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HECKE C^*-ALGEBRAS, SCHLICHTING COMPLETIONS AND MORITA EQUIVALENCE

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Abstract The Hecke algebra $\mathcal{H}$ of a Hecke pair (G, H) is studied using the *Schlichting completion* $(\bar{G}, \bar{H})$, which is a Hecke pair whose Hecke algebra is isomorphic to $\mathcal{H}$ and which is topologized so that $\bar{H}$ is a compact open subgroup of $\bar{G}$. In particular, the representation theory and C^*-completions of $\mathcal{H}$ are addressed in terms of the projection $p = \chi_{\bar{H}} \in C^*(\bar{G})$ using both Fell's and Rieffel's imprimitivity theorems and the identity $\mathcal{H} = pC_c(\bar{G})p$. An extended analysis of the case where H is contained in a normal subgroup of G (and in particular the case where G is a semi-direct product) is carried out, and several specific examples are analysed using this approach.

Keywords: Hecke algebra; totally disconnected group; group C^*-algebra; Morita equivalence

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1. Introduction

The notion of an abstract Hecke algebra was introduced by Shimura in the 1950s and has its origins in Hecke's earlier work on elliptic modular forms. A *Hecke pair* (G, H) comprises a group G and a subgroup H for which every double coset is a finite union of left cosets, and the associated *Hecke algebra*, generated by the characteristic functions of double cosets, reduces to the group $*$-algebra of G/H when H is normal.

There is an extensive literature on Hecke algebras and Hecke subgroups, most commonly treating pairs of semi-simple groups such as $(\mathrm{PSL}(n, \mathbb{Q}), \mathrm{PSL}(n, \mathbb{Z}))$. Bost and Connes [**5**] introduced Hecke algebras to operator algebraists with (among other things) the realization that solvable groups give interesting number-theoretic examples of spontaneous symmetry breaking.

A number of authors, partly in an attempt to understand [**5**] (see Remark 7.2 for references) have studied Hecke C^*-algebras as crossed products by semigroup actions. Here we give a different construction, using what we call the *Schlichting completion* $(\bar{G}, \bar{H})$, based in part upon recent work of Tzanev [**33**]. (A slight variation on this construction appears in [**12**].) The idea is that $\bar{H}$ is a compact open subgroup of $\bar{G}$ such that the Hecke algebra of $(\bar{G}, \bar{H})$ is naturally identified with the Hecke algebra $\mathcal{H}$ of (G, H). The characteristic function p of $\bar{H}$ is a projection in the group C^*-algebra $A := C^*(\bar{G})$, and $\mathcal{H}$ can be identified with $pC_c(\bar{G})p \subseteq A$; thus, closure of $\mathcal{H}$ in A coincides with the corner pAp, which is Morita–Rieffel equivalent to the ideal $\overline{ApA}$. (This is Morita–Rieffel equivalence in its most basic form: one of the motivating examples in [**30**] was that Godement's study of a group $\bar{G}$ with a 'large' compact subgroup $\bar{H}$ can be explained by the fact that pAp and $\overline{ApA}$ have the same representation theory. In this more general situation $\bar{H}$ need not be open, so $p \in M(A)$.) We also require a variant of Rieffel's theory due to Fell, allowing us to relate representations of $\mathcal{H}$ to certain representations of G using a bimodule which is not quite a 'pre-imprimitivity bimodule' in Rieffel's sense. We shall describe situations in which the ideal $\overline{ApA}$ can be identified using crossed products.

Our thesis is that Schlichting completions can be used to efficiently study the representation theory of Hecke algebras, and we focus on the following phenomena:

(1) sometimes pAp is the enveloping C^*-algebra $C^*(\mathcal{H})$ of the Hecke algebra $\mathcal{H}$, and

(2) sometimes the projection p is full in A, making the C^*-completion pAp of $\mathcal{H}$ Morita–Rieffel equivalent to the group C^*-algebra A.

Earlier approaches to these issues (see, for example, [**5, 14, 20, 21, 23**]) depend upon the fact that the semigroup $T := \{t \in G \mid tHt^{-1} \supseteq H\}$ in their cases satisfies $G = T^{-1}T$; this is equivalent to the family $\{xHx^{-1} \mid x \in G\}$ of conjugates of H being directed downward, and we investigate this directedness condition in more detail. We also show that in order to have $C^*(\mathcal{H}) = pAp$ it is sufficient that G have a normal subgroup which contains H as a normal subgroup.

There are other aspects of Hecke algebras, not treated here, which we believe will be best studied using our approach, such as the treatment of KMS states in [**5, 27**], homology and K-theory in [**25, 33**] and the 2-prime analogue of the Bost–Connes algebra studied in [**24**]. The generalized Hecke algebras in [**9**] can also be studied in a similar fashion.

We begin in § 2 by recording our conventions regarding Hecke algebras. In § 3 we introduce *Hecke groups* of permutations; the central objects of interest are permutation groups which are closed in the topology of pointwise convergence. This lays the foundation for the study of Hecke pairs and their *Schlichting completions* in § 4. In § 4 we also give three alternative descriptions of the Schlichting completion: as an inverse limit, as the weak (equivalently, strong) closure of G in the quasi-regular representation on $\ell^2(G/H)$, and as the spectrum of a certain commutative Hopf C^*-algebra.

In § 5 we give the main technical properties of the projection $p = \chi_{\bar{H}}$. In § 6 we use the imprimitivity theorems of Fell and Rieffel to relate *positive* representations of the Hecke algebra $\mathcal{H}$ and *smooth* representations of G.

The semigroup T is studied in § 7 and is used in Theorems 7.4 and 7.5 to show that if $G = T^{-1}T$, then both phenomena (1) and (2) occur, recovering results of [**14, 23**].

Section 8 concerns a special situation involving a semi-direct product, which appears in many examples in the Hecke-C^*-algebra literature. In particular, we give a direct proof that the Hecke C^*-algebra is isomorphic to a full corner in a transformation group

C^*-algebra without using the theory of semigroup actions (as, for example, is done in [**21**]); we also show that the existence of a directing semigroup T is not needed in general. In addition we give an alternate analysis in terms of a certain transformation groupoid studied in [**2**]. The full justification of the main result of § 8 is deferred until § 9, where it is given in a more general context involving the twisted crossed products of Green [**13**].

The semigroup T is closely related to (and in some cases the same as) the one which appears in the semigroup crossed products of some authors mentioned above, although for us the semigroup crossed products play no role. In § 10 we show how our techniques can be used to easily recover the dilation result of [**23**]. Finally, in § 11 we illustrate our results with a number of examples. It turns out that even finite groups pose unanswered questions. While the rational '$ax + b$' group treated in [**5**] exhibits both phenomena (1) and (2) above (namely $C^*(\mathcal{H}) = pAp$ and p is full in A), we shall see that the rational Heisenberg group behaves quite differently.

2. Preliminaries

We mostly follow [**17**] for Hecke algebras; here we record our conventions. If H is a subgroup of a group G and $x \in G$, we define

$$H_x := H \cap xHx^{-1}.$$

Note that the map $hH_x \mapsto hxH$ of H/H_x into HxH/H is a bijection.

If every double coset of H in G contains only finitely many left cosets, i.e. if

$$L(x) := |HxH/H| = [H : H_x] < \infty \quad \text{for all } x \in G,$$

then H is a *Hecke subgroup* of G and (G, H) is a *Hecke pair*. A compact open subgroup of a topological group is obviously a Hecke subgroup, and Tzanev's theorem (see [**33**, Proposition 4.1] and also Proposition 4.6 and Theorem 4.8 below) shows that a Hecke pair (G, H) can always be densely embedded in an essentially unique Hecke pair $(\bar{G}, \bar{H})$ with $\bar{H}$ a compact open subgroup of $\bar{G}$. The subspace

$$\mathcal{H} = \mathcal{H}(G, H) := \operatorname{span}\{\chi_{HxH} \mid x \in G\}$$

of the vector space of complex functions on G becomes a $*$-algebra, called the *Hecke algebra* of the pair (G, H), with operations defined by

$$f * g(x) = \sum_{yH \in G/H} f(y)g(y^{-1}x)$$
$$f^*(x) = \overline{f(x^{-1})}\Delta(x^{-1}),$$

where Δ is the 'modular function' of the pair: this is a homomorphism $\Delta : G \to \mathbb{Q}^+$ defined by $\Delta(x) := L(x)/L(x^{-1})$. Warning: some authors do not include the factor of Δ in the involution; for us it arises naturally when we embed $\mathcal{H}$ in a certain C^*-algebra.

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Also, we eschew the term 'almost normal subgroup' (used by some authors for 'Hecke subgroup') since it already has at least one other meaning in the algebraic literature.

For some computations it is convenient to have formulae for the operations on the generators:

$$\chi_{HxH} * \chi_{HyH} = \sum_{\substack{zH\in HxH/H,\\ wH\in HyH/H}} \chi_{zwH} = \sum_{wH\in HyH/H} \chi_{HxwH}\frac{L(x)}{L(xw)},$$

$$\chi^*_{HxH} = \chi_{Hx^{-1}H}\Delta(x).$$

The formula for the adjoint is obvious. To verify the first formula for the convolution, note that for $u \in G$ we have

$$\begin{aligned}\chi_{HxH} * \chi_{HyH}(u) &= \sum_{zH\in G/H} \chi_{HxH}(z)\chi_{HyH}(z^{-1}u) = \sum_{zH\in HxH/H} \chi_{HyH}(z^{-1}u)\\ &= \sum_{zH\in HxH/H}\ \sum_{wH\in HyH/H} \chi_{wH}(z^{-1}u) = \sum_{\substack{zH\in HxH/H,\\ wH\in HyH/H}} \chi_{zwH}(u).\end{aligned}$$

For the second convolution formula, consider the projection

$$\Phi : c_c(G/H) \to c_c(H \setminus G/H)$$

defined by

$$\Phi(\chi_{xH}) = \frac{1}{L(x)}\chi_{HxH}$$

(where elements of both $c_c(G/H)$ and $c_c(H \setminus G/H)$ are identified with appropriately invariant functions on G). We have

$$\begin{aligned}\chi_{HxH} * \chi_{HyH} = \Phi(\chi_{HxH} * \chi_{HyH}) &= \sum_{\substack{zH\in HxH/H,\\ wH\in HyH/H}} \Phi(\chi_{zwH})\\ &= \sum_{\substack{zH\in HxH/H,\\ wH\in HyH/H}} \frac{1}{L(xw)}\chi_{HxwH}\end{aligned}$$

(because in choosing representatives z of cosets $zH \in HxH/H$ we can take $z \in Hx$)

$$= \sum_{wH\in HyH/H} \frac{L(x)}{L(xw)}\chi_{HxwH}.$$

χ_H is a unit for $\mathcal{H}$, and it is easy to check that $\mathcal{H}$ becomes a normed $*$-algebra with the 'ℓ^1-norm' defined by

$$\|f\|_1 = \sum_{xH\in G/H} |f(x)|. \tag{2.1}$$

One reason for our definition of f^* is that then $\|f^*\|_1 = \|f\|_1$. Note that $\|\chi_{HxH}\|_1 = L(x)$ for all $x \in G$.

Remark 2.1. $\mathcal{H}$ can also be considered as the $*$-algebra of a hypergroup [**4**, Chapter 1], so [**5**] gives an example of a discrete hypergroup having a non-trivial modular function.

3. Hecke groups

In § 4 we will give a careful development of a certain completion $(\bar{G}, \bar{H})$ of a Hecke pair (G, H), due largely to Tzanev [**33**], who built upon the work of Schlichting [**31**]. But it seems to us that the proper place to begin is not with Hecke pairs, but rather in the general context of permutation groups.

Let X be a set, and let $\operatorname{Map} X$ denote the set of maps from X to itself, equipped with the product topology (that is, the topology of pointwise convergence) arising from the discrete topology on X. Clearly, $\operatorname{Map} X$ is Hausdorff. Furthermore, let $\operatorname{Per} X$ be the set of bijections of X onto itself, with the relative topology from $\operatorname{Map} X$.

Lemma 3.1. $\operatorname{Map} X$ *is a topological semigroup;* $\operatorname{Per} X$ *is a topological group.*

Proof. Let $\phi_i \to \phi$ and $\psi_i \to \psi$ in $\operatorname{Map} X$. Then, for each $s \in X$, $\psi_i(s) = \psi(s)$ eventually, so $\phi_i\psi_i(s) = \phi\psi(s)$ eventually. Thus, $\phi_i\psi_i \to \phi\psi$, so multiplication is continuous in $\operatorname{Map} X$. If $\phi_i \to \phi$ in $\operatorname{Per} X$, then, for each $s \in X$, eventually $\phi_i\phi^{-1}(s) = s$, and hence $\phi_i^{-1}(s) = \phi^{-1}(s)$. It follows that $\phi_i^{-1} \to \phi^{-1}$, so the involution on $\operatorname{Per} X$ is also continuous. □

Remark 3.2. Although we will not need this fact, $\operatorname{Per} X$ is complete with respect to the two-sided uniformity. To see this, suppose that $\{\phi_i\}$ is a Cauchy net in the two-sided uniformity. Then, for each $s \in X$, eventually $\phi_j\phi_i^{-1}(s) = s = \phi_i^{-1}\phi_j(s)$, i.e. $\phi_i(s) = \phi_j(s)$ and $\phi_i^{-1}(s) = \phi_j^{-1}(s)$. So we can define two functions by $\phi(s) = \lim \phi_i(s)$ and $\psi(s) = \lim \phi_i^{-1}(s)$. Then for large i we have $\psi\phi(s) = \phi_i^{-1}\phi(s) = \phi_i^{-1}\phi_i(s) = s$ and, similarly, $\phi\psi(s) = s$. So $\phi = \lim \phi_i \in \operatorname{Per} X$.

Interestingly, $\operatorname{Per} X$ is in general *not* complete with respect to either one-sided uniformity; see Example 3.4 for an illustration of this.

Recall that $\operatorname{Map} X$, being a product, may be viewed as an inverse limit: let $\mathcal{F}$ denote the family of finite subsets of X, and for each $F \in \mathcal{F}$ let $\operatorname{Map}(F, X)$ denote the set of maps from F to X, with the product topology. For $E \subseteq F$, define $\pi_E^F : \operatorname{Map}(F, X) \to \operatorname{Map}(E, X)$ by restriction: that is, $\pi_E^F(\phi) = \phi|_E$. Then $\{\operatorname{Map}(F, X), \pi_E^F\}$ is an inverse system, and $\operatorname{Map} X$ is identified as a topological space with the inverse limit $\varprojlim_{F\in\mathcal{F}} \operatorname{Map}(F, X)$, with the canonical projections $\pi_F : \operatorname{Map} X \to \operatorname{Map}(F, X)$ being the restriction maps: $\pi_F(\phi) = \phi|_F$.

It will be important for us to know that we can play the same game with any subset S of $\operatorname{Map} X$: for each $F \in \mathcal{F}$ put $S|_F = \{\phi|_F \mid \phi \in S\}$, and for $E \subseteq F$ define $\pi_E^F : S|_F \to S|_E$ by restriction. Then again we have an inverse system, and we can identify the inverse limit $\varprojlim_{F\in\mathcal{F}} S|_F$ with a subspace of $\varprojlim_{F\in\mathcal{F}} \operatorname{Map}(F, X)$, since $S|_F \subseteq \operatorname{Map}(F, X)$ for each F. To be precise, under the identification of $\varprojlim_{F\in\mathcal{F}} \operatorname{Map}(F, X)$ with $\operatorname{Map} X$ described above, we have

$$\varprojlim_{F\in\mathcal{F}} S|_F = \{\phi \in \operatorname{Map} X \mid \phi|_F \in S|_F \text{ for all } F \in \mathcal{F}\}.$$

It follows from the definition of the product topology that this inverse limit is just the closure $\bar{S}$ of S in $\operatorname{Map} X$. For convenient reference we formalize this as follows.

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Lemma 3.3. *For any subset S of* $\operatorname{Map} X$, $\bar{S} = \varprojlim_{F \in \mathcal{F}} S|_F$.

Now let Γ be a subgroup of $\operatorname{Per} X$, and for each $F \in \mathcal{F}$ consider the open subgroup Γ_F of Γ defined by

$$\Gamma_F = \{\phi \in \Gamma \mid \phi|_F = \mathrm{id}\}.$$

While the set $\Gamma|_F = \{\phi|_F \mid \phi \in \Gamma\}$ of restrictions is not necessarily a group, it has a transitive action of Γ on the left. From this we see that the map $\pi_F : \Gamma \to \Gamma|_F$ is constant on each coset $\phi\Gamma_F$ and therefore induces a Γ-equivariant homeomorphism between the discrete spaces Γ/Γ_F and $\Gamma|_F$. With this identification, for $E \subseteq F$ the bonding map $\pi^F_E : \Gamma/\Gamma_F \to \Gamma/\Gamma_E$ is given by $\pi^F_E(\phi\Gamma_F) = \phi\Gamma_E$. Thus, we get

$$\bar{\Gamma} = \varprojlim_{F \in \mathcal{F}} \Gamma/\Gamma_F.$$

Of course, the subgroups Γ_F are in general not normal in Γ; the above inverse limit is a purely topological one. In fact, in general $\bar{\Gamma}$ will not be contained in $\operatorname{Per} X$, because if X is infinite, $\operatorname{Per} X$ is not closed in $\operatorname{Map} X$.

Example 3.4. Let $X = \mathbb{N}$ and, for each n, define $\phi_n \in \operatorname{Per} X$ by

$$\phi_n(s) = \begin{cases} s+1 & \text{if } s < n, \\ 0 & \text{if } s = n, \\ s & \text{if } s > n. \end{cases}$$

Then $\phi_n \to \sigma$ in $\operatorname{Map} X$, where σ is the shift map $s \mapsto s+1$. Since σ is not in $\operatorname{Per} X$, $\operatorname{Per} X$ is not closed in $\operatorname{Map} X$. (This also shows that $\operatorname{Per} X$ is not complete with respect to either one-sided uniformity, since $\{\phi_n\}$ is Cauchy for the left uniformity, and $\{\phi_n^{-1}\}$ is Cauchy for the right.)

The following definition introduces a condition on Γ which guarantees that $\bar{\Gamma} \subseteq \operatorname{Per} X$.

Definition 3.5. A group $\Gamma \subseteq \operatorname{Per} X$ is called a *Hecke group on* X if for all $s, t \in X$ the orbit $\Gamma_s(t)$ is finite, where

$$\Gamma_s = \{\phi \in \Gamma \mid \phi(s) = s\}$$

is the stability subgroup of Γ at s.

Observe that whenever r and s are in the same Γ-orbit, Γ_r will have finite orbits if and only if Γ_s does, so it is enough to check that $\Gamma_s(t)$ is finite for a single s from each Γ-orbit in X. Also, the condition on $\Gamma_s(t)$ is equivalent to $\Gamma_s \cap \Gamma_t$ having finite index in Γ_s.

Also note that for any subgroup Γ of $\operatorname{Per} X$, each stability subgroup Γ_s is by definition open in Γ in the relative (product) topology.

Proposition 3.6. *Let Γ be a Hecke group on X, and let $\bar{\Gamma}$ be the closure of Γ in* $\operatorname{Map} X$. *Then $\bar{\Gamma}$ is a locally compact, totally disconnected, closed subgroup of* $\operatorname{Per} X$. *For each $s \in X$, $\overline{\Gamma}_s = (\bar{\Gamma})_s$ is compact and open in $\bar{\Gamma}$.*

Proof. We first show that $\bar{\Gamma} \subseteq \operatorname{Per} X$. Fix $\phi \in \bar{\Gamma}$. Then, for any $r, s \in X$ with $r \neq s$, there exists $\psi \in \Gamma$ such that $\psi(t) = \phi(t)$ for all $t \in \{r, s\}$. Since ψ is injective, we have $\psi(r) \neq \psi(s)$, whence $\phi(r) \neq \phi(s)$, so ϕ is also injective.

Now fix $s \in X$. Choose $\gamma \in \Gamma$ such that $\gamma(s) = \phi(s)$, and put

$$F = \Gamma_s \gamma^{-1}(s) \cup \{s\},$$

a finite subset of X. Now choose $\psi \in \Gamma$ such that $\psi(t) = \phi(t)$ for all $t \in F$. Then, in particular, $\psi(s) = \phi(s) = \gamma(s)$, so $\psi^{-1}\gamma \in \Gamma_s$. It follows that $\psi^{-1}(s) = \psi^{-1}\gamma\gamma^{-1}(s)$ is in F, so

$$\phi\psi^{-1}(s) = \psi\psi^{-1}(s) = s.$$

Therefore, ϕ is onto.

To see that each $\bar{\Gamma}_s$ is compact, note that $\Gamma_s \subseteq \prod_{t \in X} \Gamma_s(t)$, which is compact by the Tychonoff theorem. For the openness, note that $\operatorname{Map}_s X := \{\phi \in \operatorname{Map} X \mid \phi(s) = s\}$ is a closed and open subset of $\operatorname{Map} X$, so

$$\bar{\Gamma}_s = \overline{\Gamma \cap \operatorname{Map}_s X} = \bar{\Gamma} \cap \operatorname{Map}_s X = (\bar{\Gamma})_s$$

is evidently an open subset of $\bar{\Gamma}$.

Finally, since $\bar{\Gamma}$ has a compact neighbourhood of the identity (namely any $\overline{\Gamma}_s$), it is locally compact, and of course $\bar{\Gamma}$ is totally disconnected because $\operatorname{Map} X$ is totally disconnected. □

Definition 3.7. A group $\Gamma \subseteq \operatorname{Per} X$ is called a *Schlichting group on* X if every stability subgroup of Γ is compact in Γ. If Γ is a Hecke group on X, the closure $\bar{\Gamma}$ of Γ in $\operatorname{Map} X$ is a Schlichting group on X, which we call the *Schlichting completion* of Γ.

Our motivation for choosing the name Schlichting comes from [**31**]. Every Schlichting group Γ on X is a Hecke group on X. To see this, fix $s, t \in X$ and, for each $u \in \Gamma_s(t)$, let $U_u = \{\phi \in \Gamma_s \mid \phi(t) = u\}$. Then the collection $\{U_u \mid u \in \Gamma_s(t)\}$ is a disjoint open cover of Γ_s, and hence must be finite. But the map $u \mapsto U_u$ is injective, so the orbit $\Gamma_s(t)$ must be finite as well. Furthermore, every Schlichting group on X is locally compact (having a compact neighbourhood of the identity), and hence complete, so is in particular closed in $\operatorname{Map} X$. Thus, every Schlichting group is its own Schlichting completion. In fact, the Schlichting groups on X are precisely the Hecke groups on X which are closed in $\operatorname{Map} X$.

For any Hecke group Γ, the Schlichting completion $\bar{\Gamma}$ coincides with the usual completion of Γ as a topological group (since Γ is dense in $\bar{\Gamma}$ and $\bar{\Gamma}$ is complete). Thus, we have the following abstract characterization of $\bar{\Gamma}$ (cf. [**6**, Chapter 3, § 3.3, Proposition 5]).

Proposition 3.8. *Let Γ be a Hecke group on X, and let $\bar{\Gamma}$ be its Schlichting completion. Every continuous homomorphism σ of Γ into a complete Hausdorff group L has a unique extension to a continuous homomorphism $\bar{\sigma}$ of $\bar{\Gamma}$ into L.*

If σ is in fact a topological group isomorphism of Γ onto a dense subgroup of L, then $\bar{\sigma}$ will be a topological group isomorphism of $\bar{\Gamma}$ onto L.

Interestingly, not every subgroup Γ of $\operatorname{Per} X$ which is closed in $\operatorname{Map} X$ is a Hecke group on X, even when Γ acts transitively on X.

Example 3.9. Let $X = \mathbb{Z} \times \mathbb{Z}_2$, and let Γ be the subgroup of $\operatorname{Per} X$ generated by the permutations

$$\phi(x,a) = \begin{cases} (x+1,a) & \text{if } a = 0, \\ (x,a) & \text{if } a = 1, \end{cases}$$

$$\eta(x,a) = \begin{cases} (x,1) & \text{if } a = 0, \\ (x,0) & \text{if } a = 1. \end{cases}$$

Then Γ acts transitively on X, and $\Gamma_{(0,0)}(0,1) = \mathbb{Z} \times \{1\}$, so Γ is not a Hecke group on X.

To see that Γ is closed in $\operatorname{Map} X$, first note that any $\gamma \in \Gamma$ is determined by its values on $F = \{(0,0),(0,1)\}$. If (γ_n) is a sequence in Γ which converges to ξ in $\operatorname{Map} X$, we can choose N such that $n \geqslant N$ implies $\gamma_n = \xi$ on F; but then $\gamma_n = \xi = \gamma_N$ on all of X for all such n, so the sequence is eventually constant. In particular, $\xi = \gamma_N \in \Gamma$.

4. Schlichting pairs

We now apply the permutation-group techniques of the preceding section to the study of Hecke pairs, recovering Tzanev's construction in [**33**]. The results imply in particular that for every reduced Hecke pair (G,H) there is a pair $(\bar{G},\bar{H})$ consisting of a locally compact, totally disconnected group $\bar{G}$ and a compact open subgroup $\bar{H}$ of $\bar{G}$ such that G is dense in $\bar{G}$, H is dense in $\bar{H}$, and the Hecke algebra of $(\bar{G},\bar{H})$ is isomorphic to the Hecke algebra of (G,H).

Let G be a group, and let H be a subgroup of G. Define $\theta : G \to \operatorname{Per} G/H$ by

$$\theta(x)(yH) = xyH \quad \text{for } x \in G,\ yH \in G/H, \tag{4.1}$$

and put $\Gamma = \theta(G)$. Note that $\theta^{-1}(\Gamma_{xH}) = xHx^{-1}$ for each $xH \in G/H$.

Lemma 4.1. *With notation as above, (G,H) is a Hecke pair if and only if Γ is a Hecke group on G/H.*

Proof. The lemma follows immediately from the observation that

$$\Gamma_{xH}(yH) = xHx^{-1}(yH) = x(Hx^{-1}yH)$$

for each $x,y \in G$. □

Note that $\ker\theta = \bigcap_{x\in G} xHx^{-1}$, so θ will be injective if and only if the pair (G,H) is *reduced* in the sense that $\bigcap_{x\in G} xHx^{-1} = \{e\}$. If (G,H) is not reduced, then the pair $(G/\ker\theta, H/\ker\theta)$ will be a reduced Hecke pair, which is called the *reduction* of (G,H). Replacing a given Hecke pair by its reduction gives an isomorphic Hecke algebra, so it does no harm to restrict our attention to reduced Hecke pairs.

Hypothesis 4.2. We assume from now on that our Hecke pairs are reduced.

Since the family $\{\Gamma_{xH} \mid xH \in G/H\}$ is a neighbourhood sub-base at the identity of Γ, the inverse images $\{xHx^{-1} \mid x \in G\}$ give a neighbourhood sub-base at the identity for a group topology on G with respect to which θ is continuous.

Definition 4.3. The group topology on G generated by the collection $\{xHx^{-1} \mid x \in G\}$ is called the *Hecke topology* of the pair (G, H).

Because $\overline{\{e\}} = \bigcap_{x\in G} xHx^{-1}$, the Hecke topology will be Hausdorff if and only if (G, H) is reduced. A given group topology on G will be stronger than the Hecke topology if and only if H is a member of the given topology.

Definition 4.4. A reduced Hecke pair (G, H) is called a *Schlichting pair* if H is compact and open in the Hecke topology on G.

Note that a reduced Hecke pair (G, H) is a Schlichting pair if and only if $\Gamma = \theta(G)$ is a Schlichting group on G/H: since (G, H) is reduced, $\theta : G \to \Gamma$ will be a homeomorphism which carries each conjugate xHx^{-1} to the stabilizer subgroup Γ_{xH}.

Proposition 4.5. *If G is a topological group and H is a compact open subgroup of G such that*

$$\bigcap_{x\in G} xHx^{-1} = \{e\},$$

then the given topology on G coincides with the Hecke topology, so (G, H) is a Schlichting pair.

Proof. Since H is open in the given topology on G, the identity map $\mathrm{id} : G \to G$ is a continuous bijection from the given topology to the Hecke topology. Since H is compact in the given topology and the Hecke topology is Hausdorff, $\mathrm{id}|_H$ is a homeomorphism; and since H is open in both topologies, it follows that id is a homeomorphism. □

Proposition 4.6. *If (G, H) is a Hecke pair, then $(\overline{\theta(G)}, \overline{\theta(H)})$ is a Schlichting pair, where θ is as defined in (4.1) and the closures are taken in* $\mathrm{Map}\, G/H$*.*

Proof. Set $\Gamma = \theta(G)$, which is a Hecke group on G/H by Lemma 4.1. Note that $\Gamma_H = \theta(H)$. Proposition 3.6 tells us that $(\overline{\Gamma})_H = \overline{\Gamma}_H$ is a compact open subgroup of $\overline{\Gamma}$. Thus, the transitive action of $\overline{\Gamma}$ on G/H is isomorphic to the canonical action on $\overline{\Gamma}/\overline{\Gamma}_H$. Since $\overline{\Gamma}$ acts faithfully on G/H, it does so also on $\overline{\Gamma}/\overline{\Gamma}_H$, and this proves that the pair $(\overline{\Gamma}, \overline{\Gamma}_H)$ is reduced. The result now follows from Proposition 4.5. □

Definition 4.7. For any Hecke pair (G, H), the pair $(\overline{\theta(G)}, \overline{\theta(H)})$ is called the *Schlichting completion* of (G, H).

When (G, H) is reduced, we will suppress the map θ in the notation for the Schlichting completion. Thus, $\bar{G}$ is a locally compact, totally disconnected group and $\bar{H}$ is a compact open subgroup.

The following uniqueness theorem, essentially due to Tzanev [**33**, Proposition 4.1], gives an abstract characterization of the relation between a reduced Hecke pair and its Schlichting completion. We now give a different proof from that in [**33**].

Theorem 4.8. *Let (G,H) be a reduced Hecke pair and let $(\bar{G},\bar{H})$ be its Schlichting completion. If (L,K) is a Schlichting pair and σ is a homomorphism of G into L such that $\sigma(G)$ is dense in L and $\sigma(H)\subseteq K$, there exists a unique continuous homomorphism $\bar{\sigma}$ of $\bar{G}$ into L such that $\bar{\sigma}\circ\theta=\sigma$.*

If we further assume that $H=\sigma^{-1}(K)$, then $\bar{\sigma}$ will be a topological group isomorphism of $\bar{G}$ onto L and of $\bar{H}$ onto K.

Proof. By Lemma 4.1, $\varGamma=\theta(G)$ is a Hecke group on G/H, and $\bar{G}=\bar{\varGamma}$ is its Schlichting completion. L is a complete Hausdorff group because (L,K) is a Schlichting pair. Thus, for the first part it suffices, by Proposition 3.8, to prove that σ is continuous for the Hecke topologies of G and L, and for the second part it suffices to show that the continuous extension $\bar{\sigma}$ is also injective and open for the Hecke topologies of $\bar{G}$ and L.

So, first assume that $\overline{\sigma(G)}=L$ and $\sigma(H)\subseteq K$. A typical sub-basic open neighbourhood of e in L is of the form xKx^{-1} for $x\in L$. Since $\sigma(G)$ is dense in L and K is open in L, there exists $y\in G$ such that $xK=\sigma(y)K$; hence, $xKx^{-1}=\sigma(y)K\sigma(y)^{-1}$. Thus, $\sigma(yHy^{-1})\subseteq xKx^{-1}$, showing that σ is continuous.

For the other part, further assume that $H=\sigma^{-1}(K)$. We must show that the above continuous extension $\bar{\sigma}$ is injective and open.

We have $\sigma(G)\cap K=\sigma(H)$. Thus, since $\sigma(G)$ is dense and K is open and closed, we have

$$K=K\cap\overline{\sigma(G)}=\overline{K\cap\sigma(G)}=\overline{\sigma(H)}.$$

Since $\bar{H}$ is compact, so is $\bar{\sigma}(\bar{H})$. Thus, $\overline{\sigma(H)}\subset\bar{\sigma}(\bar{H})$. By continuity we have

$$\bar{\sigma}(\bar{H})\subset\overline{\bar{\sigma}(H)}=\overline{\sigma(H)}.$$

It follows that $\bar{\sigma}(\bar{H})=K$. Similarly, in the notation of the second paragraph of the proof we have $\bar{\sigma}(y\bar{H}y^{-1})=xKx^{-1}$. This shows that $\bar{\sigma}$ is open.

To show $\bar{\sigma}$ is injective, we need to know $\bar{H}=\bar{\sigma}^{-1}(K)$. Since $\bar{\sigma}^{-1}(K)$ is closed and contains $\sigma^{-1}(K)=H$, we have $\bar{H}\subset\bar{\sigma}^{-1}(K)$. For the opposite inclusion, let $x\in\bar{G}$, and assume that $\bar{\sigma}(x)\in K$. Choose $y\in G$ such that $x\bar{H}=y\bar{H}$. Then

$$K=\bar{\sigma}(x)K=\bar{\sigma}(x)\bar{\sigma}(\bar{H})=\bar{\sigma}(x\bar{H})=\bar{\sigma}(y\bar{H})=\sigma(y)\bar{\sigma}(\bar{H})=\sigma(y)K,$$

so $y\in\sigma^{-1}(K)=H$; hence, $x\bar{H}=\bar{H}$ and therefore $x\in\bar{H}$.

Thus, we do have $\bar{\sigma}^{-1}(K)=\bar{H}$, and so, because $\bar{\sigma}(\bar{G})=L$ and both

$$\bigcap_{x\in\bar{G}}x\bar{H}x^{-1}\quad\text{and}\quad\bigcap_{y\in L}yKy^{-1}$$

are trivial, it is easy to see that $\bar{\sigma}$ must be injective. □

It follows from Theorem 4.8 that every Schlichting pair is (isomorphic to) its own Schlichting completion.

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Proposition 4.9. *Let (G, H) be a reduced Hecke pair, and let $(\bar{G}, \bar{H})$ be its Schlichting completion. Then the following maps are bijections:*

(i) $xH \mapsto x\bar{H} : G/H \to \bar{G}/\bar{H}$,

(ii) $xHx^{-1} \mapsto x\bar{H}x^{-1} : \{xHx^{-1} \mid x \in G\} \to \{x\bar{H}x^{-1} \mid x \in \bar{G}\}$,

(iii) $HxH \mapsto \bar{H}x\bar{H} : H \setminus G/H \to \bar{H} \setminus \bar{G}/\bar{H}$.

Moreover, the map in (i) *is equivariant for the left G-actions.*

Proof. Suppose that $x \in G$ but $x \notin H$. Then $xH \neq H$, so $\{\phi \in \operatorname{Map} X \mid \phi(H) = xH\}$ is an open neighbourhood of x which does not meet H; thus, $x \notin \bar{H}$. In other words, $G \cap \bar{H} = H$, and it follows from this that the map in (i) is injective. For surjectivity, each $z\bar{H}$ is open in $\bar{G}$, so there exists $x \in G$ with $x \in z\bar{H}$, whence $x\bar{H} = z\bar{H}$. Equivariance is obvious.

Surjectivity in (ii) follows from that of (i). For injectivity, if $x \in G$ and $x\bar{H}x^{-1} = \bar{H}$, we have

$$H = G \cap \bar{H} = G \cap x\bar{H}x^{-1} = x(G \cap \bar{H})x^{-1} = xHx^{-1}.$$

Surjectivity in (iii) also follows from (i). For injectivity, suppose that $x, y \in G$ such that $\bar{H}x\bar{H} = \bar{H}y\bar{H}$. Then $x\bar{H}y^{-1} \cap \bar{H}$ is non-empty and open in $\bar{G}$; by density, we can choose $h \in x\bar{H}y^{-1} \cap H$, and it follows that $xH = hyH$, whence $HxH = HyH$. □

4.1. Schlichting completions as inverse limits

Suppose that (G, H) is a reduced Hecke pair, and let $F \subseteq G/H$ be finite. Identifying G with the associated Hecke group on G/H, we have

$$G_F = \bigcap_{xH \in F} xHx^{-1}$$

(as the notation implies, it is only necessary to choose one representative from each coset in F). Thus, each G_F is just the intersection of finitely many conjugates of H. From the discussion following Lemma 3.3 we have the following result.

Proposition 4.10. *For any reduced Hecke pair (G, H), the Schlichting completion is a topological inverse limit:*

$$\bar{G} = \varprojlim_{\substack{F \subseteq G/H \\ \text{finite}}} G/G_F.$$

Remark 4.11. (i) Since the subgroups G_F of G are in general non-normal, it is not at all obvious from the above description that $\bar{G}$ is a group. But note that if $F \subseteq G/H$ is finite, then the set $F' = HF \subseteq G/H$ is finite and H-invariant, so $H_{F'}$ is normal in H; thus, $\bar{H} = \varprojlim H/H_F$ is an inverse limit of groups. It is a non-trivial exercise to work out the formulae for the product and inverse in $\bar{G}$ using the standard notation of inverse limits.

(ii) As remarked following Definition 3.7, the Schlichting completion $\bar{G}$ is just the completion of G in the two-sided uniformity arising from the Hecke topology on G. But again, some of the properties of $\bar{G}$ are not obvious from this description.

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4.2. Schlichting completions via Hopf algebras

The group structure on $\bar{G} = \varprojlim G/G_F$ can also be obtained from a Hopf algebra structure on

$$\mathcal{A} := C_0(\bar{G}) = \varinjlim_{\substack{F\subseteq G/H\\ \text{finite}}} c_0(G/G_F) = \overline{\bigcup_{\substack{F\subseteq G/H\\ \text{finite}}} c_c(G/G_F)}.$$

For this it is useful to consider the dense subalgebra of *smooth functions* with respect to the Schlichting topology:

$$\mathcal{A}_0 = C_c^\infty(G) := \bigcup_{\substack{F\subseteq G/H\\ \text{finite}}} c_c(G/G_F),$$

i.e. $\mathcal{A}_0$ is the set of all complex functions f on G with finite range and such that $f(xs) = f(x)$ for all s in some G_F. The co-multiplication and antipode on $\mathcal{A}_0$ are given by the maps

$$\delta(f)(s,t) = f(st) \quad \text{and} \quad \nu(f)(s) = f(s^{-1}). \tag{4.2}$$

Proposition 4.12. *$\mathcal{A}_0$ is a multiplier Hopf algebra (as defined in [**35**]); i.e. for $f, g \in \mathcal{A}_0$ we have $\nu(f) \in \mathcal{A}_0$, $\delta(f)(g \otimes 1) \in \mathcal{A}_0 \odot \mathcal{A}_0$ and functions of this form span $\mathcal{A}_0 \odot \mathcal{A}_0$. The co-unit is given by $\varepsilon(f) = f(e)$ and left Haar measure by $\mu(\chi_{xG_F}) = [G_F : H \cap G_F] \cdot [H : H \cap G_F]^{-1}$.*

Here '$\odot$' means the algebraic tensor product. The proof is somewhat technical, but straightforward. $\mathcal{A}$ is the uniform closure of $\mathcal{A}_0$, so the maps δ and ν from (4.2) and ε from Proposition 4.12 extend to $\mathcal{A}$ and we have the following result.

Theorem 4.13. *$(\mathcal{A}, \delta, \nu)$ is a commutative Hopf C^*-algebra. The group structure on $\operatorname{spec}(\mathcal{A}) = \varprojlim G/G_F$ is the same as in Proposition 4.10.*

Here we leave the proof to the reader; one checks that the maps δ and ν on $\mathcal{A}$ satisfy [**34**, Theorem 3.8], so $\operatorname{spec}(\mathcal{A})$ is a locally compact group, and one has to check that the product is the same as the one coming from $\operatorname{Per}(G/H)$.

4.3. Schlichting completions via quasi-regular representations

Another approach is as follows: look at the quasi-regular representation $x \mapsto \lambda_H(x)$ of G on $\ell^2(G/H)$ and let $\bar{G}$ be the closure of $\lambda_H(G)$ in the weak (or strong) operator topology. The proof that this gives the same result as the other approaches is once again left to the reader.

Remark 4.14. Although we have chosen the names 'Hecke topology', 'Schlichting completion', etc., other names could also be appropriate, since similar constructions have been studied by many people for a long time.

5. The fundamental projection p

The Schlichting completion is useful because the Hecke algebra $\mathcal{H}$ of a Hecke pair (G,H) can be identified with a $*$-subalgebra of the convolution $*$-algebra $C_c(\bar{G}) \subseteq C^*(\bar{G})$. In fact, the characteristic function $\chi_{\bar{H}}$ turns out to be a projection in $C_c(\bar{G})$ (see below), and $\mathcal{H}$ is (identified with) the corresponding corner $\chi_{\bar{H}} C_c(\bar{G}) \chi_{\bar{H}}$ (Corollary 5.4). This brings a great deal of well-developed machinery into play which would not otherwise be available, since, in general, $\chi_H \notin C^*(G)$.

In this section we consider a reduced Hecke pair (G,H) and its Schlichting completion $(\bar{G},\bar{H})$. We normalize the left Haar measure μ on $\bar{G}$ so that $\mu(\bar{H}) = 1$, and we use this to define the (usual) convolution and involution on $C_c(\bar{G}) \subseteq A$:

$$f * g(x) = \int_{\bar{G}} f(t) g(t^{-1}x)\, \mathrm{d}t \quad \text{and} \quad f^*(x) = \overline{f(x^{-1})} \Delta_{\bar{G}}(x^{-1}),$$

where $\Delta_{\bar{G}}$ is the modular function on $\bar{G}$. We make sense of expressions of the form xf and fx for $x \in \bar{G}$ and $f \in C_c(\bar{G})$ by identifying $\bar{G}$ with its image in the multiplier algebra $M(A)$ (and similarly for other groups), so that

$$(xf)(s) = f(x^{-1}s) \quad \text{and} \quad (fx)(s) = f(sx^{-1}) \Delta_{\bar{G}}(x^{-1})$$

for all $s \in \bar{G}$.

Note that, since $\bar{H}$ is compact, we have $\Delta_{\bar{G}}(h) = 1$ for all $h \in \bar{H}$, and it follows that

$$L(x)\mu(\bar{H}) = \mu(\bar{H}x\bar{H}) = R(x)\mu(\bar{H}x) = L(x^{-1})\mu(\bar{H}x)$$

for each $x \in \bar{G}$; thus, the somewhat mysterious modular function Δ appearing in [**5**] (and in § 2) is simply $\Delta_{\bar{G}}$, and we will no longer differentiate the two in our notation.

We now define

$$p = \chi_{\bar{H}} \quad \text{and} \quad A = C^*(\bar{G}).$$

Thus, p is a projection (by which we mean $p = p^* = p^2$) in $C_c(\bar{G})$, and hence in A. Rieffel's theory immediately tells us that Ap is an $\overline{ApA} - pAp$ imprimitivity bimodule. (Here and elsewhere when we write $\overline{ApA}$ we mean the closed span of the products, yielding a closed two-sided ideal of A.) But before pursuing this further, we must acquire a little expertise with the projection p.

Lemma 5.1. *For each $x \in \bar{G}$,*

(i) $xp = \chi_{x\bar{H}}$,

(ii) $px = \Delta(x)^{-1}\chi_{\bar{H}x}$, *and*

(iii) $xpx^{-1} = \Delta(x)\chi_{x\bar{H}x^{-1}}$.

Moreover, there exist $y, z \in G$ such that $yp = xp$, $pz = px$ and $ypy^{-1} = xpx^{-1}$.

Proof. Items (i)–(iii) follow from elementary calculations, and then the last statement is immediate from Proposition 4.9. □

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Lemma 5.2.

(i) $pC_c(\bar{G}) = \operatorname{span}_{x\in G} px$;

(ii) $C_c(\bar{G})p = \operatorname{span}_{x\in G} xp$;

(iii) $pC_c(\bar{G})p = \operatorname{span}_{x\in G} pxp$;

(iv) $C_c(\bar{G})pC_c(\bar{G}) = \operatorname{span}_{x,y\in G} xpy$.

In (iv) we intend for '$C_c(\bar{G})pC_c(\bar{G})$' to mean the linear span of the products.

Proof. By direct calculation, pf is constant on right cosets of $\bar{H}$ for $f \in C_c(\bar{G})$. Thus,

$$pC_c(\bar{G}) = \operatorname*{span}_{x\in G} \chi_{\bar{H}x} = \operatorname*{span}_{x\in G} px,$$

proving (i). Then (ii) follows by taking adjoints, and (i), (ii) imply (iii), (iv). □

Lemma 5.3. *Let π be a (continuous unitary) representation of $\bar{G}$ on a Hilbert space V, and suppose that $\xi \in V$ has finite $\bar{H}$-orbit. Let*

$$\bar{H}_{\pi,\xi} := \{h \in \bar{H} \mid \pi(h)\xi = \xi\}. \tag{5.1}$$

Then

$$\pi(p)\xi = [\bar{H} : \bar{H}_{\pi,\xi}]^{-1} \sum_{h\bar{H}_{\pi,\xi}\in\bar{H}/\bar{H}_{\pi,\xi}} \pi(h)\xi.$$

Proof. We have $\mu(\bar{H}_{\pi,\xi}) = [\bar{H} : \bar{H}_{\pi,\xi}]^{-1}$, so

$$\begin{aligned}
\pi(p)\xi &= \int_{\bar{H}} \pi(k)\xi \,\mathrm{d}k \\
&= \sum_{h\bar{H}_{\pi,\xi}\in\bar{H}/\bar{H}_{\pi,\xi}} \int_{h\bar{H}_{\pi,\xi}} \pi(k)\xi \,\mathrm{d}k \\
&= \sum_{h\bar{H}_{\pi,\xi}\in\bar{H}/\bar{H}_{\pi,\xi}} \int_{\bar{H}_{\pi,\xi}} \pi(hk)\xi \,\mathrm{d}k \\
&= \sum_{h\bar{H}_{\pi,\xi}\in\bar{H}/\bar{H}_{\pi,\xi}} \pi(h) \int_{\bar{H}_{\pi,\xi}} \xi \,\mathrm{d}k \\
&= \sum_{h\bar{H}_{\pi,\xi}\in\bar{H}/\bar{H}_{\pi,\xi}} \mu(\bar{H}_{\pi,\xi})\pi(h)\xi \\
&= [\bar{H} : \bar{H}_{\pi,\xi}]^{-1} \sum_{h\bar{H}_{\pi,\xi}\in\bar{H}/\bar{H}_{\pi,\xi}} \pi(h)\xi.
\end{aligned}$$

□

Recall that for $x \in \bar{G}$ we have defined $\bar{H}_x$ to be $\bar{H} \cap x\bar{H}x^{-1}$.

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Corollary 5.4. *For all $x \in \bar{G}$,*

$$pxp = \frac{1}{L(x)} \sum_{h\bar{H}_x \in \bar{H}/\bar{H}_x} hxp = \frac{1}{L(x)} \chi_{\bar{H}x\bar{H}}.$$

The $*$-algebras $pC_c(\bar{G})p$ and $\mathcal{H}(\bar{G}, \bar{H})$ are identical, and (the restriction of) the L^1-norm on $C_c(\bar{G})$ coincides with the ℓ^1-norm on $\mathcal{H}$ defined by (2.1). In particular, $\|pxp\|_1 = 1$ for each $x \in \bar{G}$.

Proof. Let λ be the left regular representation of $\bar{G}$, and view $xp \in C_c(\bar{G})$ as an element of $L^2(\bar{G})$. For $h \in \bar{H}$ we have

$$\lambda(h)xp = xp \iff \chi_{hx\bar{H}} = \chi_{x\bar{H}} \iff h \in x\bar{H}x^{-1}.$$

Thus, $\bar{H}_{\lambda,xp} = \bar{H}_x$ so the first assertion follows from Lemma 5.3 and the identity $L(x) = [\bar{H} : \bar{H}_x]$.

Lemmas 5.1 and 5.2 (iii) now give $pC_c(\bar{G})p = \operatorname{span}\{pxp \mid x \in G\} = \operatorname{span}\{\chi_{\bar{H}x\bar{H}} \mid x \in \bar{G}\} = \mathcal{H}(\bar{G}, \bar{H})$, and it is clear from their definitions that the involutions on both $*$-algebras agree. For the convolution, first note that, since $\bar{H}$ is open, $\bar{G}/\bar{H}$ is discrete, so

$$\int_{\bar{G}} f(t)\,\mathrm{d}t = \sum_{y\bar{H} \in \bar{G}/\bar{H}} \int_{\bar{H}} f(yh)\,\mathrm{d}h$$

for $f \in C_c(\bar{G})$. Any f and g in $\mathcal{H}$ are left- and right-$\bar{H}$-invariant, so, since $\mu(\bar{H}) = 1$, it follows that, for any $x \in \bar{G}$,

$$\begin{aligned}\int_{\bar{G}} f(t)g(t^{-1}x)\,\mathrm{d}t &= \sum_{y\bar{H} \in \bar{G}/\bar{H}} \int_{\bar{H}} f(yh)g(h^{-1}y^{-1}x)\,\mathrm{d}h \\ &= \sum_{y\bar{H} \in \bar{G}/\bar{H}} f(y)g(y^{-1}x).\end{aligned}$$

Similarly, for $f \in \mathcal{H}$ we have

$$\int_{\bar{G}} |f(t)|\,\mathrm{d}t = \sum_{y\bar{H} \in \bar{G}/\bar{H}} \int_{\bar{H}} |f(yh)|\,\mathrm{d}h = \sum_{y\bar{H} \in \bar{G}/\bar{H}} |f(y)|.$$

□

Remark 5.5. Lemma 5.3 holds, with the same proof, for continuous representations on complete locally convex topological vector spaces. Using the more general version would let us avoid putting $C_c(\bar{G})$ into $L^2(\bar{G})$ in the proof of Corollary 5.4.

6. C^*-completions

We begin this section with a streamlined summary of Fell's abstract imprimitivity theorem, which we then apply to our Hecke context.

6.1. Fell's version of Morita equivalence

Let E and D be $*$-algebras and let X be an $E-D$ bimodule. Suppose we have inner products in the sense of Fell:

$$E \xleftarrow{{}_{\mathrm{L}}\langle\cdot,\cdot\rangle} X \times X \xrightarrow{\langle\cdot,\cdot\rangle_{\mathrm{R}}} D,$$

which are appropriately sesquilinear (with respect to the one-sided module structures), Hermitian in the sense that $\langle f,g\rangle = \langle g,f\rangle^*$, and compatible in the sense that ${}_{\mathrm{L}}\langle f,g\rangle h = f\langle g,h\rangle_{\mathrm{R}}$ for $f,g,h \in X$.

Definition 6.1. X is an $E-D$ *imprimitivity bimodule* if either

(i) $\operatorname{span}\langle X,X\rangle_{\mathrm{R}} = D$ and $\operatorname{span}{}_{\mathrm{L}}\langle X,X\rangle = E$, or

(ii) D and E are Banach $*$-algebras, $\overline{\operatorname{span}}\langle X,X\rangle_{\mathrm{R}} = D$ and $\overline{\operatorname{span}}\,{}_{\mathrm{L}}\langle X,X\rangle = E$.

Fell and Doran would call imprimitivity bimodules of type (i) above *strict* [**10**, Definition XI.6.2] and those of type (ii) *topologically strict* [**10**, Definition XI.7.1]. We will present the elementary theory of these two types in a unified fashion for convenience.

For our purposes the most important examples of imprimitivity bimodules arise from a projection p in a $*$-algebra B, and we take $D = pBp$, $X = Bp$ and $E = BpB$ (or $\overline{BpB}$ if B is a Banach $*$-algebra and we want a bimodule of type (ii)), with bimodule operations given by multiplication within B and inner products

$${}_{\mathrm{L}}\langle f,g\rangle = fg^*, \qquad \langle f,g\rangle_{\mathrm{R}} = f^*g.$$

In Fell's theory, as opposed to Rieffel's, it is important to note that there is no positivity condition on the inner products. Rather, positivity is a condition attributable to individual representations.

Definition 6.2. Given an $E-D$ imprimitivity bimodule X, a representation π of D is $\langle\cdot\,,\cdot\rangle_{\mathrm{R}}$*-positive* if

$$\pi(\langle f,f\rangle_{\mathrm{R}}) \geqslant 0 \quad \text{for all } f \in X,$$

and similarly for ${}_{\mathrm{L}}\langle\cdot\,,\cdot\rangle$ and representations of E.

Positive representations of D can be induced via X to positive representations of E in direct analogy with Rieffel's inducing process, and we have Fell's abstract imprimitivity theorem, below.

Theorem 6.3 (Fell and Doran [10, Theorems XI.6.15 and XI.7.2]). *If X is an $E-D$ imprimitivity bimodule, then induction via X gives a category equivalence between the ${}_{\mathrm{L}}\langle\cdot\,,\cdot\rangle$-positive representations of E and the $\langle\cdot\,,\cdot\rangle_{\mathrm{R}}$-positive representations of D.*

Definition 6.4. The inner product $\langle\cdot\,,\cdot\rangle_{\mathrm{R}}$ on an $E-D$ imprimitivity bimodule X is *positive* if one of the following properties holds:

(i) for each $f \in X$ there exist $g_1, \dots, g_n \in D$ such that

$$\langle f, f\rangle_{\mathrm{R}} = \sum_1^n g_i^* g_i;$$

(ii) D is a Banach $*$-algebra, and for each $f \in X$ and $\varepsilon > 0$ there exist $g_1, \dots, g_n \in D$ such that

$$\left\| \langle f, f\rangle_{\mathrm{R}} - \sum_1^n g_i^* g_i \right\| < \varepsilon.$$

A similar definition applies to ${}_{\mathrm{L}}\langle\cdot\,,\cdot\rangle$.

Observe that in the case $E = BpB$, $X = Bp$, $D = pBp$ mentioned above, the left inner product ${}_{\mathrm{L}}\langle\cdot\,,\cdot\rangle$ is automatically positive since $X \subseteq E$.

Proposition 6.5. *Let B and C be C^*-algebras and let Y be a $C-B$ imprimitivity bimodule with positive inner products. Suppose that ${}_E X_D \subseteq {}_C Y_B$ densely, and $C = C^*(E)$. Then*

(i) *a representation of D extends to B if and only if it is $\langle\cdot\,,\cdot\rangle_{\mathrm{R}}$-positive,*

(ii) *$B = C^*(D)$ if and only if every representation of D is $\langle\cdot\,,\cdot\rangle_{\mathrm{R}}$-positive,*

(iii) *if $\langle\cdot\,,\cdot\rangle_{\mathrm{R}}$ is positive on X, then $B = C^*(D)$.*

Proof. It suffices to show (i), for then (ii) will follow immediately, and (iii) follows from (ii) because if $\langle\cdot\,,\cdot\rangle_{\mathrm{R}}$ is positive on X, then every representation of D is $\langle\cdot\,,\cdot\rangle_{\mathrm{R}}$-positive (and similarly for ${}_{\mathrm{L}}\langle\cdot\,,\cdot\rangle$). Let π be a representation of D. First assume that π is $\langle\cdot\,,\cdot\rangle_{\mathrm{R}}$-positive. Induce π across the imprimitivity bimodule X to get a representation ρ of E. Then ρ extends uniquely to a representation $\bar{\rho}$ of C. Induce $\bar{\rho}$ across Y to get a representation τ of B. On the other hand, we can induce ρ back across X to get a representation λ of D. Since X is dense in Y, we have $\tau|_D = \lambda$. Since X is an imprimitivity bimodule, by Fell's abstract imprimitivity theorem λ is unitarily equivalent to π. Thus, since λ extends to a representation of B, so does π.

Conversely, assume that π extends to a representation $\bar{\pi}$ of B. Then, since $\langle\cdot\,,\cdot\rangle_{\mathrm{R}}$ is positive on Y and $X \subseteq Y$, we have $\pi(\langle f, f\rangle_{\mathrm{R}}) = \bar{\pi}(\langle f, f\rangle_{\mathrm{R}}) \geqslant 0$ for all $f \in X$. Thus, π is $\langle\cdot\,,\cdot\rangle_{\mathrm{R}}$-positive on X. □

6.2. Application to Hecke algebras

For the remainder of this section, we will let G be a locally compact group and let H be a compact open subgroup of G such that (G, H) is a reduced Hecke pair. As usual, the Haar measure on G is normalized so that $p = \chi_H$ is a projection in $C_{\mathrm{c}}(G)$, and the Hecke algebra $\mathcal{H}$ of (G, H) is identified with $pC_{\mathrm{c}}(G)p$ as in Corollary 5.4. Also recall from § 4 that every Hecke algebra arises from such a pair.

For convenience, we let

$$C_{\mathrm{c}} := C_{\mathrm{c}}(G), \quad L^1 := L^1(G) \quad \text{and} \quad A := C^*(G).$$

Thus, we have the following inclusions of imprimitivity bimodules:

$$_{C_{\mathrm{c}}pC_{\mathrm{c}}}(C_{\mathrm{c}}p)_{\mathcal{H}} \subseteq {}_{\overline{L^1pL^1}}(L^1p)_{pL^1p} \subseteq {}_{\overline{ApA}}(Ap)_{pAp}.$$

Remarks 6.6.

(i) ${}_{\mathrm{L}}\langle\cdot\,,\cdot\rangle$ is positive on all three bimodules, because in each case we have $X \subseteq E$.

(ii) $\langle\cdot\,,\cdot\rangle_{\mathrm{R}}$ is positive on ${}_{\overline{ApA}}(Ap)_{pAp}$ because A is a C^*-algebra.

(iii) By density, if $\langle\cdot\,,\cdot\rangle_{\mathrm{R}}$ is positive on $C_{\mathrm{c}}p$, then it is also positive on L^1p.

(iv) Similarly, if $C^*(\mathcal{H}) = pAp$ then also $C^*(pL^1p) = pAp$, because $\mathcal{H} \subseteq pL^1p \subseteq pAp$.

Theorem 6.7. *Let H be a compact open subgroup of a locally compact group G such that (G, H) is a reduced Hecke pair. Then with the above notation we have*

$$C^*(C_{\mathrm{c}}pC_{\mathrm{c}}) = C^*(\overline{L^1pL^1}) = \overline{ApA}.$$

In preparation for the proof of Theorem 6.7, we introduce a certain type of representation of G, as follows.

Definition 6.8. A representation π of G on a Hilbert space V is H-*smooth* if

$$\overline{\operatorname{span}}\, \pi(G)V_{\pi,H} = V,$$

where $V_{\pi,H} = \{\xi \in V \mid \pi(h)\xi = \xi \text{ for all } h \in H\}$.

We pause to justify that our use of 'smooth' is consistent with the traditional one as, for example, in [**32**]. If π is a bounded continuous representation of G on a Banach space V, then every vector $\xi \in \operatorname{span} \pi(G)V_{\pi,H}$ has the property that $x \mapsto \pi(x)\xi$ is constant on a compact, open subgroup of G, i.e. ξ is a *smooth* vector in the sense of [**32**]. Thus, if π is H-smooth in our sense, the vectors that are smooth as in [**32**] are dense in V. (The main objects in [**32**] are *admissible* representations, which means that $V_{\pi,H}$ also is finite dimensional; this is a concept we will not need.)

Proposition 6.9. *A continuous representation of G is H-smooth if and only if its integrated form is non-degenerate on $\overline{ApA}$.*

Proof. Let π be a continuous representation of G on a Hilbert space V, and let π also denote the integrated form. Since $V_{\pi,H} = \pi(p)V$, the result follows from the computation

$$\begin{aligned}\pi(\overline{ApA})V &= \overline{\pi(C_{\mathrm{c}}pC_{\mathrm{c}})V} \\ &= \overline{\operatorname{span} \pi(GpG)V} \\ &= \overline{\operatorname{span} \pi(G)\pi(p)V} \\ &= \overline{\operatorname{span} \pi(G)V_{\pi,H}}.\end{aligned}$$

□

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Corollary 6.10. *The projection p is full in A if and only if every continuous representation of G is H-smooth.*

Proof. By the preceding proposition, p is full if and only if every representation of A is non-degenerate on the closed ideal $\overline{ApA}$, equivalently, if and only if $A = \overline{ApA}$, since A is a C^*-algebra. □

Proof of Theorem 6.7. Since $C_cpC_c \subseteq \overline{L^1pL^1} \subseteq \overline{ApA}$, it suffices to show that every (non-degenerate) representation π of C_cpC_c on a Hilbert space V extends to $\overline{ApA}$.

We claim that there is an H-smooth representation σ of G on V such that

$$\sigma(x)\pi(f)\xi = \pi(xf)\xi \quad \text{for all } s \in G,\ f \in C_cpC_c,\ \xi \in V.$$

First we show that for fixed $x \in G$ the above formula gives a well defined linear map $\sigma(x)$ on the dense subspace $\operatorname{span}\pi(C_cpC_c)V$ of V: let $f_1, \dots, f_n \in C_cpC_c$ and $\xi_1, \dots, \xi_n \in V$, and assume that $\sum_1^n \pi(f_i)\xi_i = 0$. Then

$$\begin{aligned}\left\| \sum_1^n \pi(xf_i)\xi_i \right\|^2 &= \left\langle \sum_1^n \pi(xf_i)\xi_i, \sum_1^n \pi(xf_j)\xi_j \right\rangle \\ &= \sum_{i,j} \langle \pi(f_i^* x^{-1} x f_j)\xi_i, \xi_j \rangle \\ &= \sum_{i,j} \langle \pi(f_i^* f_j)\xi_i, \xi_j \rangle \\ &= \left\| \sum_1^n \pi(f_i)\xi_i \right\|^2 \\ &= 0.\end{aligned}$$

Thus, $\sigma(x)$ is well defined, and then the above computation also shows that $\sigma(x)$ is isometric, and hence extends uniquely to an isometry on V. In fact $\sigma(x)$ must be unitary since the map $\sigma : G \to \mathcal{L}(V)$ is multiplicative and $\sigma(e) = 1$.

We still need to verify that σ is H-smooth. But from the definition of σ we see that $\pi(p)V \subseteq V_{\sigma,H}$, so

$$\begin{aligned}\operatorname{span}\sigma(G)V_{\sigma,H} &\supseteq \operatorname{span}\sigma(G)\pi(p)V = \operatorname{span}\pi(Gp)V = \operatorname{span}\pi(GpG)V \\ &= \pi(C_cpC_c)V \quad \text{(by Lemma 5.2 (iv))},\end{aligned}$$

which is dense in V.

We have thus verified the claim. By Proposition 6.9 the integrated form of σ, which we also denote by σ, is non-degenerate on the ideal $\overline{ApA}$ of A. We show that $\sigma|_{C_cpC_c} = \pi$. Since $C_cpC_c = \operatorname{span}_{x,y \in G} xpy$, it suffices to show that $\sigma(p) = \pi(p)$: for $f \in C_cpC_c$ and

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$\xi \in V$ we have

$$\begin{aligned}\sigma(p)\pi(f)\xi &= \int_H \sigma(h)\pi(f)\xi \, \mathrm{d}h \\ &= \int_H \pi(hf)\xi \, \mathrm{d}h \\ &= \int_H \pi(h)\pi(f)\xi \, \mathrm{d}h = \pi(p)\pi(f)\xi,\end{aligned}$$

which implies $\sigma(p) = \pi(p)$ by linearity, continuity and density. □

Note that Theorem 6.7 allows us to translate Corollary 6.5 into the present context, as follows.

Corollary 6.11.

(i) *A representation of $\mathcal{H}$ or pL^1p extends to pAp if and only if it is $\langle\cdot\,,\cdot\rangle_{\mathrm{R}}$-positive.*

(ii) *For $D = \mathcal{H}$ or pL^1p, we have $C^*(D) = pAp$ if and only if every representation of D is $\langle\cdot\,,\cdot\rangle_{\mathrm{R}}$-positive.*

(iii) *For $D = \mathcal{H}$ or pL^1p and $X = C_{\mathrm{c}}p$ or L^1p respectively, if $\langle\cdot\,,\cdot\rangle_{\mathrm{R}}$ is positive on X, then $C^*(D) = pAp$.*

Together with Fell's imprimitivity theorem, Theorem 6.7 also gives the following result.

Corollary 6.12. *For $D = \mathcal{H}$ or pL^1p, and $X = C_{\mathrm{c}}p$ or L^1p respectively, induction via X gives a category equivalence between the representations of $\overline{ApA}$ and the $\langle\cdot\,,\cdot\rangle_{\mathrm{R}}$-positive representations of D.*

Theorem 6.13. *Let H be a compact open subgroup of a locally compact group G such that the Hecke pair (G, H) is reduced, and suppose that H is normal in some closed normal subgroup N of G. Then $\langle\cdot\,,\cdot\rangle_{\mathrm{R}}$ is positive on $C_{\mathrm{c}}p$, and hence $C^*(\mathcal{H}) = pAp$.*

Proof. We must show that if $f = \sum_1^n c_i x_i p$ with $c_i \in \mathbb{C}$, $x_i \in G$, then f^*f is of the form $\sum_1^n g_i^* g_i$ with $g_i \in \mathcal{H}$. Note that $\{x_i p x_i^{-1}\}_1^n$ are commuting projections in A (because H is compact, open, and normal in N); let q be their least upper bound in the projections of A.

We will prove by induction on n that q is a sum of elements of the form gpg^* with $g \in C_{\mathrm{c}}$. This is obvious for $n = 1$, so assume that $n > 1$ and the $\sup q'$ of $\{x_i p x_i^{-1}\}_1^{n-1}$ has the desired form. Then so does

$$q = q' + (1 - q')x_n p x_n^{-1} = \sup\{q', x_n p x_n^{-1}\}.$$

For each i, since $q \geqslant x_i p x_i^{-1}$, we have $q x_i p = x_i p$. Thus, $qf = f$, so

$$f^*f = f^*q^*qf = f^*qf$$

is a sum of elements of the form f^*g^*pgf with $g \in C_{\mathrm{c}}$, and hence is a sum of elements of the form h^*h with $h \in \mathcal{H}$. □

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It is *not* true in general that $C^*(\mathcal{H})$, when it exists, necessarily coincides with pAp; thus, Theorem 4.2 (iii) of [**33**] is wrong. The problem in Tzanev's proof (and in an earlier version of the present paper) is that the equality $C^*(pL^1p) = pC^*(L^1)p$ fails in general. Tzanev has recently informed us of work (personal communication) showing that, for any prime q, the pair $(\mathrm{PSL}(3, \mathcal{Q}_q), \mathrm{PSL}(3, \mathcal{Z}_q))$ provides a counterexample—more precisely, in this example we do not know whether $C^*(\mathcal{H})$ exists, but we do know that $C^*(pL^1p) \neq pC^*(L^1)p$.

However, the problem does not arise if G is Hermitian. Recall from [**28**] that a $*$-algebra is *Hermitian* if every self-adjoint element has real spectrum, and G is called Hermitian if $L^1(G)$ is.

Theorem 6.14. *If G is Hermitian, then $C^*(pL^1p) = pAp$.*

We need a preparatory lemma.

Lemma 6.15. *Let B be a Hermitian Banach $*$-algebra.*

(i) *If D is a Banach $*$-subalgebra of B, then the largest C^*-semi-norm on B restricts to the largest C^*-semi-norm on D.*

(ii) *If p is a projection in B (i.e. if $p = p^* = p^2$), then*

$$C^*(pBp) = pC^*(B)p,$$

where we identify p with its image in $C^*(B)$.

Proof. (i) Since B is Hermitian and D is closed, by [**28**, Theorem 11.4.4] every representation of D on a Hilbert space extends to a representation of B on a possibly larger Hilbert space. The result follows.

(ii) It follows from (i) that the closure of the image of pBp in $C^*(B)$, namely $pC^*(B)p$, is an enveloping C^*-algebra for pBp. □

Proof of Theorem 6.14. This follows from the above lemma, since $A = C^*(L^1)$. □

Questions 6.16.

(i) When is $C^*(pL^1p) = pAp$? Hermitianness of G is certainly unnecessary (see, for example, Example 11.9).

(ii) If $C^*(\mathcal{H})$ exists, must it be pAp? Whenever we have been able to show that $C^*(\mathcal{H})$ exists, we have in fact found that $C^*(\mathcal{H}) = pAp$.

(iii) More generally, if $C^*(\mathcal{H})$ exists, what can be said concerning the surjections

$$C^*(\mathcal{H}) \to C^*(pL^1p) \to pAp?$$

(iv) If p is full, must $C^*(\mathcal{H})$ exist? It is easy to find examples, for instance, with finite groups, where $C^*(\mathcal{H})$ exists and p is not full.

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We now indicate how the above general theory can be used when (G, H) is the Schlichting completion of an arbitrary reduced Hecke pair (G_0, H_0). First of all, by the results in § 5 we can compute with the imprimitivity bimodule ${}_{C_c p C_c}(C_c p)_{\mathcal{H}}$ completely in terms of the uncompleted pair, since

$$C_c p C_c = \operatorname{span} G_0 p G_0, \quad C_c p = \operatorname{span} G_0 p \quad \text{and} \quad \mathcal{H} = \operatorname{span} p G_0 p.$$

Next, we can compute in $pL^1 p$ in terms of the uncompleted pair. To see how, recall that the double coset spaces $H_0 \setminus G_0/H_0$ and $H \setminus G/H$ can be canonically identified. Let $\ell^1(H_0 \setminus G_0/H_0)$ denote the completion of $\mathcal{H}$ in the ℓ^1-norm from (2.1):

$$\|f\|_1 = \sum_{xH_0 \in G_0/H_0} |f(x)|.$$

The L^1-norm on C_c restricts on $\mathcal{H}$ to give exactly the ℓ^1-norm, so $\ell^1(H_0 \setminus G_0/H_0)$ may be identified with $pL^1 p$, as observed by Tzanev [**33**].

Finally, H_0-smooth representations of G_0 are defined just as in Definition 6.8 (but no continuity is assumed), and we have the following.

Proposition 6.17. *If (G, H) is the Schlichting completion of a reduced Hecke pair (G_0, H_0), then a representation of G_0 is H_0-smooth if and only if it extends to a continuous H-smooth representation of G.*

Proof. Using density and continuity, it is easy to see that the restriction to G_0 of every H-smooth representation of G is H_0-smooth. It remains to show that every H_0-smooth representation π of G_0 on a Hilbert space V extends to an H-smooth representation of G. For this it suffices to show that π is in fact continuous for the Hecke topology of the pair (G_0, H_0) and the strong operator topology on the unitary group of V, for then π will extend uniquely to a continuous representation of G, which will obviously be H-smooth. Let $x \to e$ in the Hecke topology. We must show that $\pi(x)\xi \to \xi$ in norm for all $\xi \in V$. Since $\pi(G_0)$ is bounded in the operator norm, by linearity and density it suffices to show that if $y \in G_0$ and $\xi \in V_{\pi, H_0}$, then $\pi(x)\pi(y)\xi \to \pi(y)\xi$. But in fact we eventually have $x \in yH_0 y^{-1}$, and hence $\pi(x)\pi(y)\xi = \pi(y)\xi$, because $yH_0 y^{-1}$ is a neighbourhood of e in the Hecke topology. □

Combining Proposition 6.17 with Corollary 6.10 and Proposition 6.9 gives the following corollaries.

Corollary 6.18. *If (G, H) is the Schlichting completion of a reduced Hecke pair (G_0, H_0), then*

(i) *p is full in A if and only if every representation of G_0 which is continuous in the Hecke topology is H_0-smooth, and*

(ii) *restriction from G to G_0 gives a bijection between representations of $\overline{ApA}$ and H_0-smooth representations of G_0.*

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Corollary 6.19. *If (G_0, H_0) is a reduced Hecke pair, then there is a category equivalence between the H_0-smooth representations of G_0 and the $\langle\cdot\,,\cdot\rangle_{\mathrm{R}}$-positive representations of $\mathcal{H}$.*

Proof. This follows from the above corollary and Corollary 6.12. □

We recover Hall's equivalence [**14**, Theorem 3.25], as follows.

Corollary 6.20. *If (G_0, H_0) is a reduced Hecke pair such that the $\mathcal{H}$-valued inner product on $C_{\mathrm{c}}p$ is positive, then there is a category equivalence between the H_0-smooth representations of G_0 and the representations of $\mathcal{H}$.*

7. The directing semigroup

Let (G, H) be a reduced Hecke pair, with Schlichting completion $(\bar{G}, \bar{H})$. As usual, we set $A = C^*(\bar{G})$ and $p = \chi_{\bar{H}} \in A$. In this section we give a condition, formulated in terms of the following semigroup T, which ensures that $C^*(\mathcal{H}) = pAp$ and that p is full in A.

Definition 7.1. We say (G, H) is *directed* if $G = T^{-1}T$, where

$$T := \{t \in G \mid tHt^{-1} \supseteq H\}.$$

Remark 7.2. In many papers (see, for example, [**1**, **7**, **18**–**23**]), a crossed product by a certain action related to this semigroup T has been used in a crucial way to study Hecke algebras. For us the semigroup crossed product plays no role (although we can easily recover some of the main results of those papers); our interest in the semigroup T arises from Theorems 7.4 and 7.5 below.

We chose the term 'directed' because of the following result.

Lemma 7.3. *The following are equivalent:*

(i) *the pair (G, H) is directed;*

(ii) *G is directed upward by the pre-order $x \leqslant y \iff yx^{-1} \in T$;*

(iii) *the family $\{xHx^{-1} \mid x \in G\}$ of conjugates of H is directed downward in the sense that the intersection of any two of them contains a third.*

Proof. The equivalence (i) $\iff$ (ii) is probably folklore (see, for example, Lemma 2.1 in [**7**], and also Theorem 1.2 in [**18**] for the forward implication); for the convenience of the reader we give the outline of the argument: if (G, H) is directed, then for all $x, y \in G$ there exist $s, t \in T$ such that $s^{-1}t = xy^{-1}$, and then $x, y \leqslant sx = ty$, while, conversely, if G is directed upward by $\leqslant$, then for all $x \in G$ there exist $s, t \in T$ such that $e, x \leqslant sx = t$, and then $x = s^{-1}t$.

For (ii) $\iff$ (iii), just note that $x \leqslant y$ if and only if $x^{-1}Hx \supseteq y^{-1}Hy$. □

Note that if $x = s^{-1}t$ with $s, t \in T$, then $x^{-1}Hx \supset t^{-1}Ht$. Thus, the above lemma implies that if (G, H) is directed, then the family $\{t^{-1}Ht \mid t \in T\}$ is also directed downward.

We remark that our formulations of the Hecke $*$-algebra $\mathcal{H}$, the $\mathcal{H}$-valued inner product on $C_c(\bar{G})$ and directedness of (G, H) are slightly different from Hall's (see [**14**, §§ 2.2, 3.4.1, 4.1]), so for the reader's convenience we include the proof of the following, which includes [**14**, Lemma 4.4 and Corollary 4.6] (for similar results, see also [**19**, Proposition 1.4] and [**7**, Proposition 2.8]).

Theorem 7.4. *If the Hecke pair (G, H) is directed, then $\langle\cdot\,,\cdot\rangle_{\mathrm{R}}$ is positive on $C_c(\bar{G})p$, and hence $C^*(\mathcal{H}) = pAp$.*

Proof. We only need to prove the positivity, for then the other part follows immediately from the general theory of § 6. Let $c_1, \dots, c_n \in \mathbb{C}$ and $x_1, \dots, x_n \in G$, so that $\sum_1^n c_i x_i p$ is a typical element of $C_c(\bar{G})p$. By directedness we can choose a common upper bound y for $x_1^{-1}, \dots, x_n^{-1}$. Thus, for each i we have $yx_i \in T$, so that $yx_i p = pyx_i p$. Then

$$\begin{aligned}\left\langle \sum_{i=1}^n c_i x_i p, \sum_{j=1}^n c_j x_j p \right\rangle &= \sum_{i,j} \overline{c_i} c_j p x_i^{-1} x_j p \\ &= \sum_{i,j} \overline{c_i} c_j p x_i^{-1} y^{-1} y x_j p \\ &= \sum_{i,j} (c_i p y x_i p)^* c_j p y x_j p \\ &= \left(\sum_{i=1}^n c_i p y x_i p \right)^* \sum_{j=1}^n c_j p y x_j p,\end{aligned}$$

so we are done since $\sum_1^n c_i p y x_i p \in \mathcal{H}$. □

Theorem 7.5. *If the Hecke pair (G, H) is directed, then p is full in A.*

Proof. We first verify the following claims:

(i) $(\bar{G}, \bar{H})$ is also directed;

(ii) $\bar{T} = \{t \in \bar{G} \mid t\bar{H}t^{-1} \supseteq \bar{H}\}$;

(iii) $T = G \cap \bar{T}$;

(iv) $\bigcap_{t \in T} t^{-1}\bar{H}t = \{e\}$.

For (i), given $x \in \bar{G}$ we can choose $y \in G$ such that $x\bar{H} = y\bar{H}$, and then

$$x \in y\bar{H} \subseteq T^{-1}T\bar{H} \subseteq \bar{T}^{-1}\bar{T}.$$

For (ii), let R denote the right-hand side. We first show that $T = G \cap R$: first let $t \in T$. Then

$$t^{-1}Ht \subseteq H \subseteq \bar{H},$$

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so $t^{-1}\bar{H}t \subseteq \bar{H}$, and hence $t \in R$. Thus, $T \subseteq G \cap R$. For the opposite containment, let $t \in G \cap R$. Then

$$t^{-1}Ht \subseteq t^{-1}\bar{H}t \subseteq \bar{H},$$

so

$$t^{-1}Ht \subseteq G \cap \bar{H} = H,$$

and hence $t \in T$.

Now, since $\bar{H}$ is closed, so is R. On the other hand, $t \in R$ implies that $t\bar{H} \subseteq R$, so R is a union of cosets of the open subgroup $\bar{H}$, and is therefore open. Since G is dense in $\bar{G}$ and R is open in $\bar{G}$, $G \cap R$ is dense in R. Thus,

$$R = \bar{R} = \overline{G \cap R} = \bar{T}.$$

(iii) follows immediately from the above proof of (ii).

For (iv), first note that

$$\bigcap_{t\in\bar{T}} t^{-1}\bar{H}t \subseteq \bigcap_{x\in\bar{G}} x^{-1}\bar{H}x = \{e\},$$

since $(\bar{G}, \bar{H})$ is directed and reduced. Now, for each $t \in \bar{T}$ there exists $s \in G$ such that $s^{-1}\bar{H}s = t^{-1}\bar{H}t$, and then $s \in G \cap \bar{T} = T$. It follows that

$$\{t^{-1}\bar{H}t \mid t \in T\} = \{t^{-1}\bar{H}t \mid t \in \bar{T}\},$$

and hence $\bigcap_{t\in T} t^{-1}\bar{H}t = \{e\}$, as desired.

We have thus verified claims (i)–(iv). Now, we have

$$C_c(\bar{G})pC_c(\bar{G}) = \operatorname*{span}_{x,y\in G} xpy \supseteq \operatorname*{span}_{x\in G,t\in T} xt^{-1}pt.$$

Since $\bigcap_{t\in T} t^{-1}\bar{H}t = \{e\}$, the family $\{t^{-1}\bar{H}t \mid t \in T\}$ is a neighbourhood sub-base at e in $\bar{G}$. Since $(\bar{G}, \bar{H})$ is also directed, this sub-base is actually a base, because it is directed downward. Consequently, $\{t^{-1}pt\}_{t\in T}$ is an approximate identity for $C_c(\bar{G})$ in the inductive-limit topology, and hence also for A. Therefore, ApA is dense in A, so the theorem follows. □

Directedness is certainly not necessary for the conclusions of either of Theorems 7.4 or 7.5. For example, when G is finite, $C^*(\mathcal{H}) = pAp$ is automatic, directedness is impossible (unless G is the trivial group) and fullness is possible (see Example 11.2). In fact, we leave it to the conscientious reader to verify that when G is finite the projection p is full if and only if $\sum_{x\in G} xpx^{-1}$ is invertible. It seems an interesting problem to describe the finite pairs (G, H) for which p is full.

The next corollary recovers [**14**, Corollary 4.5] and [**12**, Theorem 6.10] and (essentially) includes [**23**, Theorem 3.1].

Corollary 7.6. *If the Hecke pair (G, H) is directed, then there are category equivalences among the continuous representations of $\bar{G}$, the H-smooth representations of G, and the representations of $\mathcal{H}$.*

Proof. Combine fullness of p with the general theory of § 6. □

8. Semi-direct product

In this section we examine the C^*-algebra $\overline{ApA}$ in the special case that $G = N \rtimes Q$ is a semi-direct product and the normal subgroup N is abelian and contains H (with (G, H) a reduced Hecke pair). We will defer part of the proof of the main result until the next section, where we will handle a more general situation (assuming only that $H \subseteq N \lhd G$). The present section applies to Examples 11.2, 11.4, 11.5 and 11.6, some of which have also been studied in [**3**, **5**, **7**, **21**, **27**].

Taking closures, $\bar{N}$ is an abelian normal subgroup of $\bar{G}$ containing $\bar{H}$. Since $\bar{N}$ is open in $\bar{G}$ and G is dense in $\bar{G}$, the map $xN \mapsto x\bar{N}$ gives an isomorphism $G/N \cong \bar{G}/\bar{N}$. Thus, we may write $\bar{G} = \bar{N} \rtimes Q$. One of the most elementary examples of the crossed product construction is that

$$A = C^*(\bar{G}) \cong C^*(\bar{N}) \times_\alpha Q,$$

where $\alpha_x(n) = xnx^{-1}$ for $x \in Q, n \in \bar{N}$. The Fourier transform gives

$$A \cong C_0(\hat{\bar{N}}) \times_\beta Q,$$

where

$$\beta_x(g)(\phi) = g(\phi \circ \alpha_x)$$

for $g \in C_0(\hat{\bar{N}})$, $\phi \in \hat{\bar{N}}$, $x \in Q$. Note that β corresponds to the natural action of Q by homeomorphisms of $\hat{\bar{N}}$ given by $x \cdot \phi = \phi \circ \alpha_{x^{-1}}$.

Let us look at this a little more closely. We make the convention that the Fourier transform of a group element x is the function whose value at a character ϕ is $\phi(x)$. Then the Fourier transform of $\chi_{\bar{H}}$ is $\chi_{\bar{H}^\perp}$. The open set

$$\Omega = \bigcup_{x \in Q} (xHx^{-1})^\perp$$

is the smallest Q-invariant subset of $\hat{\bar{N}}$ containing the compact open subset $\bar{H}^\perp = H^\perp$.

Theorem 8.1. *Let $G = N \rtimes Q$ be a semi-direct product with N abelian, let H be a Hecke subgroup of G contained in N and let β be the above action of Q on $C_0(\Omega)$. Then*

(i) $\overline{ApA} \cong C_0(\Omega) \times_\beta Q$,

(ii) $\langle \cdot\,, \cdot \rangle_{\mathrm{R}}$ *is positive on* $C_{\mathrm{c}}(\bar{G})p$*, so* $C^*(\mathcal{H}) = pAp$ *is Morita equivalent to* $C_0(\Omega) \times_\beta Q$,

(iii) p *is full in* A *if and only if* $\Omega = \hat{\bar{N}}$.

Proof. We defer the proof of (i) to the next section. Parts (ii) and (iii) follow immediately from (i) and Theorem 6.13. □

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8.1. Comparison with the groupoid approach

We now show how this semi-direct product construction can be cast in the framework of Arzumanian and Renault's groupoid [**2**]. For this we regard the action of Q on Ω as a transformation group. The associated transformation groupoid is

$$\mathcal{G} = \{(\phi, x, \psi) \in \Omega \times Q \times \Omega \mid \phi = x \cdot \psi\},$$

with multiplication

$$(\phi, x, \psi)(\psi, y, \nu) = (\psi, xy, \nu).$$

Then the groupoid C^*-algebra is canonically a crossed product:

$$C^*(\mathcal{G}) \cong C_0(\Omega) \times_\beta Q.$$

Let $\mathcal{G}(H^\perp)$ denote the reduction of the groupoid $\mathcal{G}$ to the compact open subset $H^\perp$ of the unit space Ω:

$$\mathcal{G}(H^\perp) = \{(\phi, x, \psi) \in \mathcal{G} \mid \phi, \psi \in H^\perp\}.$$

Since $H^\perp$ meets every orbit in Ω, i.e. Ω is the saturation of $H^\perp$ in the unit space, [**26**, Example 2.7] gives us a groupoid equivalence $\mathcal{G} \sim \mathcal{G}(H^\perp)$, and hence a Morita–Rieffel equivalence $C^*(\mathcal{G}) \sim C^*(\mathcal{G}(H^\perp))$.

Proposition 8.2. *With the above notation,* $C^*(\mathcal{G}(H^\perp)) \cong pAp$.

Proof. We borrow from the next section the isomorphism $\theta : \overline{ApA} \to C_0(\Omega) \times_\beta Q$, which appears in (9.1). Composing with the isomorphism $C_0(\Omega) \times_\beta Q \cong C^*(\mathcal{G})$, we get an isomorphism $\zeta : \overline{ApA} \to C^*(\mathcal{G})$, which we shall show takes pAp onto $C^*(\mathcal{G}(H^\perp))$. But this is easy: we have $\zeta(p) = \chi_{H^\perp}$, and

$$\chi_{H^\perp} C^*(\mathcal{G}) \chi_{H^\perp} = C^*(\mathcal{G}(H^\perp)).$$

□

A special case of the above situation is worked out in [**2**, § 6], where Arzumanian and Renault give a groupoid whose C^*-algebra is the Hecke C^*-algebra of Bost and Connes [**5**]: it is the groupoid

$$\left\{\left(x, \frac{m}{n}, y\right) \in \mathcal{Z} \times \mathbb{Q}_+^* \times \mathcal{Z} \;\middle|\; mx = ny\right\},$$

where $\mathcal{Z}$ denotes the integers in the ring $\mathcal{A}$ of finite adeles, and $\mathbb{Q}_+^*$ denotes the multiplicative group of positive rational numbers.

This groupoid is the restriction to the compact open subset $\mathcal{Z}$ of the unit space of the transformation groupoid associated to the canonical action of $\mathbb{Q}_+^*$ on $\mathcal{A}$ (cf. Example 11.4), so that the Arzumanian–Renault result is 'equivalent to' our observation that pAp is the enveloping C^*-algebra of $\mathcal{H}$. To see this, assume (as is the case in the Bost–Connes example) that $Q = S^{-1}S$, where $S = T/N$, and use the identity

$$\mathcal{G}(H^\perp) = \{(\phi, s^{-1}t, \psi) \mid \phi, \psi \in H^\perp;\; s, t \in S;\; s \cdot \phi = t \cdot \psi\}.$$

9. Crossed products

In this section we give the full justification for Theorem 8.1 in the more general context of a reduced Hecke pair (G, H) such that H is contained in some normal subgroup N of G.

Taking closures in the Schlichting completion $\bar{G}$, we have $\bar{H} \subseteq \bar{N} \triangleleft \bar{G}$. We continue to let $A = C^*(\bar{G})$ and $p = \chi_{\bar{H}}$, and we introduce the notation

$$B := C^*(\bar{N}).$$

The action of $\bar{G}$ on B, and all other actions arising from the action of $\bar{G}$ on $\bar{N}$ by conjugation, will be denoted Ad.

This action is twisted over $\bar{N}$ in the sense of [**13**] (the twisting map is just the canonical embedding of $\bar{N}$ in $M(C^*(\bar{N}))$) and the twisted crossed product $B \times_{\bar{N}} \bar{G}$ is isomorphic to $A = C^*(\bar{G})$. This isomorphism $\theta : B \times_{\bar{N}} \bar{G} \to A$ is determined by

$$\theta(\pi(b)u(f)) = bf \quad \text{for } b \in B,\ f \in C_c(\bar{G}), \tag{9.1}$$

where (π, u) is the canonical covariant homomorphism of $(B, \bar{G})$ into $M(B \times_{\bar{N}} \bar{G})$ (see [**13**, Corollary of Proposition 1]). Our next result shows that, under this isomorphism, the ideal $\overline{ApA}$ of A corresponds to the twisted crossed product of an invariant ideal of B.

Theorem 9.1. *Let (G, H) be a reduced Hecke pair, and suppose that N is a normal subgroup of G which contains H. Then*

$$I = \overline{\operatorname{span}}\{xpx^{-1}n \mid x \in G,\ n \in N\} = \overline{\operatorname{span}}\{xpx^{-1}n \mid x \in \bar{G},\ n \in \bar{N}\}$$

is an Ad*-invariant ideal of B such that $I \times_{\bar{N}} \bar{G} \cong \overline{ApA}$.*

Proof. The equality of the two closed spans defining I follows from Lemma 5.1, which implies that for each $x \in \bar{G}$ and $n \in \bar{N}$ there exist $y \in G$ and $m \in N$ such that $ypy^{-1}m = xpx^{-1}n$.

Now, since $\bar{N}$ is normal in $\bar{G}$, $xpx^{-1}n = \Delta(x)\chi_{x\bar{H}x^{-1}n}$ is in $C_c(\bar{N})$ for each $x \in \bar{G}$ and $n \in \bar{N}$, so I is in fact contained in B, and hence I is a closed subspace of B. Moreover, since $(xpx^{-1}n)^* = n^{-1}xpx^{-1} = (n^{-1}x)p(n^{-1}x)^{-1}n^{-1}$, we have $I^* = I$. I is clearly Ad-invariant, since for $x, y \in \bar{G}$ and $n \in \bar{N}$ we have

$$\operatorname{Ad} x(ypy^{-1}n) = (xy)p(xy)^{-1}(xnx^{-1}) \in I.$$

Clearly, if $z \in I$ and $m \in N$, then $zm \in I$. Since $I = I^*$, we also have $mz \in I$. From this it follows that I is an ideal in $C^*(N)$.

Regarding $I \times_{\bar{N}} \bar{G}$ as an ideal of $B \times_{\bar{N}} \bar{G}$ in the usual way, we now claim that the isomorphism θ defined in (9.1) takes $I \times_{\bar{N}} \bar{G}$ onto $\overline{ApA}$. With canonical maps (π, u) as in (9.1), we have

$$\begin{aligned}\theta(I \times_{\bar{N}} \bar{G}) &= \theta(\overline{\operatorname{span}}\{\pi(xpx^{-1}n)u(f) \mid x \in G,\ n \in N,\ f \in C_c(\bar{G})\}) \\ &= \overline{\operatorname{span}}\{xpx^{-1}nf \mid x \in G,\ n \in N,\ f \in C_c(\bar{G})\}.\end{aligned}$$

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Temporarily fix $x \in G$. Then, for all $n \in N, f \in C_c(\bar{G})$, Lemma 5.2 gives

$$xpx^{-1}nf \in xpC_c(\bar{G}) = \operatorname*{span}_{y \in G} xpy.$$

On the other hand, for all $y \in G$,

$$xpy = xp\chi_{\bar{H}}y = xp\chi_{\bar{H}y}\Delta(y)^{-1} \in xpC_c(\bar{G}) = xpx^{-1}nC_c(\bar{G}).$$

Thus,

$$\overline{\operatorname{span}}\{xpx^{-1}nf \mid x \in G,\ n \in N,\ f \in C_c(\bar{G})\} = \overline{\operatorname*{span}_{x,y \in G}}\, xpy = \overline{ApA},$$

and we are done. □

Via restriction to $G \subseteq \bar{G}$, we get an action (I, G, Ad) which is twisted over N.

Theorem 9.2. *With the hypotheses and notation of Theorem 9.1, we have $I \times_N G \cong \overline{ApA}$, and therefore the C^*-completion pAp of the Hecke algebra $\mathcal{H}$ is Morita–Rieffel equivalent to the twisted crossed product $I \times_N G$.*

Proof. By Theorem 9.1, we need only show that $I \times_N G \cong I \times_{\bar{N}} \bar{G}$. Let $(\sigma, v) : (I, \bar{G}) \to M(I \times_{\bar{N}} \bar{G})$ and $(\mu, w) : (I, G) \to M(I \times_N G)$ be the canonical covariant homomorphisms. The crux of the matter is the following claim: $w : G \to M(I \times_N G)$ extends to a continuous homomorphism $\bar{w} : \bar{G} \to M(I \times_N G)$. Given the claim, we will have homomorphisms

$$\sigma \times v|_G : I \times_N G \to M(I \times_{\bar{N}} \bar{G}),$$
$$\mu \times \bar{w} : I \times_{\bar{N}} \bar{G} \to M(I \times_N G),$$

which routine computations show are inverses of each other.

To establish the claim, by Proposition 6.17 (whose proof applies to representations on Banach space as well as Hilbert space) it suffices to show that $w : G \to M(I \times_N G)$ is H-smooth. Note that

$$\mu(p)w(G) \subseteq (I \times_N G)_H,$$

since $w|_H = \mu|_H$ and $hp = p$ for all $h \in H$. Because (μ, w) preserves the twist, we have

$$I \times_N G = \overline{\operatorname{span}}\{\mu(xpx^{-1})w(y) \mid x, y \in G\}.$$

Since

$$\mu(xpx^{-1})w(y) = w(x)\mu(p)w(x^{-1})w(y) = w(x)\mu(p)w(x^{-1}y),$$

and $\mu(p)w(x^{-1}y) \in (I \times_N G)_H$, we have

$$\overline{\operatorname{span}}\, w(G)(I \times_N G)_H = I \times_N G,$$

so w is H-smooth. □

Note that if H is normal in N (in addition to the hypotheses of Theorem 9.2), then $C^*(\mathcal{H}) = pAp$ by Theorem 6.13, and I is the closed G-invariant ideal of $C^*(\bar{N})$ generated by the central projection p.

Suppose, in the situation of Theorems 9.1 and 9.2, that N is abelian. Then $C^*(\bar{N}) \cong C_0(\hat{\bar{N}})$ via the Fourier transform, so we get an isomorphism $C^*(\bar{N}) \times_N G \cong C_0(\hat{\bar{N}}) \times_N G$ of twisted crossed products. The open set

$$\Omega = \bigcup_{x \in G} (xHx^{-1})^{\perp}$$

is the smallest subset of $\hat{\bar{N}}$ which contains $H^{\perp}$ and is invariant under the induced action of G on $\hat{\bar{N}}$.

Corollary 9.3. *Let (G,H) be a reduced Hecke pair and let N be an abelian normal subgroup of G which contains H. Then $\overline{ApA} \cong C_0(\Omega) \times_N G$, and hence p is full if and only if $\Omega = \hat{\bar{N}}$.*

Proof. By Theorem 9.2, we need only show that the Fourier transform $\hat{I}$ of the ideal I is $C_0(\Omega)$. Now $\hat{I}$ is an ideal of $C_0(\hat{\bar{N}})$, and hence is of the form $C_0(M)$, where M is an open subset of $\hat{\bar{N}}$. Since I is densely spanned by the functions $xpx^{-1}n = \chi_{x\bar{H}x^{-1}n}\Delta(x)$ for $x \in G$ and $n \in N$, $\hat{I}$ is densely spanned by the Fourier transforms $\hat{n}\chi_{(xHx^{-1})^{\perp}}\Delta(x)$. The support of such a function is the compact open subset $(xHx^{-1})^{\perp}$ of $\hat{\bar{N}}$, and it follows that $M = \Omega$. □

To see how Theorem 8.1 (i) follows from Corollary 9.3, suppose that $G = N \rtimes Q$ is a semi-direct product, where N is abelian and contains H. Then the twisted crossed product $C_0(\Omega) \times_N G$ becomes the ordinary crossed product $C_0(\Omega) \times_{\beta} Q$, where

$$\Omega = \bigcup_{x \in G} (xHx^{-1})^{\perp} = \bigcup_{x \in Q} (xHx^{-1})^{\perp}$$

and β is as in § 8.

10. Semigroup action

In this section, even though we did not need semigroup actions for our main results, we show how our techniques can be used to recover the dilation result of [**23**].

Keep the notation from the preceding sections: (G,H) is a reduced Hecke pair, $T = \{t \in G \mid tHt^{-1} \supseteq H\}$, $B = C^*(\bar{N})$, and $H \subseteq N \lhd G$. But now impose the further restriction that H be normal in N. Then the map $nH \mapsto n\bar{H} = \overline{nH}$ of N/H onto $\bar{N}/\bar{H}$ is an isomorphism. Since $\bar{H}$ is normal in $\bar{N}$, the projection p is central in B, so $pB \lhd B$. Moreover, the map $n\bar{H} \mapsto np$ extends to an isomorphism

$$\varphi : C^*(\bar{N}/\bar{H}) \xrightarrow{\cong} pB \subseteq C^*(\bar{N}).$$

(φ is obviously a homomorphism of $C^*(\bar{N}/\bar{H})$ onto pB, and the canonical map $C^*(\bar{N}) \to C^*(\bar{N}/\bar{H})$ is a left inverse.) In what follows we implicitly use φ to identify $C^*(\bar{N}/\bar{H})$ with $pB \subseteq C^*(\bar{N})$.

The following lemma is a special case of [**19**, Theorem 1.9]. Our techniques involving the Schlichting completion make the proof significantly shorter, and hence perhaps of independent interest.

Lemma 10.1. *If $t \in T$, then the automorphism $\operatorname{Ad} t$ of $C^*(\bar{N})$ maps $C^*(\bar{N}/\bar{H})$ into itself, giving rise to a semigroup action*

$$\operatorname{Ad} : T \to \operatorname{End} C^*(\bar{N}/\bar{H}).$$

Proof. For $t \in T, n \in N$ we have

$$\operatorname{Ad} t(\chi_{n\bar{H}}) = \chi_{tn\bar{H}t^{-1}} \Delta(t).$$

Since $t\bar{H}t^{-1} \supseteq \bar{H}$, $tn\bar{H}t^{-1}$ is a finite union of left cosets in $\bar{N}/\bar{H}$. Thus,

$$\chi_{tn\bar{H}n^{-1}} = \sum_{k\bar{H} \subseteq tn\bar{H}n^{-1}} \chi_{k\bar{H}} \in C^*(\bar{N}/\bar{H}).$$

□

Corollary 10.2. *Let $i : C^*(N/H) \xrightarrow{\cong} C^*(\bar{N}/\bar{H})$ be the C^*-isomorphism arising from the group isomorphism $N/H \cong \bar{N}/\bar{H}$. Then the identity*

$$\operatorname{Ad} t \circ i = i \circ \beta_t \quad \textit{for all } t \in T$$

defines a semigroup action $\beta : T \to \operatorname{End} C^*(N/H)$ such that

$$\beta_t(\chi_{nH}) = \chi_{tnHt^{-1}} \Delta(t) \quad \textit{for all } n \in N \textit{ and } t \in T.$$

The following result includes [**23**, Theorem 2.5], although there the semigroup is (in our notation) T/N and the minimal automorphic dilation is an action of G/N. In our version, we have a group action $(I, G, \operatorname{Ad})$, where, as in Theorem 9.1, I is the closed ideal of $C^*(\bar{N})$ generated by $\{xpx^{-1} \mid x \in G\}$.

Theorem 10.3. *If (G, H) is a (reduced) directed Hecke pair such that $H \lhd N \lhd G$ for some N, then $I = C^*(\bar{N})$. Moreover, the group action $(C^*(\bar{N}), G, \operatorname{Ad})$ is the minimal automorphic dilation of the semigroup action $(C^*(N/H), T, \beta)$ in the sense of [**18**].*

Proof. We have

$$I \supseteq \operatorname*{span}_{x\in G, n \in N} xpx^{-1}n \supseteq \operatorname*{span}_{t \in T, n\in N} t^{-1}ptn = \operatorname*{span}_{t\in T, n \in N} \chi_{t^{-1}\bar{H}tn}.$$

By an argument similar to that of Theorem 7.5, the latter span is dense in $C^*(\bar{N})$, proving the first part.

For the other part, we have already observed (Corollary 10.2) that the embedding $i : C^*(N/H) \to C^*(\bar{N}/\bar{H}) \subseteq C^*(\bar{N})$ satisfies $\operatorname{Ad} t \circ i = i \circ \beta_t$ for all $t \in T$, so that Ad is a dilation of β. By [**18**] it remains to show that

$$\overline{\operatorname*{span}}_{t \in T} (\operatorname{Ad} t)^{-1}(i(C^*(N/H))) = C^*(\bar{N}).$$

For $t \in T, n \in N$ we have

$$\operatorname{Ad} t^{-1}(i(\chi_{nH})) = \operatorname{Ad} t^{-1}(\chi_{n\bar{H}}) = \chi_{t^{-1}n\bar{H}t}\Delta(t)^{-1}$$

and (again arguing as in Theorem 7.5) these elements have dense span in $C^*(\bar{N})$. □

11. Examples

We shall here illustrate the different concepts with a number of examples. Even finite groups give interesting insights. In other examples we have stuck to matrix groups over $\mathbb{Q}$ and $\mathbb{Z}$, but the same techniques apply to matrix groups over other fields, as for example in [**1**, **8**, **25**]. Some arguments are only sketched, and we leave many details to the reader.

Example 11.1. We start with perhaps the simplest example (largely due to [**33**]) of a Hecke pair having none of the good properties mentioned in Theorems 7.4 and 7.5. Let

$$G = \mathbb{Z} \rtimes \mathbb{Z}_2 = \langle a, b \mid b^2 = 1,\ bab = a^{-1} \rangle$$

be the infinite dihedral group, and take $H = \langle b \rangle \cong \mathbb{Z}_2$. Note that, since H is finite, (G, H) coincides with its Schlichting completion. A short calculation shows that the double coset of a typical element $a^n h$ of G (where $n \in \mathbb{Z}, h \in H$) is

$$Ha^n hH = Ha^n H = a^n H \cup a^{-n} H.$$

So, letting

$$\phi_n = \begin{cases} \chi_H & \text{if } n = 0, \\ \frac{1}{2}\chi_{Ha^nH} & \text{if } n > 0, \end{cases}$$

we get a linear basis for the Hecke algebra $\mathcal{H}$ satisfying $\|\phi_n\|_1 = 1$ and

$$\phi_m * \phi_n = \tfrac{1}{2}(\phi_{m+n} + \phi_{m-n}) \quad \text{for all } m \geqslant n \geqslant 0.$$

Let c be a non-zero complex number. Then the maps $\pi_c : \mathcal{H} \to \mathbb{C}$ defined on the generators by

$$\pi_c(\phi_n) = \tfrac{1}{2}(c^n + c^{-n})$$

are easily checked to give us all the characters on $\mathcal{H}$. π_c is self-adjoint if and only if $c \in \mathbb{R}$ or $|c| = 1$, and π_c is ℓ^1-bounded if and only if $|c| = 1$. Since $\|\pi_c(\phi_n)\| \to \infty$ as $c \to \infty$, $\mathcal{H}$ does not have a greatest C^*-norm.

Moreover, the one-dimensional representation of G determined by $a \mapsto 1$ and $b \mapsto -1$ has no non-zero H-fixed vectors. Consequently, not all representations of G are H-smooth so, by Corollary 6.10, p is not full in A.

Note that this example is very far from being directed since, if H is finite, the 'directing semigroup' reduces to $T = H$. Tzanev [**33**] has shown that in this example the C^*-completion $pC^*(G)p$ of the Hecke algebra $\mathcal{H}$ is isomorphic to $C[-1, 1]$.

If $|c| = 1$, then π_c extends to a character of pAp, so here we see directly that $C^*(pL^1(G)p) = pAp$; it also follows from Theorem 6.14, since G is Hermitian by [**28**, Theorem 12.5.18a].

Example 11.2. The following even simpler example shows that p being full does not imply that (G,H) is directed. It belongs to § 8: take $N=\mathbb{Z}_2\times\mathbb{Z}_2$ and $H=\mathbb{Z}_2\times\{0\}$, and let $Q=\mathbb{Z}_3$ act so that the generator corresponds to the matrix $\left(\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\right)$. Then $\hat{N}\cong\mathbb{Z}_2\times\mathbb{Z}_2$ and $H^\perp\cong\{0\}\times\mathbb{Z}_2$ with the same action of Q. One may easily check that $\Omega=\bigcup_g gH^\perp=\hat{N}$, so p is full, but (G,H) is not directed since H is finite. Note that G is the symmetry group of the tetrahedron.

Remark 11.3. By taking direct products, other combinations of properties can be exhibited, e.g. there are infinite groups G for which p is full, but (G,H) is not directed.

Example 11.4. Let us next look at the by-now-classical example studied in [**5**, Proposition 3.6] and [**3**], which started much of recent work on Hecke algebras. It is the rational '$ax+b$'-group so, in the notation of § 8, $N=(\mathbb{Q},+)$ and $Q=(\mathbb{Q}^\times,\cdot)$ acts by multiplication:

$$(x,k)\mapsto xk \quad \text{for } x\in\mathbb{Q}^\times,\ k\in\mathbb{Q}.$$

As the Hecke subgroup we take $H=\mathbb{Z}\subseteq N$. We may identify these groups as

$$G=\left\{\begin{pmatrix}a&b\\0&1\end{pmatrix}\,\middle|\, a\in\mathbb{Q}^\times, b\in\mathbb{Q}\right\},$$

$$N=\left\{\begin{pmatrix}1&b\\0&1\end{pmatrix}\,\middle|\, b\in\mathbb{Q}\right\},$$

$$H=\left\{\begin{pmatrix}1&m\\0&1\end{pmatrix}\,\middle|\, m\in\mathbb{Z}\right\}.$$

So with obvious identifications we have for $x\in\mathbb{Q}^\times$ that $xHx^{-1}=x\mathbb{Z}\subseteq\mathbb{Q}$. Therefore, the subgroups $\{x\mathbb{Z}\mid x\in\mathbb{Q}^\times\}$ are both upward and directed downward (in particular, the pair (G,H) is directed): given $x,y\in\mathbb{Q}^\times$, there are $s,t\in\mathbb{Q}^\times$ such that

$$x\mathbb{Z}\cap y\mathbb{Z}=s\mathbb{Z} \quad\text{and}\quad x\mathbb{Z}+y\mathbb{Z}=t\mathbb{Z}.$$

From this and Proposition 4.10 it follows that

$$\bar{N}=\varprojlim_{x\in\mathbb{Q}^+}\mathbb{Q}/x\mathbb{Z}=\mathcal{A} \quad\text{and}\quad \bar{H}=\varprojlim_{x\in\mathbb{Q}^+}\mathbb{Z}/x\mathbb{Z}=\mathcal{Z}.$$

These are the finite adeles $\mathcal{A}$ and the integer adeles $\mathcal{Z}$, respectively, with $\mathbb{Q}^\times$ acting by multiplication. From this or Theorem 4.8 we have

$$\bar{G}=\left\{\begin{pmatrix}a&b\\0&1\end{pmatrix}\,\middle|\, a\in\mathbb{Q}^\times,\ b\in\mathcal{A}\right\}.$$

(Note that the Hecke topology is the same as the one coming from $(\mathbb{Q}^+,\mathcal{A})$; so (G,H) is a Schlichting pair, $\bar{H}\cap G=H$, and G is dense in $\bar{G}$.)

We get $\bar{H}^\perp=\mathcal{Z}^\perp\cong\mathcal{Z}$ inside $\hat{\mathcal{A}}\cong\mathcal{A}$, and we see directly that $\Omega=\bigcup_{x\in Q^+}x\mathcal{Z}=\mathcal{A}$, so Theorem 8.1 (iii) tells us the projection p is full in $C^*(\bar{G})$; however, this also follows from Theorem 7.5. Thus, we obtain the result of [**21**] that the C^*-completion $C^*(\mathcal{H}) = pC^*(\bar{G})p$ of this Hecke algebra is Morita–Rieffel equivalent to $C^*(\bar{G})$. Our approach here shows that this can be obtained directly without the theory of semigroup actions and dilations. The ideal structure of this C^*-algebra was determined in [**21**] (see also [**5**, **27**]).

As to the other properties studied in §§ 6 and 7, since (G, H) is directed and $H \lhd N \lhd G$ we also see that $\langle \cdot\,, \cdot \rangle_{\mathrm{R}}$ is positive on $C_{\mathrm{c}}(\bar{G})p$ by Theorem 6.13, and there are category equivalences among the continuous representations of $\bar{G}$, the H-smooth representations of G, and the representations of $\mathcal{H}$ by Corollary 7.6. Jenkins showed in [**15**] that the discrete group G contains a free semigroup, so G is not Hermitian. We do not know whether $\bar{G}$ is Hermitian.

Example 11.5. We shall look briefly at the generalization of Example 11.4 obtained by Brenken in [**7**]. Here $N = \mathbb{Q}^n$, $H = \mathbb{Z}^n$ and Q is a subgroup of $\mathrm{GL}(n, \mathbb{Q})$ with the usual action on $\mathbb{Q}^n$. (Brenken assumes that Q is abelian, but this is not important in the following.) It is usually straightforward to check whether H is a Hecke subgroup of $G = N \rtimes Q$. We assume that $\bigcap_{x \in Q} xHx^{-1} = \{0\}$ to make the pair (G, H) reduced. Section 8 applies, so the inner product $\langle \cdot\,, \cdot \rangle_{\mathrm{R}}$ is positive on $C_{\mathrm{c}}(\bar{G})p$, and hence $C^*(\mathcal{H}) = pAp$. One can check whether or not (G, H) is directed from the equality $Q \cap T^{-1} = Q \cap \mathrm{GL}(n, \mathbb{Z})$. The topology defined by $\{xHx^{-1} \mid x \in Q\}$ is quite often the same as the one determined by $\{x_1\mathbb{Z} \times \cdots \times x_n\mathbb{Z} \mid x_i \in \mathbb{Q}^+\}$, in which case $\bar{N} = \mathcal{A}^n$ and $\bar{H} = \mathcal{Z}^n$ with the same action of Q. The set $\varOmega$ is also easily determined, and one can then check whether p is full. If Q is the group $\mathrm{GL}(n, \mathbb{Q})^+$ of matrices with positive determinant, then p is full, and we recover [**19**, Proposition 2.4].

Example 11.6. Brenken's examples are motivated by Galois theory, i.e. one is looking at Example 11.4, but replacing $\mathbb{Q}$ by other number fields. We illustrate this by looking at quadratic number fields, so let d be a square-free integer. As in Example 11.4 we get

$$G = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \,\middle|\, a, b \in \mathbb{Q}(\sqrt{d}),\ a \neq 0 \right\},$$

$$N = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \,\middle|\, b \in \mathbb{Q}(\sqrt{d}) \right\},$$

$$H = \left\{ \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} \,\middle|\, m \in \mathbb{Z}[\sqrt{d}] \right\}.$$

We leave it to the reader to check that here we get the similar result:

$$\bar{G} = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \,\middle|\, a \in \mathbb{Q}(\sqrt{d}),\ a \neq 0,\ b \in \mathcal{A}[\sqrt{d}] \right\}.$$

One checks that (G, H) is directed, so again the projection p is full in $A = C^*(\bar{G})$ and the completion $C^*(\mathcal{H}) = pAp$ of the Hecke algebra is Morita–Rieffel equivalent to A. Example 2.1 of [**19**] can be treated similarly.

Example 11.7. We shall illustrate the results of § 9, where $H \subseteq N \lhd G$, but G is not necessarily a semi-direct product, in the special case of abelian N. In this example, $C^*(\mathcal{H}) = pAp$ but the projection p is not full in A; the same phenomenon can be obtained from Example 11.5 by letting Q be a nilpotent subgroup of $\mathrm{GL}(n, \mathbb{Q})$.

To save space we introduce the notation

$$[u, v, w] := \begin{pmatrix} 1 & v & w \\ 0 & 1 & u \\ 0 & 0 & 1 \end{pmatrix}.$$

We would like to take G to be the rational Heisenberg group (i.e. the group of all matrices as above with $u, v, w \in \mathbb{Q}$) and H as the integer subgroup with $u, v, w \in \mathbb{Z}$. But then the pair (G, H) would not be reduced, so we instead take the quotient by $\bigcap_g gHg^{-1} = \{[0, 0, w] \mid w \in \mathbb{Z}\}$ and therefore look at

$$G = \{[u, v, w] \mid u, v \in \mathbb{Q},\ w \in \mathbb{Q}/\mathbb{Z}\};$$

just remember that when multiplying two such matrices everything in the third component from $\mathbb{Q}$ is mapped into $\mathbb{Q}/\mathbb{Z}$. We then take

$$H = \{[u, v, 0] \mid u, v \in \mathbb{Z}\},$$

and from the formula

$$[x, y, z][u, v, w][x, y, z]^{-1} = [u, v, w + yu - xv] \tag{11.1}$$

it is easy to see that H is a Hecke subgroup. In fact, with $g = [x, y, z]$ we have

$$H \cap gHg^{-1} \supseteq H_{x,y} := \{[u, v, 0] \mid u \in \mathbb{Z} \cap y^{-1}\mathbb{Z},\ v \in \mathbb{Z} \cap x^{-1}\mathbb{Z}\}.$$

The sets $\{H_{x,y} \mid x, y \in \mathbb{Z} \setminus \{0\}\}$ will be a neighbourhood base at e in the Hecke topology, so the completion is given by

$$\begin{aligned} \bar{G} &= \varprojlim G/H_{x,y} \\ &= \varprojlim \{[u, v, w] \mid u \in \mathbb{Q}/y\mathbb{Z},\ v \in \mathbb{Q}/x\mathbb{Z},\ w \in \mathbb{Q}/\mathbb{Z}\} \\ &= \{[u, v, w] \mid u, v \in \mathcal{A},\ w \in \mathbb{Q}/\mathbb{Z}\}. \end{aligned}$$

The product is still given by matrix multiplication; just remember that this time anything in the third component from $\mathcal{A}$ is mapped into $\mathcal{A}/\mathcal{Z} \cong \mathbb{Q}/\mathbb{Z}$. We see that

$$\bar{H} = \{[u, v, 0] \mid u, v \in \mathcal{Z}\}.$$

We shall take as N take the normalizer of H in G:

$$N = \{[u, v, w] \mid u, v \in \mathbb{Z},\ w \in \mathbb{Q}/\mathbb{Z}\}.$$

This is an abelian normal subgroup of G, and

$$\bar{N} = \{[u,v,w] \mid u,v \in \mathcal{Z},\ w \in \mathbb{Q}/\mathbb{Z}\}.$$

We have

$$\hat{\bar{N}} = \{(p,q,r) \mid p,q \in \mathbb{Q}/\mathbb{Z},\ r \in \mathcal{Z}\} \quad \text{and} \quad \bar{H}^{\perp} = \{(0,0,r) \mid r \in \mathcal{Z}\}.$$

The action of G on N by $(g,n) \mapsto gng^{-1}$ (see (11.1)) defines a transpose action on $\hat{\bar{N}}$ given by

$$[x,y,z] \cdot (p,q,r) = (p + yr, q - xr, r). \tag{11.2}$$

From all this it follows that

$$\Omega = \bigcup_{g \in G} g\bar{H}^{\perp} = \{(yr, -xr, r) \mid x,y \in \mathbb{Q},\ r \in \mathcal{Z}\}.$$

This is a proper subset of $\hat{\bar{N}}$, so by Corollary 9.3 p is not full; hence, (G,H) is not directed. In fact, $T = N$, so here the pair (G,H) is as far as possible from being directed. By studying the orbits of the action of G on Ω, one can again determine the structure of the crossed product using the techniques of [**21**]. We have $C^*(\mathcal{H}) = pAp$ by Theorem 6.13, and also by Theorem 6.14, since $\bar{G}$ is Hermitian by [**28**, Theorem 12.5.17].

Example 11.8. The classical Hecke pair is given by $G = \mathrm{PSL}(2,\mathbb{Q})$ and $H = \mathrm{PSL}(2,\mathbb{Z})$. There is a vast literature of Hecke algebras related to this and other semi-simple groups, and we shall briefly describe how this relates to our presentation. To make things a little simpler we look at the q-adic version with $G = \mathrm{PSL}(2,\mathbb{Z}[1/q])$ for some prime number q and $H = \mathrm{PSL}(2,\mathbb{Z})$. Similar computations as in earlier examples show that for $x \in G$ there is $n \in \mathbb{Z}$ such that

$$H \cap xHx^{-1} \supseteq \mathrm{PSL}(2, q^n\mathbb{Z}) := \{a \in \mathrm{PSL}(2,\mathbb{Z}) \mid a \equiv I \bmod q^n\}.$$

From this it follows that $\bar{H} = \varprojlim \mathrm{PSL}(2,\mathbb{Z})/\mathrm{PSL}(2,q^n\mathbb{Z}) = \mathrm{PSL}(2,\mathcal{Z}_q)$ and, from Theorem 4.8, that $\bar{G} = \mathrm{PSL}(2,\mathcal{Q}_q)$. Here $\mathcal{Q}_q = \varprojlim_n \mathbb{Z}[1/q]/q^n\mathbb{Z}$ is the q-adic completion of $\mathbb{Q}$ and $\mathcal{Z}_q = \varprojlim_n \mathbb{Z}/q^n\mathbb{Z}$ is the q-adic integers. Note that $(\mathrm{PSL}(2,\mathcal{Q}_q), \mathrm{PSL}(2,\mathcal{Z}_q))$ is a Schlichting pair and that the Hecke topology is the same as the one coming from $\mathcal{Q}_q$.

The projection $p = \chi_{\bar{H}} \in C^*(\bar{G})$ is not full because there are representations T in the principal (continuous) series of $\mathrm{PSL}(2,\mathcal{Q}_q)$ (cf. [**11**, Chapter 2.3, pp. 157 ff.]) with $T(p) = 0$.

The structure of the Hecke algebra is well documented; we will do a quick review. Taking

$$x_n = \begin{pmatrix} q^n & 0 \\ 0 & q^{-n} \end{pmatrix},$$

one has $\bar{G} = \bigcup_{n \geqslant 0} \bar{H}x_n\bar{H}$, and with $\phi_n = px_np = L(x_n)^{-1}\chi_{Hx_nH}$ we have

$$\phi_n * \phi_1 = \frac{q}{q+1}\phi_{n+1} + \frac{q-1}{q(q+1)}\phi_n + \frac{1}{q(q+1)}\phi_{n-1}. \tag{11.3}$$

Hall [**14**] has shown that the characters of $\mathcal{H}$ are given by

$$\pi_z(\phi_m) = \frac{1-qz}{(q+1)(1-z)}\left(\frac{z}{q}\right)^m + \frac{q-z}{(q+1)(1-z)}\left(\frac{1}{qz}\right)^m \tag{11.4}$$

for $z \neq 1$, and

$$\pi_1(\phi_m) = \frac{2m(q-1)+q+1}{(q+1)q^m}. \tag{11.5}$$

Note that Hall worked with the pair $(\mathrm{SL}(2,\mathbb{Q}), \mathrm{SL}(2,\mathbb{Z}))$, of which PSL is the reduction, and that z and $1/z$ give the same character. From this it follows that $\mathcal{H}$ is isomorphic to the polynomial ring $\mathbb{C}[z+1/z]$ (and therefore also to the Hecke algebra of Example 11.1), so $\mathcal{H}$ has no universal C^*-completion. π_z is self-adjoint if and only if $z \in \mathbb{R}$ or $z \in \mathbb{T}$, and π_z is L^1-bounded if and only if $1/q \leqslant |z| \leqslant q$. So $pL^1(\bar{G})p$ has non-self-adjoint characters and is therefore not Hermitian (a different proof of this can be found in [**16**]); hence, $\bar{G}$ is non-Hermitian.

Here pAp is a commutative C^*-algebra, and the situation is quite the opposite of the other examples: pAp is an algebra which is easy to describe (determined by its Gel'fand spectrum) and we can use this information to describe $\overline{ApA}$. For instance, $\overline{ApA}$ is continuous trace with trivial Dixmier–Douady invariant (see, for example, [**29**]); in particular, it is liminal.

We do not quite know whether $C^*(pL^1(\bar{G})p) = pAp$ in this case. Since $pL^1(\bar{G})p$ is commutative, to show $C^*(pL^1(\bar{G})p) = pAp$ it would suffice to prove that for every self-adjoint character π_z of $pL^1(\bar{G})p$ there is an irreducible representation T of $\mathrm{PSL}(2,\mathcal{Q}_q)$ such that

$$T(px_mp) = \pi_z(\phi_m)T(p).$$

If $z \in \mathbb{T}$, it follows from [**11**, Chapter 2.3, pp. 174 ff.] that this is obtained with T a representation from the principal series. If $1/q \leqslant z \leqslant q$, similar (though much longer and boring) computations show that this can be obtained with T a representation from the supplementary series. We have not settled the case $-q \leqslant z \leqslant -1/q$; it seems that in this case there are no irreducible representations T of $\mathrm{PSL}(2,\mathcal{Q}_q)$ such that $T(px_mp) = \pi_z(\phi_m)T(p) \neq 0$. If this is true, it will follow that $C^*(pL^1(\bar{G})p) \neq pAp$. (As a test, one could check the case $z = -q$.)

For the similar case of $\mathrm{PSL}(3,\mathcal{Q}_q)$, we have already remarked in § 6 that $C^*(pL^1(\bar{G})p) \neq pAp$.

Example 11.9. Let us finish with another example of Hall [**14**]: take $G = \mathrm{PSL}(2,\mathcal{Q}_q)$, and consider the Iwahori subgroup

$$H = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}(2,\mathcal{Z}_q) \;\middle|\; c \in q\mathcal{Z}_q \right\}.$$

Hall has shown [**14**, Theorem 6.10] that $\langle\cdot\,,\cdot\rangle_{\mathrm{R}}$ on $C_{\mathrm{c}}(\bar{G})p$ is positive in this case, but (G,H) is not directed, thus showing that the converse of Theorem 7.4 fails.

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