mathbb{Q})$ acts transitively on Ξ_{V_1}, $X^*(T_{\Phi_1}) \otimes \mathbb{Q} \subset X^*(T_{\Phi_2}) \otimes \mathbb{Q}$. By theorem 5.4, A_1 is potentially torsion infinite for A_2 over K. ∎

REMARK 5.10. In the proof, we choose $r = m$ and $s = n = \sum_{\beta\in\Xi_{\rho_2}} mc_\beta^+$ for convenience. However, sometimes smaller m and n can be chosen. See Examples 5.12 and 5.7.

https://doi.org/10.1017/nmj.2023.19 Published online by Cambridge University Press

Before displaying some concrete examples, let us prove the following lemma. Recall our discussion of nondegenerate abelian varieties (5.1.3).

Lemma 5.11. *Let A_1 and A_2 be two CM abelian varieties. Suppose that A_i has a CM type (E,Φ_i) and A_2 is nondegenerate. Then $X^*(T_{\Phi_1})\otimes\mathbb{Q}\subseteq X^*(T_{\Phi_2})\otimes\mathbb{Q}$. In particular, if A_1 and A_2 are nondegenerate, then $X^*(T_{\Phi_1})\otimes\mathbb{Q}=X^*(T_{\Phi_2})\otimes\mathbb{Q}$.*

Proof. First, we let E^+ be the totally real subfield of E and let $\mathbb{U}_1^{E^+}$ be the norm one subtorus of T^{E^+}. For any CM type (E,Φ),

$$\mathbb{U}_1^{E^+}\subset(\ker(N_{E,\Phi}))^\circ,$$

because the restriction of $N_{E,\Phi}$ to T^{E^+} is simply $N_{E^+/\mathbb{Q}}$; and a dimension count shows that equality holds if and only if (E,Φ) is nondegenerate.

Under the hypotheses of the lemma, we have the following commutative diagram:

$$\begin{tikzcd} 0 \arrow[r] & X^*(T_{\Phi_1})\otimes\mathbb{Q} \arrow[r, hook] & X^*(T^E)\otimes\mathbb{Q} \arrow[r, two heads] \arrow[d, equal] & X^*(\ker(N_{E,\Phi_1}))\otimes\mathbb{Q} \arrow[r] \arrow[d, two heads, "f"] & 0 \\ 0 \arrow[r] & X^*(T_{\Phi_2})\otimes\mathbb{Q} \arrow[r, hook] & X^*(T^E)\otimes\mathbb{Q} \arrow[r, two heads] & X^*(\ker(N_{E,\Phi_2}))\otimes\mathbb{Q} \arrow[r] & 0 \end{tikzcd},$$

where the surjection f is induced by the inclusion $(\ker(N_{E,\Phi_2}))^\circ=\mathbb{U}_1^{E^+}\hookrightarrow(\ker(N_{E,\Phi_1}))^\circ$. This implies that $X^*(T_{\Phi_1})\otimes\mathbb{Q}\subseteq X^*(T_{\Phi_2})\otimes\mathbb{Q}$. ∎

Example 5.12. Let E be $\mathbb{Q}(\zeta_7)$. Then $\mathrm{Gal}(E/\mathbb{Q})\cong\langle\sigma|\sigma^6=1\rangle$. There are two isomorphism classes of CM types for E:

$$\Phi_1=\{1,\sigma^2,\sigma^4\}\qquad\qquad \Phi_2=\{1,\sigma,\sigma^2\}.$$

Let A_i be an abelian variety with CM type (E,Φ_i). Then A_1 is geometrically isogenous to the third power of an elliptic curve with CM by $\mathbb{Q}(\sqrt{-7})$, while A_2 is nondegenerate. By Lemma 5.11, A_1 is potentially torsion-infinite for A_2. In fact, we have

$$\begin{aligned}1+\sigma^2+\sigma^4&=(1+\sigma+\sigma^2)+(\sigma^2+\sigma^3+\sigma^4)+(\sigma^4+\sigma^5+1)-\chi\\&=\sum_{\tau\in\Phi_2}\tau+\sigma^2(\sum_{\tau\in\Phi_2}\tau)+\sigma^4(\sum_{\tau\in\Phi_2}\tau)-\chi.\end{aligned}$$

This means that $H^1(A_1,\mathbb{Q})^\vee\otimes H^3(A_2,\mathbb{Q})(1)$ contains a torsion-infinite Hodge class from A_1 to A_2.

Example 5.13. We return to the setting of Example 5.7, with $E=\mathbb{Q}(\zeta_{11})$. Consider A_2 and A_3 with CM types $(E,\Phi_2=\{1,\sigma^6,\sigma^2,\sigma^3,\sigma^4\})$ and $(E,\Phi_3=\{1,\sigma^3,\sigma^6,\sigma^9,\sigma^2\})$. Since A_2 and A_3 are primitive, by Lemma 5.11, $X^*(T_{\Phi_2})\otimes\mathbb{Q}=X^*(T_{\Phi_3})\otimes\mathbb{Q}$. Considering the relation between $X^*(T_{\Phi_2})$ and $X^*(T_{\Phi_3})$, we have

$$X^*(T_{\Phi_2})\subseteq X^*(T_{\Phi_3}).$$

More precisely,

$$\begin{aligned}1+\sigma^6+\sigma^2+\sigma^3+\sigma^4=&(1+\sigma^3+\sigma^6+\sigma^9+\sigma^2)+(\sigma^2+\sigma^5+\sigma^8+\sigma+\sigma^4)\\&+(\sigma^4+\sigma^7+1+\sigma^3+\sigma^6)-\chi\end{aligned}$$

https://doi.org/10.1017/nmj.2023.19 Published online by Cambridge University Press

and

$$\begin{aligned}3(1+\sigma^3+\sigma^6+\sigma^9+\sigma^2) = 2(1+\sigma^6+\sigma^2+\sigma^3+\sigma^4)+(\sigma^7+\sigma^3+\sigma^9+1+\sigma)\\ +(\sigma^9+\sigma^5+\sigma+\sigma^2+\sigma^3)+2(\sigma^6+\sigma^2+\sigma^8+\sigma^9+1)\\ +(\sigma^3+\sigma^9+\sigma^5+\sigma^6+\sigma^7)-2\chi.\end{aligned}$$

So A_2 and A_3 are potentially torsion-infinite for each other. Moreover, $H^1(A_3,\mathbb{Q})^\vee \otimes H^3(A_2,\mathbb{Q})(1)$ contains a torsion-infinite Hodge class from A_3 to A_2, and $H^3(A_2^3,\mathbb{Q})^\vee \otimes H^7(A_3^2,\mathbb{Q})(2)$ contains a torsion-infinite Hodge class from A_2^3 to A_3^2.

5.3 Low-dimensional abelian varieties

In [23, §2], Moonen and Zarhin list all the possible Hodge groups for absolute simple abelian varieties with dimension $g \leq 3$. We will follow their classification and use the notation (g,Type) to denote an absolutely simple abelian variety with dimension g and the indicated endomorphism type in the Albert classification. For instance, $(2,\mathrm{I}V(2,1))$ refers to an absolutely simple CM abelian surface.

THEOREM 5.14. *Suppose A and B are absolutely simple abelian varieties over a common number field, and assume that they are nonisogenous over $\mathbb{C}$. Suppose that $\dim A \leq \dim B \leq 3$. Then A and B are mutually essentially torsion finite except for the following cases:*

(a) *A is a CM elliptic curve, and B is of type $(3,IV(3,1))$, that is, B is a simple CM abelian threefold. Then B is essentially torsion finite for A; and A is potentially torsion infinite for B exactly when there is an embedding of $\mathbb{Q}$-algebras $\mathrm{End}^0(A) \hookrightarrow \mathrm{End}^0(B)$.*
(b) *A is a CM elliptic curve, and B is of type $(3,IV(1,1))$. Then B is essentially torsion finite for A; and A is potentially torsion infinite for B exactly when there is an isomorphism of $\mathbb{Q}$-algebras $\mathrm{End}^0(A) \cong \mathrm{End}^0(B)$.*
(c) *$[A,B]$ is of type $[(3,IV(3,1)),(3,IV(3,1))]$, that is, both of them are CM abelian threefolds. Then the essential torsion finiteness depends on the CM types of A and B as in Theorem 5.4.*

Proof. Our proof contains two parts. In the first part, we will assume that the pair (A,B) is *not* one of the cases (a), (b), or (c). The analysis of the special situations is carried out in the second part.

Recall that, if

$$\mathrm{sMT}(A\times B) = \mathrm{sMT}(A)\times \mathrm{sMT}(B), \tag{5.9}$$

then A and B are mutually torsion finite (Corollary 4.15). Since $\dim A \leq \dim B$, (5.9) holds in each of the following cases.

1. Suppose both A and B are of odd relative dimension, and they are not both of type IV. Then [10, Th. IA] states that (5.9) holds.
2. Suppose A is a CM elliptic curve and condition (a) does not hold. Then (5.9) is a consequence of [23, Prop. (3.8)]. In particular, (5.9) holds if B is a surface (since a geometrically simple abelian surface in characteristic zero does not admit an action by a quadratic imaginary field—this result of Shimura informs [23, (2.2)]) or a non-type IV threefold.

3. If $\dim A = \dim B = 2$, then (5.9) follows from [23, (5.4) and (5.5)].
4. If $\dim A = 2$ and $\dim B = 3$, then (5.9) follows from [23, Th. (0.2)(iv)].

Hence, we are left with two situations to discuss. For expository ease, we will let $A_1 = A$ and $A_2 = B$ in the following discussion.

Case 1. Suppose the pair is of type [(3,IV(1,1)), (3,IV(1,1))], and that A_1 is potentially torsion infinite for A_2; we will show that A_1 and A_2 are geometrically isogenous. We start by describing the Mumford–Tate groups of each A_i although ultimately we will analyze their *special* Mumford–Tate groups, in order to exploit the fact that isogenous one-dimensional algebraic tori are actually isomorphic. Recall that if G is a reductive group with derived group G' and connected center Z, then Z is a torus and G is canonically isomorphic to $G' \times Z/(G' \cap Z)$.

For $i = 1,2$, the endomorphism algebra $F_i := \mathrm{End}^0(A_i)$ is an imaginary quadratic field. The Mumford–Tate group G_i of A_i is a unitary similitude group in three variables attached to the quadratic extension $F_i/\mathbb{Q}$, which we denote $\mathrm{GU}_{F_i}(3)$, and the Hodge group sG_i is the unitary group $\mathrm{U}_{F_i}(3)$. The center of G_i is $T^{F_i} = \mathrm{Res}_{F_i/\mathbb{Q}}\mathbb{G}_m$; the connected center Z_i of sG_i is the norm one torus $T^{F_i,1} = \mathrm{Res}^{(1)}_{F_i/\mathbb{Q}}\mathbb{G}_m \cong \mathrm{U}_{F_i}(1)$; and we have exact sequences

$$\begin{tikzcd}
1 \arrow[r] & \mathrm{SU}_{F_i}(3) \arrow[r] \arrow[d, "="] & sG_i \cong \mathrm{U}_{F_i}(3) \arrow[r, "\det"] \arrow[d, hook] & T^{F_i,1} \arrow[r] \arrow[d, hook] & 1 \\
1 \arrow[r] & \mathrm{SU}_{F_i}(3) \arrow[r] & G_i \cong \mathrm{GU}_{F_i}(3) \arrow[r] & T^{F_i} \arrow[r] & 1
\end{tikzcd}$$

The restriction $\delta_i := \det|_{Z_i}$ is $[3]_{Z_i}$, the cubing map. Moreover, $H_1(A_i,\mathbb{Q})$ is the standard representation of G_i (see, e.g., [23, (2.3)]).

Note that $\dim G_1 = \dim G_2$. Under the assumption that A_1 is potentially torsion-infinite for A_2, we have $\dim G_{12} = \dim G_2$ (Theorem 4.10). Therefore, $\dim G_{12} = \dim G_1$ as well, and thus A_1 and A_2 are mutually potentially torsion-infinite.

The isogenies $\pi_i : G_{12} \to G_i$ induce isomorphisms of Lie algebras $\mathfrak{g}_{12} \to \mathfrak{g}_i$. We thus have an isomorphism of $\mathbb{Q}$-Lie algebras $\mathfrak{gu}_{F_1}(3) \cong \mathfrak{gu}_{F_2}(3)$, and so $F_1 \cong F_2$. We relabel this common quadratic field F and proceed.

For each i, the inclusion $H_1(A_i,\mathbb{Q}) \hookrightarrow H_1(A_1 \times A_2,\mathbb{Q})$ is F-linear. Therefore, we have commutative diagrams

$$\begin{tikzcd}
 & G_{12} \arrow[dd, two heads, "\pi_i"] \\
T^F \arrow[ur, hook] \arrow[dr, hook] & \\
 & G_i
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
 & sG_{12} \arrow[dd, two heads, "\pi_i"] \\
T^{F,1} \arrow[ur, hook] \arrow[dr, hook] & \\
 & sG_i
\end{tikzcd},$$

where the right-hand diagram is the restriction of the left-hand diagram to Hodge groups.

Fix some i, and consider the isogeny of Hodge groups $\pi_i : sG_{12} \to sG_i$. Let $M_i = \pi_i^{-1}(\mathrm{SU}_F(3))^\circ$. Since $\mathrm{SU}_F(3)$ is simply connected, $M_i \cong \mathrm{SU}_F(3)$ maps isomorphically onto its image. Let $d_{12} : sG_{12} \to sG_{12}/M_i$ be the projection. The quotient sG_{12}/M_i is a smooth geometrically connected group which is isogenous to the one-dimensional torus $T^{F,1}$, and

https://doi.org/10.1017/nmj.2023.19 Published online by Cambridge University Press

thus is isomorphic to $T^{F,1}$. Similarly, the connected center Z_{12} of sG_{12} is isomorphic to $T^{F,1}$, and we have a commutative diagram:

$$\begin{tikzcd}
& sG_{12} \arrow[r, two heads, "d_{12}"] \arrow[dd, two heads, "\pi_i"] & T^{F,1} \arrow[dd, two heads, "\bar{\pi}_i"] \\
T^{F,1} \arrow[ur, hook] \arrow[dr, hook] & & \\
& sG_i \arrow[r, two heads, "\det"] & T^{F,1}
\end{tikzcd}$$

Let $\delta_{12} = d_{12}|_{Z_{12}}$, and note that $\ker\delta_{12} = Z_{12} \cap \mathrm{SU}_F(3)$. Since $\bar{\pi}_i \circ \delta_{12} = \delta_1 = [3]$ and $\ker[3]$ is simple, exactly one of $\bar{\pi}_i$ and δ_{12} is an isomorphism. So either:

- δ_{12} is an isomorphism. Then $Z_{12} \cap \mathrm{SU}_F(3) = \{1\}$, and so $sG_{12} \cong \mathrm{SU}_F(3) \times \mathrm{U}_F(1)$, and π_i is the canonical projection; or
- $\bar{\pi}_i$ is an isomorphism. Then $Z_{12} \cap \mathrm{SU}_F(3) = \ker[3]$, $sG_{12} \cong \mathrm{U}_F(3)$, and π_i is an isomorphism.

Of course, the isomorphism class of sG_{12} is independent of the choice of i; and we have seen that each π_i is determined, up to isomorphism, by the isomorphism class of sG_{12}. Therefore, $sG_{12} \to sG_i \to \mathrm{GL}_{V_i}$ is independent of i, and so A_1 and A_2 are isogenous (Lemma 4.1). (After the fact, using Lemma 4.1(c), we recognize that the second case happens, that is, that $sG_{12} \cong U_F(3)$.)

Case 2. If the pair is of type $[(3,\mathrm{IV}(1,1)),(3,\mathrm{IV}(3,1))]$, then the Hodge group of A_1 has been explained in Case 1. Note in particular that the center of $\mathrm{sMT}(A_1)$ is $U_{F_1}(1)$, a one-dimensional torus.

Now consider A_2. It has complex multiplication by a CM field E_2. Since $\dim A_2 = 3$ is prime, the CM type is nondegenerate, that is, $\dim \mathrm{sMT}(A_2) = 3$.

In particular, there is no isogeny from the center of $\mathrm{sMT}(A_1)$ to $\mathrm{sMT}(A_2)$. By [23, Lem. 3.6], A_1 and A_2 satisfy (5.9), and thus are mutually essentially torsion finite. This finishes the first part of the proof.

It remains to discuss cases (a)–and (c). Of course, there is nothing to prove for case (c). As for (a) and (b), since

$$\dim \mathrm{MT}(A_1 \times A_2) \geq \dim \mathrm{MT}(A_2) > 2 = \dim \mathrm{MT}(A_1),$$

we immediately deduce that A_2 is essentially torsion finite for A_1.

Moreover, by [23, Prop. (3.8)], $\mathrm{sMT}(A_1 \times A_2) = \mathrm{sMT}(A_1) \times \mathrm{sMT}(A_2)$ if and only if there is no embedding $\mathrm{End}^0(A_1) \hookrightarrow \mathrm{End}^0(A_2)$; and this is equivalent to the essential torsion finiteness of A_1 for A_2 (Corollary 4.12). ∎

Acknowledgments. We thank Yuan Ren for bringing this interesting question to our attention. Lian Duan thanks Ken Ribet for useful suggestions; Xiyuan Wang thanks Stefan Patrikis for many enlightening discussions; and we all thank Davide Lombardo for helpful comments on a draft of this article. This article benefited greatly from a referee's extraordinarily close reading; we are grateful for their efforts.

References

[1] G. Ancona, A. Huber, and S. Pepin Lehalleur, *On the relative motive of a commutative group scheme*, Algebr. Geom. **3** (2016), 150–178.

[2] K. A. Behrend, *Derived l-adic categories for algebraic stacks*, Mem. Amer. Math. Soc. **163** (2003), viii + 93.

[3] G. Böckle, W. Gajda, and S. Petersen, *Independence of ℓ-adic representations of geometric Galois groups*, J. Reine Angew. Math. **736** (2018), 69–93.

[4] A. Cadoret and B. Moonen, *Integral and adelic aspects of the Mumford–Tate conjecture*, J. Inst. Math. Jussieu **19** (2020), 869–890.

[5] C.-L. Chai, B. Conrad, and F. Oort, *Complex Multiplication and Lifting Problems*, Surveys Monogr. **195**, Amer. Math. Soc., Providence, RI, 2014.

[6] J. Commelin, *The Mumford–Tate conjecture for products of abelian varieties*, Algebr. Geom. **6** (2019), 650–677.

[7] B. Conrad, "Reductive group schemes" in *Autour Des schémas en Groupes. Vol. I*, Panor. Synthèses **42/43**, Soc. Math. France, Montrouge, 2014, 93–444.

[8] M. Hindry and N. Ratazzi, *Torsion pour les variétés abéliennes de type I et II*, Algebra Number Theory **10** (2016), 1845–1891.

[9] J. E. Humphreys, *Linear Algebraic Groups*, Grad. Texts in Math. **21**, Springer, New York–Heidelberg, 1975.

[10] T. Ichikawa, *Algebraic groups associated with abelian varieties*, Math. Ann. **289** (1991), 133–142.

[11] H. Imai, *A remark on the rational points of abelian varieties with values in cyclotomic Z_p-extensions*, Proc. Japan Acad. **51** (1975), 12–16.

[12] N. M. Katz and S. Lang, *Finiteness theorems in geometric classfield theory*, Enseign. Math. (2) **27** (1981), 285–319 (1982). With an appendix by Kenneth A. Ribet.

[13] M. Larsen and R. Pink, *On l-independence of algebraic monodromy groups in compatible systems of representations*, Invent. Math. **107** (1992), 603–636.

[14] M. J. Larsen and R. Pink, *Finite subgroups of algebraic groups*, J. Amer. Math. Soc. **24** (2011), 1105–1158.

[15] A. Li, *On the torsion points of Drinfeld modules in abelian extensions*, J. Pure Appl. Algebra **176** (2002), 153–159.

[16] D. Lombardo, *On the ℓ-adic Galois representations attached to nonsimple abelian varieties*, Ann. Inst. Fourier (Grenoble) **66** (2016), 1217–1245.

[17] D. Lombardo, *Non-isogenous abelian varieties sharing the same division fields*, Trans. Amer. Math. Soc. **376** (2023), no. 4, 2615–2640.

[18] B. Mazur, *Rational points of abelian varieties with values in towers of number fields*, Invent. Math. **18** (1972), 183–266.

[19] J. S. Milne, *Algebraic Groups: The Theory of Group Schemes of Finite Type over a Field*, Cambridge Stud. Adv. Math. **170**, Cambridge Univ. Press, Cambridge, 2017.

[20] J. S. Milne, *Complex multiplication*, 2020, https://www.jmilne.org/math/CourseNotes/CM.pdf

[21] B. Moonen, *Notes on Mumford–Tate groups*, 1999. https://www.math.ru.nl/~bmoonen/Lecturenotes/CEBnotesMT.pdf

[22] B. Moonen, *Families of motives and the Mumford–Tate conjecture*, Milan J. Math. **85** (2017), 257–307.

[23] B. J. J. Moonen and Y. G. Zarhin, *Hodge classes on abelian varieties of low dimension*, Math. Ann. **315** (1999), 711–733.

[24] D. Mumford, *Abelian Varieties*, Tata Inst. Fund. Res. Stud. Math. **5**, Published for the Tata Institute of Fundamental Research, Bombay, by Hindustan Book Agency, New Delhi, 2008, with appendices by C. P. Ramanujam and Yuri Manin, corrected reprint of the second (1974) edition.

[25] M. V. Nori, *On subgroups of $GL_n(F_p)$*, Invent. Math. **88** (1987), 257–275.

[26] M. Orr, *Lower bounds for ranks of Mumford–Tate groups*, Bull. Soc. Math. France **143** (2015), 229–246.

[27] Y. Ozeki, *Torsion points of abelian varieties with values in infinite extensions over a p-adic field*, Publ. Res. Inst. Math. Sci. **45** (2009), 1011–1031.

[28] R. Pink, *ℓ-adic algebraic monodromy groups, cocharacters, and the Mumford–Tate conjecture*, J. Reine Angew. Math. **495** (1998), 187–237.

[29] V. Platonov and A. Rapinchuk, *Algebraic Groups and Number Theory*, Pure Appl. Math. **139**, Academic Press, Boston, MA, 1994, translated from the 1991 Russian original by Rachel Rowen.

[30] H. Pohlmann, *Algebraic cycles on abelian varieties of complex multiplication type*, Ann. of Math. **2** (1968), 161–180.

[31] K. A. Ribet, *Galois action on division points of abelian varieties with real multiplications*, Amer. J. Math. **98** (1976), 751–804.

https://doi.org/10.1017/nmj.2023.19 Published online by Cambridge University Press

[32] K. A. Ribet, *Hodge classes on certain types of abelian varieties*, Amer. J. Math. **105** (1983), 523–538.
[33] D. Rössler and T. Szamuely, *Cohomology and torsion cycles over the maximal cyclotomic extension*, J. Reine Angew. Math. **752** (2019), 211–227.
[34] J.-P. Serre, *Abelian l-adic representations and elliptic curves*. McGill University lecture notes written with the collaboration of Willem Kuyk and John Labute, W. A. Benjamin, New York–Amsterdam, 1968.
[35] J.-P. Serre, *Propriétés galoisiennes des points d'ordre fini des courbes elliptiques*, Invent. Math. **15** (1972), 259–331.
[36] J.-P. Serre, *Local Fields*, Grad. Texts in Math. **67**, Springer, New York–Berlin, 1979, translated from the French by Marvin Jay Greenberg.
[37] J.-P. Serre, "Lettres à ken Ribet du 1/1/1981 et du 29/1/1981" in *Œuvres—Collected Papers. Vol. IV*, Springer Collected Works in Mathematics, Springer, Heidelberg, 2013.
[38] J.-P. Serre, *Un critère d'indépendance pour Une famille de représentations ℓ-adiques*, Comment. Math. Helv. **88** (2013), 541–554.
[39] J. P. Serre and J. Tate, *Good reduction of abelian varieties*, Ann. Math. **88** (1968), 492–517.
[40] T. Shioda, *Algebraic cycles on abelian varieties of Fermat type*, Math. Ann. **258** (1981/82), 65–80.
[41] A. Silverberg, *Fields of definition for homomorphisms of abelian varieties*, J. Pure Appl. Algebra **77** (1992), 253–262.
[42] S. Tankeev, *Cycles on simple abelian varieties of prime dimension*, Izv. Akad. Nauk SSSR Ser. Mat. **46** (1982), 155–170.
[43] S. Tankeev, *Cycles on simple abelian varieties of prime dimension over number fields*, Izv. Akad. Nauk SSSR Ser. Mat. **51** (1987), 1214–1227.
[44] A. Vasiu, *Some cases of the Mumford–Tate conjecture and Shimura varieties*, Indiana Univ. Math. J. **57** (2008), 1–75.
[45] C.-F. Yu, *A note on the Mumford–Tate conjecture for CM abelian varieties*, Taiwan. J. Math. **19** (2015), 1073–1084.
[46] Y. G. Zarhin, *Endomorphisms and torsion of abelian varieties*, Duke Math. J. **54** (1987), 131–145.

Jeffrey D. Achter
Department of Mathematics
Colorado State University
Fort Collins, Colorado 80523
USA
j.achter@colostate.edu

Lian Duan
Institute of Mathematical Sciences
ShanghaiTech University
No. 393, Middle Huaxia Road
Pudong New District, Shanghai 201210
China
duanlian@shanghaitech.edu.cn

Xiyuan Wang
Department of Mathematics
The Ohio State University
100 Math Tower
231 West 18th Avenue
Columbus, Ohio 43210
USA
wang.15476@osu.edu

https://doi.org/10.1017/nmj.2023.19 Published online by Cambridge University Press