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THE LOGARITHMIC FUNCTION AND TRACE ZERO ELEMENTS IN FINITE VON NEUMANN FACTORS

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Abstract

In this short note we present a common characterisation of the logarithmic function and the subspace of all trace zero elements in finite von Neumann factors.

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1. Introduction

On the set $\mathbb{P}_n$ of all $n \times n$ positive definite complex matrices we have the identity $\mathrm{Tr}(\log(A)) = \log(\det A)$ for $A \in \mathbb{P}_n$, where Tr and det stand for the usual trace functional and the determinant function, respectively. By the multiplicativity of the determinant, we deduce the identity

$$\mathrm{Tr}(\log(ABA) - (2\log A + \log B)) = 0, \quad A, B \in \mathbb{P}_n.$$

Consequently, the linear span of all matrices $\log(ABA) - (2\log A + \log B)$ for $A, B \in \mathbb{P}_n$ is included in the linear space of all trace zero matrices. In the present note we use this property to give a common characterisation of the logarithmic function and the space of all trace zero elements in the case of factor von Neumann algebras.

Our aim is to prove the following statement.

Theorem 1.1. *Let $\mathcal{A}$ be a von Neumann factor and $f : (0, \infty) \to \mathbb{R}$ be a nonconstant continuous function. Let $\mathcal{A}_+^{-1}$ denote the set of all positive invertible elements in $\mathcal{A}$ and set*

$$\mathcal{S}_f^{\mathcal{A}} = \overline{\mathrm{span}}\{f(ABA) - (2f(A) + f(B)) : A, B \in \mathcal{A}_+^{-1}\}.$$

Then either $\mathcal{S}_f^{\mathcal{A}} = \mathcal{A}$ or $\mathcal{S}_f^{\mathcal{A}} \subsetneq \mathcal{A}$. In the latter case $\mathcal{A}$ is finite, $f = a\log$ holds with some real number $a \neq 0$ and $\mathcal{S}_f^{\mathcal{A}}$ equals the space of all trace zero elements of $\mathcal{A}$.

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Here $\overline{\text{span}}$ stands for the closed linear span relative to the norm topology in $\mathcal{A}$. The above statement can be viewed as a common characterisation of the logarithmic function and the space of all trace zero elements (and hence the trace itself) in factor von Neumann algebras of finite type.

2. Preliminaries

For the proof we need some preliminary preparations which follow. We call a linear functional l on an algebra $\mathcal{A}$ tracial if it satisfies $l(XY) = l(YX)$ for any $X, Y \in \mathcal{A}$. If $\mathcal{A}$ is a *-algebra, a linear functional $h : \mathcal{A} \to \mathbb{C}$ is called Hermitian if $h(X^*) = \overline{h(X)}$ holds for all $X \in \mathcal{A}$.

Assume now that $\mathcal{A}$ is a C^*-algebra. For a tracial bounded linear functional l on $\mathcal{A}$, defining

$$l_1(X) = \frac{1}{2}(l(X) + \overline{l(X^*)}), \quad l_2(X) = \frac{1}{2i}(l(X) - \overline{l(X^*)}), \quad X \in \mathcal{A},$$

gives Hermitian tracial bounded linear functionals l_1, l_2 such that $l = l_1 + il_2$.

It is well known that every Hermitian bounded linear functional h on the C^*-algebra $\mathcal{A}$ can be written as $h = \varphi - \psi$, where φ, ψ are positive (bounded) linear functionals on $\mathcal{A}$ and the above decomposition, called the Jordan decomposition, is uniquely determined by the condition $\|\varphi - \psi\| = \|\varphi\| + \|\psi\|$ (see, for example, [3, Theorem 3.2.5]). If h is tracial, so are φ and ψ. To see this, for any unitary element $U \in \mathcal{A}$, define $\varphi_U(X) = \varphi(UXU^*)$ for $X \in \mathcal{A}$ and define ψ_U in a similar way. It is obvious that φ_U, ψ_U are positive linear functionals, $\|\varphi_U\| = \|\varphi\|$, $\|\psi_U\| = \|\psi\|$, $\|\varphi_U - \psi_U\| = \|\varphi - \psi\|$ and

$$\varphi(X) - \psi(X) = h(X) = h(UXU^*) = \varphi_U(X) - \psi_U(X), \quad X \in \mathcal{A}.$$

By the uniqueness of the Jordan decomposition mentioned above, it follows that $\varphi_U = \varphi$ and $\psi_U = \psi$, implying that φ, ψ are invariant under all unitary similarity transformations. But it is well known that this implies that φ, ψ are necessarily tracial. In fact, this follows from the argument given below in the paragraph containing (3.3) or see [1, Proposition 8.1.1].

We can now prove the following statement. It is certainly known, but we present the proof for the reader's convenience. Recall that in any finite von Neumann algebra there is a unique centre-valued positive linear functional which is tracial and acts as the identity on the centre. This functional is called the trace (see [1, Theorem 8.2.8]).

Proposition 2.1. *Assume that $\mathcal{A}$ is a von Neumann factor and l is a nonzero tracial bounded linear functional on $\mathcal{A}$. Then $\mathcal{A}$ is of finite type and l is a scalar multiple of the (unique) trace.*

Proof. By the previous discussion, we may assume that l is Hermitian. Consider the Jordan decomposition $l = \varphi - \psi$ of l. As we have seen above, the positive linear functionals φ, ψ are also tracial and one of them is necessarily nonzero.

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Suppose that ω is a nonzero positive tracial functional on $\mathcal{A}$. Then $\mathcal{A}$ cannot be infinite. Indeed, in such a case we would have $I = P + Q$ with some projections $P, Q \in \mathcal{A}$ both equivalent to I. Since ω clearly takes equal values on equivalent projections, from $\omega(I) = \omega(P) + \omega(Q)$ we infer that $\omega(I) = 0$. By positivity, this implies that ω vanishes on all projections, which, by continuity and the spectral theorem, would yield $\omega = 0$, which is a contradiction. Therefore, the existence of a nonzero positive tracial functional ω on $\mathcal{A}$ implies that $\mathcal{A}$ is necessarily finite and, by [1, Theorem 8.2.8], it is a constant multiple of the trace. □

3. Proof of Theorem 1.1

After these preliminaries we can now present the proof of our main result.

Proof of Theorem 1.1. Let $\mathcal{A}$ be a von Neumann factor and $f : (0, \infty) \to \mathbb{R}$ be a nonconstant continuous function. Assume that $\mathcal{S}_f^{\mathcal{A}}$ is not equal to the whole algebra $\mathcal{A}$. By the Hahn–Banach theorem, we have a nonzero bounded linear functional l on $\mathcal{A}$ such that

$$l(f(ABA) - (2f(A) + f(B))) = 0, \quad A, B \in \mathcal{A}_+^{-1}. \tag{3.1}$$

Since l is not zero and, by the spectral theorem, the closed linear span of the set of all projections in $\mathcal{A}$ equals $\mathcal{A}$, it follows that we have a projection $P \in \mathcal{A}$ such that $l(P) \neq 0$. Denote $P^\perp = I - P$. Put $A = tP + P^\perp$ and $B = sP + P^\perp$ into (3.1), where t, s are arbitrary positive real numbers. It follows from (3.1) that

$$l(((f(t^2 s) - (2f(t) + f(s)))P - 2f(1)P^\perp) = 0$$

and hence

$$((f(t^2 s) - (2f(t) + f(s)))l(P) = 2f(1)l(P^\perp)$$

for all $t, s > 0$. This implies that

$$f(t^2 s) - (2f(t) + f(s)) = -2c$$

for all $t, s > 0$ with some given real number c. This means that, for $f' = f - c$,

$$f'(t^2 s) - (2f'(t) + f'(s)) = 0, \quad t, s > 0.$$

Substituting $t = s = 1$, we obtain $f'(1) = 0$. Substituting $s = 1$, we get $f'(t^2) = 2f'(t)$ and finally

$$f'(ts) = f'(t) + f'(s), \quad t, s > 0.$$

Considering the function $t \mapsto f'(\exp(t))$, we have a continuous real function which is additive and hence linear, implying that it is a scalar multiple of the identity. Consequently, f' is a scalar multiple of the logarithmic function and $f = a \log + b$ holds with some real scalars a, b.

Clearly, a is nonzero and the equality (3.1) can be rewritten as

$$l(\log(ABA) - (2\log(A) + \log(B))) = d, \quad A, B \in \mathcal{A}_+^{-1},$$

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with $d = (2b/a)l(I)$. Inserting $A = B = I$, it follows that $d = 0$ and hence

$$l(\log(ABA)) = 2l(\log(A)) + l(\log(B))), \quad A, B \in \mathcal{A}_+^{-1}. \tag{3.2}$$

Now, the validity of (3.2) implies that the linear functional l is tracial. In fact, this is the content of [2, Lemma 15]. For the sake of completeness, we present the proof. First pick projections P, Q in $\mathcal{A}$. Let

$$A = I + tP, \quad B = I + tQ,$$

where $t > -1$ is any real number. Easy computation shows that

$$ABA = (I + tP)(I + tQ)(I + tP) = I + t(2P + Q) + t^2(P + PQ + QP) + t^3(PQP).$$

Recall that in an arbitrary unital Banach algebra, for any element a with $\|a\| < 1$,

$$\log(1 + a) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} a^n}{n}.$$

This shows that for a suitable positive real number ϵ, the elements $\log(ABA)$, $\log A$ and $\log B$ of $\mathcal{A}$ can be expressed by power series of t ($|t| < \epsilon$) with coefficients from the algebra. In particular, considering the coefficients of t^3 on both sides of the equality (3.2) and, using their uniqueness, we obtain the equation

$$\begin{aligned} &l(PQP - \tfrac{1}{2}((2P + Q)(P + PQ + QP) + (P + PQ + QP)(2P + Q)) + \tfrac{1}{3}(2P + Q)^3) \\ &\quad = l(\tfrac{1}{3}(P + Q + P)). \end{aligned}$$

Executing the operations and subtracting those terms which appear on both sides of this equation, we arrive at the equality

$$l(\tfrac{1}{3}(PQP) - \tfrac{1}{3}(QPQ)) = 0.$$

Therefore,

$$l(PQP) = l(QPQ)$$

holds for all projections $P, Q \in \mathcal{A}$.

We claim that this implies that l is tracial. To see this, select an arbitrary pair P, Q of projections in $\mathcal{A}$, define $S = I - 2P$ and compute

$$\begin{aligned} l(Q + SQS) &= \tfrac{1}{2} l((I - S)Q(I - S) + (I + S)Q(I + S)) \\ &= \tfrac{1}{2} l(4PQP + 4(I - P)Q(I - P)) \\ &= 2l(PQP + (I - P)Q(I - P)) \\ &= 2l(QPQ + Q(I - P)Q) = 2l(Q). \end{aligned}$$

Since the symmetries (that is, the self-adjoint unitaries) in $\mathcal{A}$ are exactly the elements of the form $S = I - 2P$ with some projection $P \in \mathcal{A}$, we see that $l(Q) = l(SQS)$ holds for every symmetry S and every projection Q in $\mathcal{A}$. By the continuity of the linear

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functional l and the spectral theorem, we infer that $l(X) = l(SXS)$ holds for any $X \in \mathcal{A}$ and symmetry $S \in \mathcal{A}$. This implies that

$$l(SX) = l(S(XS)S) = l(XS) \tag{3.3}$$

for every $X \in \mathcal{A}$ and symmetry $S \in \mathcal{A}$. Plainly, this shows that $l(PX) = l(XP)$ for every projection $P \in \mathcal{A}$. Finally, we conclude that $l(XY) = l(YX)$ for all $X, Y \in \mathcal{A}$, that is, l is a nonzero tracial bounded linear functional. By Proposition 2.1, it follows that the factor $\mathcal{A}$ is of finite type and l is a (nonzero) scalar multiple of the trace. In particular, $l(I) \neq 0$ and, since $0 = d = (2b/a)l(I)$, it follows that $b = 0$, yielding $f = a \log$.

To see that $\mathcal{S}_f^{\mathcal{A}}$ equals the space of all trace zero elements of $\mathcal{A}$, observe that above we have seen that any nonzero bounded linear functional l on $\mathcal{A}$ with the property $\mathcal{S}_f^{\mathcal{A}} \subset \ker l$ is necessarily a scalar multiple of the same linear functional, namely, the trace. This implies that $\mathcal{S}_f^{\mathcal{A}}$ must equal the kernel of the trace, that is, it equals the space of all trace zero elements of $\mathcal{A}$. The proof of the theorem is complete. □

We remark that one can easily find other variants of our result. Here we mention the following one. For any pair $A, B \in \mathcal{A}_+^{-1}$ of positive invertible elements, we denote by $A\#B$ their geometric mean, that is, $A\#B = A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}$.

Corollary 3.1. *Let $\mathcal{A}$ be a von Neumann factor and $f : (0, \infty) \to \mathbb{R}$ be a nonconstant continuous function. Set*

$$\mathcal{L}_f^{\mathcal{A}} = \overline{\mathrm{span}}\{f(A\#B) - (1/2)(f(A) + f(B)) : A, B \in \mathcal{A}_+^{-1}\}.$$

Then either $\mathcal{L}_f^{\mathcal{A}} = \mathcal{A}$ or $\mathcal{L}_f^{\mathcal{A}} \subsetneq \mathcal{A}$. In the latter case, $\mathcal{A}$ is finite, $f = a \log + b$ holds for some constants a, b with $a \neq 0$ and $\mathcal{L}_f^{\mathcal{A}}$ equals the set of all trace zero elements of $\mathcal{A}$.

Proof. We only sketch the proof. First observe that by replacing f by the function $f - f(1)$ we may and do assume that $f(1) = 0$. If $\mathcal{L}_f^{\mathcal{A}} \subsetneq \mathcal{A}$, then we have a nonzero bounded linear functional l on $\mathcal{A}$ such that

$$l(f(A\#B) - (1/2)(f(A) + f(B))) = 0, \quad A, B \in \mathcal{A}_+^{-1}.$$

In the same way as in the proof of Theorem 1.1,

$$f(\sqrt{ts}) - (1/2)(f(t) + f(s)) = 0$$

for any real numbers $t, s > 0$ and we easily deduce that $f = a \log$ with some scalar a. Since f is assumed to be nonconstant, it follows that $a \neq 0$ and

$$l(\log(A\#B) - (1/2)(\log(A) + \log(B))) = 0, \quad A, B \in \mathcal{A}_+^{-1}.$$

It is known that $A\#B$ is the unique solution $X \in \mathcal{A}_+^{-1}$ of the equation $XA^{-1}X = B$ (the Anderson–Trapp theorem). Therefore, the above displayed equation is equivalent to

$$l(2\log(X) - (\log(A) + \log(XA^{-1}X))) = 0, \quad A, X \in \mathcal{A}_+^{-1}$$

and, replacing A by A^{-1}, this is equivalent to

$$l(\log(XAX) - (2\log(X) + \log(A))) = 0, \quad A, X \in \mathcal{A}_+^{-1}.$$

By the proof of Theorem 1.1, we already know that this implies that the algebra $\mathcal{A}$ is finite and l is a constant multiple of the trace. The proof can now be completed easily. □

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References

[1] R. V. Kadison and J. R. Ringrose, *Fundamentals of the Theory of Operator Algebras,* Vol. II (Academic Press, New York, 1986).

[2] L. Molnár, 'General Mazur–Ulam type theorems and some applications', in: *Operator Semigroups Meet Complex Analysis, Harmonic Analysis and Mathematical Physics*, Operator Theory: Advances and Applications (eds. W. Arendt, R. Chill and Y. Tomilov), **250** (2015), 311–342.

[3] G. K. Pedersen, *C^*-Algebras and their Automorphism Groups* (Academic Press, London–New York–San Francisco, 1979).

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